MATHEMATICAL PAPERS

CAMBRIDGE UNIVERSITY PRESS WAREHOUSE,
C. F. CLAY, MANAGER.
London: FETTER LANE, E.C.
Glasgow: 50, WELLINGTON STREET.

Leipzig: F. A. BROCKHAUS.
New York: G. P. PUTNAM'S SONS.
Bombay and Calcutta: MACMILLAN & CO., LTD.

THE COLLECTED MATHEMATICAL PAPERS

OF

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VOLUME II

(1854—1873)

WITH TWO PLATES

Cambridge
At the University Press
1908

Cambridge
PRINTED BY JOHN CLAY, M.A.
AT THE UNIVERSITY PRESS.

PREFATORY NOTE.

AMONG the Papers contained in this Volume are, the Author's Lecture on Geometry, delivered before the Gresham Committee (No. 2), the Author's seven lectures on the Partition of Numbers, in outline (No. 26), the long memoir on Newton's Rule (No. 74), and the Presidential Address to the Mathematical and Physical Section of the British Association at Exeter (No. 100). The papers here numbered 87, 88, 89 and 94 were republished together with the title "Nugae Mathematicae," and are found in the British Museum Catalogue under that name.

As in the first Volume, save for obvious errors of algebraical formulae, the papers are reprinted unaltered, cross references, and—in a few cases—indications of correction, being enclosed in square brackets.

A Table of Contents is prefixed, a General Index being deferred to the last Volume.

H. F. BAKER.

ST JOHN'S COLLEGE, CAMBRIDGE.
2 *March* 1908.

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1.

ON THE DOUBLE SQUARE REPRESENTATION OF PRIME AND COMPOSITE NUMBERS.

[*York British Association Report,* (1844), Part II. p. 2.]

"THE author first alluded to what had been done by the French mathematicians; and then pointed out the manner in which he thought numbers might be conceived to be composed of squares; and concluded by mentioning some of the advantages which might be expected from this mode of considering them."

2.

A PROBATIONARY LECTURE ON GEOMETRY, DELIVERED BEFORE THE GRESHAM COMMITTEE AND THE MEMBERS OF THE COMMON COUNCIL OF THE CITY OF LONDON, 4 DECEMBER, 1854.

[PRECEDED BY A DEDICATION TO THE MEMBERS OF THE GRESHAM COMMITTEE, MENTIONED BY NAME.]

PREFACE.

To account for the manifold short-comings of the annexed Lecture, it may be excusable and is indeed needful to state the circumstances under which it was written and delivered.

The Author having declared himself a Candidate for the vacant Professorship of Geometry in Gresham College, received a notice of little more than eight and forty hours, that he would be required to deliver a Probationary Lecture on Monday the 4th inst., before the Trustees on the City side of the Gresham Trust.

Matters of pressing importance happening at that moment to absorb his whole attention, he addressed a letter to the Secretary of the Trust, containing an urgent request that he might have the delay of a week for preparation; but his application having been sent too late in the day to obtain a reply, the Author deemed it his duty (not knowing how far his absence might derange the intended proceedings, of the precise nature of which he was unaware) to arm himself with a lecture of some kind, and for better or for worse, to appear to his summons at the appointed place and time. Accordingly, under the necessity of the case, the following pages were commenced and finished at a single sitting of a few hours' duration; the Author being so pressed for time that he had not even enough of it at his disposal to write out a fair copy of the manuscript.

The Lecture, with unimportant exceptions, such as the insertion of the closing paragraph (which was felt but not spoken), the occasional retrenchment of an exuberant expression, or the toning down of an over-florid passage, is printed as it was composed and delivered.

It must not be regarded as a criterion of what the Author could produce, with sufficient leisure, and the usual aids to reflection and research at his command, and still less as a specimen of the kind of lecture which he would consider adapted to a professorial course; but as the hasty outpouring of some of the thoughts lying at the threshold of the subject, and happening at the moment of composition to be most present to his mind. However, with all its imperfections on its head, the Author has deemed that he would be wanting in

proper deference to his judges, especially to such of them as were unable to attend in their places on the day of probation, were he to fail to afford them an opportunity of considering it in print. The views put forth and the opinions expressed are, at all events, the result of the Author's own reflections and of the questioning of his own mind, and not of a foray upon the works of the standard writers on the subject.

If his example in printing his Lecture should be followed by any of the numerous body of gentlemen who lectured after him, he will have much gratification in feeling that he has been instrumental in causing the public to participate in the pleasure which he derived from the many excellent discourses which were pronounced on the occasion referred to, and which he hopes to experience again, in listening to the corps of gentlemen from the country, who remain to bring up the reserve of the little army of probationers, on the field day appointed for the 11th instant. He can say unaffectedly and knows that this opinion was shared by many of his fellow-listeners, that the marked variety in the views and manner of treatment adopted by the several lecturers following one another in rapid succession on the same subject, rendered the *concours*, held under the auspices of the Committee and in the presence of the Common Council of the City of London, on the 4th instant, in Gresham Hall, one of the most interesting exhibitions of character and mental differences at which he had ever the good fortune to be present; one, he believes, of most uncommon occurrence, if not altogether unprecedented and unique in this country.

The free and obviously improvised review of his opinions, to which each lecturer was subjected in turn at the hands of those who came after him, threw additional life and spirit into the scene. With scarcely an exception, these light arrows of criticism were untipped with venom and passed sportively to their destination, striking without wounding, or glanced harmlessly off from the impervious shield of good humour interposed to receive them. The usual right of reply under attack must, it is presumed, in this instance be reserved until a fresh vacancy occurs and the same parties re-assemble in the college*.

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* The Author will only so far forestall the arrival of the period (*quod longum absit!*) above alluded to, by protesting against the abuse of the word "practical" as employed by an ingenious lecturer who succeeded him at the desk.

To discourse fluently on things of practice is no sufficient evidence in itself of the possession of a practical mind. The first rule of practice is to do all things at the right time and in their proper places, to proportion the means to the ends and the ends to the means, above all to know what is possible, and to confine one's endeavours within the limits of the feasible.

The Author allows and has habitually acted on the principle that for the purpose of *illustrating* lectures on geometry or any other abstract science, the lecturer should lay his hands on the plough, the loom, the forge, the workshop, the mine, the sea, the stars, all things on earth or under heaven, which may help to arouse the attention or interest the imagination of his auditors. But to profess to make the mere applications of a science such as geometry, the staple of the matter to be taught within the walls of the college by the Gresham Lecturer, to undertake to comprise within a course of geometrical lectures systematic *instruction* in mechanics, astronomy and navigation, descriptive geometry, engineering and drawing, the method of interpolation, the theory of toothed wheels, the two kinds of perspective, machinery, mapping, the art of ship-building, rules for cutting the best form of screws, and for enabling the citizens of London to qualify themselves for being their own land-surveyors, is a suggestion which, with all due deference to its propounder, the author regards as one of the wildest and most visionary which ever entered into the mind or issued from the lips of a practical man.

A long life would not suffice to exhaust the circle of the applications of geometry. Sir Thomas Gresham, a true philosopher and man of practical wisdom, ordained that courses of lectures should be delivered on his Foundation, not upon the applications of the sciences but upon the sciences themselves; well knowing that he who has mastered the principles of a science will be capable of making for himself, whenever required, those specific applications of them, which the

The Author fully agrees with the sentiment expressed by one of the candidates, and emphatically assented to by several members of the Committee, a sentiment which he confidently anticipates will be found to actuate the whole honourable body of the electors, in the decision about to be pronounced by them, in the double and each so sacred character, of Trustees and Judges, which is, that without regard to any other claim than that of capacity and desert, the worthiest candidate may be the one preferred. With his whole heart he is ready to exclaim,

DETUR DIGNIORI.

LECTURE.

MR CHAIRMAN AND GENTLEMEN OF THE GRESHAM COMMITTEE*,

In compliance with your requisition, although upon a very brief and unexpected notice, I have felt it my duty not to shrink from appearing before you this day to deliver a lecture upon the science, of which the chair now stands vacant in Gresham College. A consciousness of want of sufficient opportunity for preparation on my part, and a consideration of the wearisome and laborious duty which you, Gentlemen, have undertaken to perform in listening to a succession of lecturers on the same subject, conspire to impress upon me the importance of condensation and brevity.

I do not propose to tax your patience by entering upon any elaborate discussion of the principles of geometry, its history, its methods, or its applications.

It would be a vain endeavour to seek to convey within the limits of a single lecture any adequate notion of the scope of a science which has engaged the attention and grown up amidst the accumulated labours and meditations of the greatest minds and most profound thinkers of ancient and modern times, which, for the last two or three thousand years, has pursued an almost unbroken course of development and progression, and, still flourishing in all the vigour and freshness of early youth, bids fair to furnish occupation to the reasoning and inventive faculties of mankind for ages yet to come.

I conceive that the purpose for which I have been summoned before you will be best attained and our time most profitably employed, if I confine myself to the suggestion of a

peculiar circumstances of his calling or his opportunities in life may render advisable, and that he who has surprised the citadel will have no difficulty in carrying the outworks.

In writing with reluctance and under a sense of duty, the above remarks, the author begs most distinctly to disclaim the intention of leading it to be inferred that he considers the statement of opposite views as constituting the slightest ground of disqualification in the candidate who gave expression to them. He cannot but surmise that they were hazarded under the exigency of the occasion and in the absence of sufficient time for mature reflection. It has appeared to him however to be not the less necessary on that account, seeing that they were put forth with considerable plausibility, and chime in with some confused notions of what is really useful and practical, which are too prevalent at the present day, and would if carried out be fatal to the cause of sound education, to express his own dissent from them, and to meet them with an immediate refutation. He would be doing pain to his feelings were he not to add that he entertains a high respect for the abilities of the gentleman whose opinions he has felt himself under the necessity of controverting, and that he considers him to occupy a place amongst the foremost rank of those whose election by the Committee would be received with satisfaction by the Public.

* In the place of the Lord Mayor, who was unavoidably absent, Mr Deputy Holt presided.

few general ideas, which will admit of being rendered easily intelligible to any sensible man who, without having made geometry his special study, may be disposed to form some conception of the nature of its subject-matter and the relation which it bears to the other mathematical sciences.

There are three ruling ideas, three so to say, spheres of thought, which pervade the whole body of mathematical science, to some one or other of which, or to two or all of them combined, every mathematical truth admits of being referred; these are the three cardinal notions, of Number, Space and Order*.

Arithmetic has for its object the properties of number in the abstract. In algebra, viewed as a science of operations, order is the predominating idea. The business of geometry is with the evolution of the properties and relations of Space, or of bodies viewed as existing in space; it is true that the ideas of quantity and order enter largely into the developments of the science, but its proper purpose and foundation, that which confers upon it its distinctive name and character, is the contemplation of the properties of space or of the relations of the parts of space to one another, and only through this, or the equivalent notion of infinite extension, can any approach be made to a just appreciation of its objects.

It is the province of the metaphysician to inquire into the nature of space as it exists in itself, or with relation to the human mind. The less aspiring but more satisfactory business of the geometer is to deal with space as an objective reality, and to view it in its relation to matter, and as the substratum or the condition necessary to the existence of our conception of form.

The first property which strikes the mind in dwelling upon the idea of space is its infinitude, its capacity of boundless extension. If we stretch our thoughts to the very verge of the universe we are still unable to conceive space as come to an end and are constrained to admit the existence of further space beyond.

We may next contemplate space with reference to its modes of extension. We frequently hear of space having three dimensions; that there exist subordinate forms of space in which one or more of these dimensions are wanting; we are all familiar with such forms of speech, with the ideas attaching to the terms solidity, surface, linear magnitude or direction; let us inquire how the notions which they convey may be conceived to arise.

If we imagine a solid figure indefinitely expanded and extended in all directions, we fall back upon the idea of infinite space. If this space be conceived to be subjected to an ideal separation into two parts, distinct but contiguous, the boundary of each such part will give rise to the notion of a superficies, which may be conceived as co-extensive with the space from which it is derived, and like it, infinite in extent. A continuation of this process, that is to say, the dichotomy of superficies in its turn into distinct contiguous parts, gives birth to the notion of an infinite line; a surface limits space, a line limits a surface, and is thus a limitation upon a limitation; now again, conceive a line to separate into two parts and we arrive at the notion of a point, the lowest term in the scale of geometrical being; for here our analysis comes to an end; we have arrived at a limitation of the third order and can go no further; the point admits of no division. Thus it is we become able to attach a distinct meaning to the well known axiom or definition of the old geometers,

* The subject-matter of the notion of abstract order or arrangement is undoubtedly time; thus number, space and time may be said to be the three mathematical categories giving birth to three pure mathematical theories, viz.: arithmetic, the most abstract of all, next tactic or the doctrine of aggregation, and finally geometry, or topic. Each of these again has a double aspect and admits of being pursued in a descriptive and in a quantitative direction.

that "a point is that which hath no parts." Three several times, as we have seen, the process of division, of dual separation may be applied; but we come at last to that which has no parts, and which consequently, or rather by the very force of the term, is insusceptible of further division.

To say that every solid has length, breadth and depth, and therefore that space has three dimensions, is to convey a very inadequate explanation of the fact to be accounted for; for if we consider a spherical or any other body bounded by a continuous surface without angles or edges, we see nothing to indicate the existence in it of three or any other specific number of directions of measurement; it is true that through the idea of quantity, we may compare any solid whatever with a cube of equal magnitude, which of course will possess three definite directions or dimensions of linear admeasurement; but this is at best a very indirect and imperfect mode of arriving at the notion of the property in question, the property of three-foldness, inherent as a quality in the conception of space under the most general and absolute point of view.

Having thus acquired a notion of surfaces and lines in general, it becomes important to limit our attention in the first instance to the study of the simplest forms of each, and here our intuitions evolved by the latent force of early and unremitting observation, experience and induction, present our minds with the plane and sphere, as the elementary forms of surface, and the right line and circle as the corresponding simplest forms of lines.

A plane surface should be always conceived for the purposes of the geometer as extending out indefinitely in all directions; it consists of parts capable of exact superposition each over every other, so that if two portions of a plane be supposed to be brought together they cannot contain a closed hollow between them. A plane may be folded down over itself, and if two planes coincide in three points, they must coincide throughout their whole infinite extent.

Different as a sphere and a plane surface may at first sight appear to be, they possess many properties in common; the most striking difference between them, but which turns out to be comparatively unimportant under a mathematical point of view, consists in the circumstance of the plane being free and unlimited in extent, whereas a sphere is a closed and bounded surface; but in the property of the parts of either being similar *inter se*, and capable of superposition upon one another, there is a perfect resemblance between the sphere and the plane. Nor is it at all necessary to consider the sphere as a result of the idea of the circle, or to define it as Euclid does, as produced by the revolution of a circle about its diameter; we may even form a complete notion of a sphere by regarding it as a simple whole without any express reference to a centre or radii, as a surface containing a solid figure and capable of moving in its own place, without encroaching upon the neighbouring parts of space exterior to itself.

As the plane and sphere are the simplest of all surfaces, so the right line and circle are the simplest of all lines. The right line and circle, like the plane and sphere, are each moveable in their own place, that is, they admit of their parts being shifted upon one another without any absolute change of place in the entire line. There is only one other line in nature, namely the screw line (well known as the helix or Archimedes' screw), which possesses this property of self-similarity, which is the final reason why all the simple mechanical powers exhibit only three sorts of motion, namely, rectilinear, circular, and helical; thus in the lever and toothed wheels circular motion is exemplified; in the pulley and inclined plane, rectilinear motion; and finally helical motion in the screw, such as is used in an ordinary press. I need hardly add that it is the screw of Archimedes which has lent a new power to steam navigation, and which imparts to the rifled barrel its sure and deadly aim.

The foundation of the ancient (indeed it may be said of all) geometry is laid in the contemplation of the properties of figures, capable of being drawn upon a plane, and especially of the simplest of these, namely, the right line and circle.

From the time of Thales, who flourished about 600 years before the birth of Christ, and is reputed to have been the first to bring geometry from the land of the Pharaohs to find a more genial home in Greece, down to the time of Plato, two centuries later, the attention of geometers appears to have been almost exclusively confined to the study of the properties of these simple species of form, and as derived from them, of the sphere and solid figures bounded by plane faces.

The discovery of the conic sections, attributed to Plato, first threw open the higher species of form to the contemplation of geometers*. But for this discovery, which was probably regarded in Plato's time and long after him, as the unprofitable amusement of a speculative brain, the whole course of practical philosophy at the present day, of the science of astronomy, of the theory of projectiles, of the art of navigation, might have run in a different channel; and the greatest discovery that has ever been made in the history of the world, the law of universal gravitation, with its innumerable direct and indirect consequences and applications to every department of human research and industry, might never to this hour have been elicited.

This law, as you are aware, is deduced from the motions of the heavenly bodies in their orbits; no correct system of physical astronomy, no knowledge of the forces binding together the distant parts of the universe was possible, until the form of their orbits had been correctly ascertained by observation.

It is to Kepler, Newton's precursor, that we are indebted for this important information. He it was who discovered that the motion of a planet is not circular nor derived from any combination of circular movements, as was previously supposed to be the case, from a perfection idly supposed to be inherent in that figure, which rendered it alone worthy to image the movements of the heavenly bodies. Kepler discovered that the true form of a planet's orbit is that of an oval perspective projection of a circle, familiar to the geometricians of the Platonic school under the name of an elliptic section of the cone; such also is the general form of the orbits of the moon, of the satellites to the other planets, and in a word of all the bodies in nature revolving about centres of force, subject only to deviations of more or less consequence, arising from disturbing forces for which geometry is perfectly able to account. Thus (as I have said) it was, that the way was laid open to the discovery of this great secret of nature. Little could Plato himself have imagined, when, indulging his instinctive love of the true and beautiful for their own sakes, he entered upon these refined speculations and revelled in a world of his own creation, that he was writing the grammar of the language in which it would be demonstrated in after ages that the pages of the universe are written.

As Plato and Pythagoras before him, the two greatest philosophers of ancient times, have stamped their names upon, and indissolubly associated their memories with the history of the geometry of their period, so the new geometry which has arisen in later days, and achieved still higher triumphs than its elder-born sister, may be said to have taken its origin in the methods invented by Descartes and Pascal, the great philosophical luminaries of modern times. It may be doubted whether Newton could have ever risen to the heights which he attained had not Descartes lived and written before him, and it may be difficult to pronounce the existence of which of the two, Kepler or Descartes, ought to be considered as the more essential link in the order of events prepared by

* Here the Lecturer with the aid of a model showed how the different species of plane conics may be obtained from the dissection of a solid cone.

providence, to furnish the materials to be elaborated by the genius of Newton and to fit it for its lofty appointed work.

But I must not allow myself to be tempted into the facile and seductive path of historical investigation or comparison, which would carry me far beyond the limits which I prescribed to myself at the outset of this discourse, or than I can hope to carry your indulgent attention along with me. For obvious reasons also I think it would be inexpedient to attempt in this place a description of the difference between the spirit and methods of the ancient and those of the modern schools of geometry. I shall prefer to occupy the short remaining period of the lecture with inviting your attention to a distinction which lies deeper in the subject-matter of the science itself, being drawn from its relation to the two leading attributes of space, namely, magnitude and direction. I allude to a distinction, which or the like to which, runs through every branch of mathematical speculation, and has its analogue even in the study of the natural sciences, such as chemistry, botany and anatomy.

When we have attained a certain elevation in our view of the subject, and can look down upon the territory which we have traversed to arrive there, we begin to perceive that geometry resolves itself naturally into two great divisions, geometry of position and geometry of measurement, geometry descriptive* or morphological and geometry quantitative or metrical. The ancients chiefly concerned themselves with the metrical properties of space; the more subtle and essential spirit of the science, however, probably resides in the purely descriptive part. A single proposition selected from each may serve to place the distinction between these two provinces of inquiry in a clearer light.

If we draw any two triangles upon the same base, say for instance along this floor where the wall meets it, terminating respectively in two points, (so chosen that their line of junction shall be parallel to the base line) as for instance to two points in the line running along the cornice of the room, it is easily proved that the two triangles so formed, will be of equal superficial magnitude; this would be true although the apex of one of them were taken anywhere along the actual line of the ceiling, but the other in a prolongation of the cornice stretching out a hundred miles away. Both triangles so obtained would contain the same number of square inches or square feet, although the measure of one round its periphery might be a thousand times greater than that of the other. This is an example of a metrical or quantitative proposition. Again, if we take a triangle and bisect each side and join each bisecting point with the opposite angle, it is a known property of the triangle that these three lines must meet one another, not as three lines taken at hazard would do, cutting out another triangle between them, but in one and the same point. This proposition is partly metrical and partly descriptive; it is descriptive so far as regards the property of the bisecting lines passing through the same point; quantitative in so far as the idea of a line being bisected implies a notion of the relative magnitudes of the equal parts.

Propositions however exist which are purely descriptive; as for instance, the celebrated theorem of Pascal known under the name of the Mystic Hexagram, which is, that if you take two straight lines in a plane, and draw at random other straight lines traversing in a zigzag fashion between them, from A in the first to B in the second, from B in the second to C in the first, from C in the first to D in the second, from D in the second to E in the first, from E in the first to F in the second and finally from F in the second back again to A the starting point in the first, so as to obtain $ABCDEF$ a twisted hexagon, or sort of cat's-cradle figure and if you arrange the six lines so drawn symmetrically in three couples: viz. the 1st and 4th in one couple, the 2nd and 5th in a second couple, the 3rd

* The word "descriptive" is here employed *out* of its technical sense.

and 6th in a third couple; then (no matter how the points ACE have been selected upon one of the given lines, and BDF upon the other) the three points through which these three couples of lines respectively pass, or to which they converge (as the case may be) will lie all in one and the same straight line.

This is a purely descriptive proposition, it refers solely to position, and neither invokes nor involves the idea of magnitude. The existence, I will not say of a class, but of a whole world of truths of this kind, truths undeniably geometrical in their nature, serves to show how imperfect is the definition once generally accepted of geometry (however conformable to the etymology of the word and the early history of the subject), which described it as the science of the measurement of magnitude, in a word as a science of mensuration, which is in fact only one and that a subordinate division of the science. Sciences, true sciences, spring from celestial seeds sown in a mortal soil, they outgrow the restrictions which human shortsightedness seeks to impose upon them, and spread themselves outwards and upwards to the heavens from whence they derive their birth. We may write learnedly upon the history of geometry, upon its origin, growth, and apparent tendencies; but there is that within it which baffles our predictions and sets at nought our calculations as to the uses to which it may hereafter be turned and the form which it may be finally destined to assume; that which, analogous to the vital principle in an organized being, resists the circumscription of language and defies mere verbal definition.

It has been said that to appreciate what virtue and morals mean, men must live virtuous and moral lives. It is equally true, that a knowledge of the objects of science is not to be attained by any scheme of definitions however carefully contrived. He who would know what geometry is, must venture boldly into its depths and learn to think and feel as a geometer. I believe that it is impossible to do this, to study geometry as it admits of being studied and am conscious it can be taught, without finding the reasoning invigorated, the invention quickened, the sentiment of the orderly and beautiful awakened and enhanced, and reverence for truth, the foundation of all integrity of character, converted into a fixed principle of the mental and moral constitution, according to the old and expressive adage "*abeunt studia in mores.*"

I have now only to thank you, Mr Chairman and Gentlemen, for the patient attention which you have accorded to me, and to assure you with perfect sincerity, that if I should have the honour of being selected by you for the permanent occupation of the chair which I this day fill upon trial, I shall not treat the appointment as a sinecure, nor content myself with discharging the mere duties of routine. Far otherwise! if accredited by you to teach publicly a science, the object of my passionate fondness and earliest predilection, to propagate a taste for which would be to me, not merely a labour of duty but of love, I should strive, both in and out of the lecture room, to respond to the intentions of your enlightened and munificent Founder, by imparting freely to all who might approach me for the purpose, advice, encouragement, and sympathy, in their pursuit of mathematical truth, and I should labour with unceasing diligence to evince myself a worthy successor of the many eminent men, who have previously occupied here the chair which it is my ambition to obtain.

As one who has given pledges to the world of an earnest devotion to science, who lays claim to the possession of faculties which would find or create here a fitting theatre for their development, I appeal to your public spirit. I seek, Gentlemen, at your hands to be placed in a position which shall entitle me to take a part in bringing this noble Institution into connection with the great movement of national education now in progress throughout the land, and as a professor in this place, to be permitted to dedicate the past and future labours of my life to the promotion of the general good. The privilege to be useful is the crown of honour which I covet, and which it is in your power to bestow.

3.

NOTE ON SIR JOHN WILSON'S THEOREM.

[*Cambridge and Dublin Mathematical Journal*, IX. (1854), pp. 84, 85.]

THE following is probably the best and the briefest mode of deducing Sir John Wilson's Theorem and its cognate Theorems from Fermat's. I can say nothing as to its originality.

p being any prime number, let

$$(x-1)(x-2)(x-3)\ldots\{x-(p-1)\} = x^{p-1} + A_1x^{p-2} + A_2x^{p-3} + \&c. + A_{p-1}.$$

Let x successively take the values $1, 2, 3, \ldots (p-1)$; then to modulus p, by Fermat's Theorem, we have

$$x^{p-1} + A_{p-1} \equiv 1 + A_{p-1}, \text{ say } A_0,$$

and we derive the $(p-1)$ congruences to modulus p:

$$A_0 + A_1 + A_2 + A_3 \ldots\ldots\ldots\ldots\ldots + A_{p-2} \equiv 0,$$
$$A_0 + 2^{p-2}A_1 + 2^{p-3}A_2 + 2^{p-4}A_3 \ldots + 2A_{p-2} \equiv 0,$$
$$A_0 + 3^{p-2}A_1 + 3^{p-3}A_2 + 3^{p-4}A_3 \ldots + 3A_{p-2} \equiv 0,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$A_0 + (p-1)^{p-2}A_1 + (p-1)^{p-3}A_2 + (p-1)^{p-4}A_3 \ldots + (p-1)A_{p-2} \equiv 0.$$

Now the determinant formed by the coefficients of

$$A_0, A_1, A_2, \ldots A_{p-2}$$

is $1.2.3\ldots(p-1)$ multiplied into the product of the differences of $1, 2, 3, \ldots (p-1)$, and is therefore incongruent to zero for the modulus p. Hence, there being $(p-1)$ *independent* homogeneous congruences between $(p-1)$ quantities, each of these quantities must be congruent to zero, that is

$$A_0 \equiv 0,\ A_1 \equiv 0, \ldots A_{p-2} \equiv 0 \text{ [mod. } p].$$

The congruence $A_0 \equiv 0$, that is $1 + 1.2.3\ldots(p-1) \equiv 0$ [mod. p], is evidently Sir John Wilson's Theorem. We see also (by virtue of the remaining equations) at the same time, that the sums of the binary, ternary, &c., up to the $(p-2)^{\text{ary}}$ combinations of the numbers $1, 2, 3, \ldots (p-1)$, are all severally congruent to zero to the modulus p; that is, are all divisible by that number.

4.

ON THE CALCULUS OF FORMS, OTHERWISE THE THEORY OF INVARIANTS.

[*Continued from* p. 422 *of Volume I. of this Reprint.*]

[*Cambridge and Dublin Mathematical Journal,* IX. (1854), pp. 85—103.]

SECTION VII. (*Continued.*)

BEFORE proceeding further I must guard against a misconception as to my meaning to which the modification of the title of this memoir might give birth; it is not to be understood that I regard the Theory of Invariants as coextensive with the Calculus of Forms, but only with a certain portion of that Calculus which is here exclusively treated of; the Calculus of Forms itself has for its subject-matter the whole theory of the Composition, Decomposition, and Comparison of Forms. In the theory of invariants the composition of single forms with sets of linear forms is alone considered, and the idea of invariance must be regarded as a transient idea arising out of an artificial mode of viewing the effects of composition, so as to ignore the presence in the result of factors which depend on the resultants of the linear forms employed, which resultants, although in this portion of the subject treated as mere moduli and as such generally supposed to be reduced to unity, yet in regard to the general theory are as important as the factors which are retained as the sole objects of contemplation; so that in fact the idea of invariance is but a special and it may be said accidental notion which merges in the more general notion of permanency of character in the resultant of forms compounded in a given manner out of given forms. Again, as to combinants, the idea contained in this word may, by a change in the mode of statement of the definition, be extended to functions of unlike degrees. A combinant of U, V, W, ..., all functions of the same system or systems of variables, is in fact only another name for invariants of the function $\lambda U + \mu V + \nu W +$ &c., where, over and above the sets of variables contained in U, V, W, ... there is a new correlated set of variables λ, μ, ν, &c. So now, more generally, if U, V, W, ... are of p, q, r, ... dimensions in one set of variables, of which the highest number is I, if λ is taken of $I-p$, μ of $I-q$, ν of $I-r$, &c. dimensions in the same, the functions λ, μ, ν, &c. being each the most general of their kind, any

invariant of $\lambda U + \mu V + \nu W + \ldots$ which is such as well in respect to the coefficients in $\lambda, \mu, \nu, \ldots$ which must be considered as forming a set among themselves, as also in respect to the set of variables in $U, V, W, \ldots$ will be a combinant to the system $U, V, W, \ldots$; and so, more generally, if $U, V, W, \ldots$ contain several (say i) unrelated sets or systems of sets of variables, we must form in an analogous manner

$$\lambda_1\lambda_2 \ldots \lambda_i U + \mu_1\mu_2 \ldots \mu_i V + \nu_1\nu_2 \ldots \nu_i W + \&c.,$$

and then an invariant in respect to the i given sets in $U, V, W, \ldots$ and the i new sets contained in $(\lambda_1, \mu_1, \nu_1, \ldots)$, $(\lambda_2, \mu_2, \ldots \nu_2)$, &c. $(\lambda_i, \mu_i, \nu_i, \ldots)$ will be a combinant to the system $U, V, W, \ldots$. Perhaps, however, a more immediate extension of the idea of combinants to the case supposed of i unrelated sets or systems of sets would be to take, instead of $\lambda_1\lambda_2 \ldots \lambda_i$, $\mu_1\mu_2 \ldots \mu_i$, &c., the perfectly general forms of the same degrees in each set of the variables as these quantities are respectively of the same; to use these general forms, the coefficients of which will constitute not i new sets but a single new set of variables, as the syzygetic multipliers to $U, V, W, \ldots$, and then the invariant of the corresponding conjunctive in respect to the i original sets or systems of sets, and the one new set of variables thus obtained will be a combinant to the given system of functions*. As a matter of punctilio I may here take the opportunity of observing that the process for obtaining the relation between שׂ, ט (inadvertently written טּ), and R, would have been more perfectly symmetrical to the eye had the equation for W [p. 416 of Vol. I.] been written $\tau(z^2 - y^2) = W$ in lieu of $\sigma(y^2 - z^2) = W$. I now return to take up the subject from the point where it was brought to a close in the last number of the *Journal*.

Let us consider what the equation (A)† becomes when U, V, W become the first partial derivatives (quâ x, y, z) of a single homogeneous cubic function ψ, so that

$$U = \frac{d\psi}{dx}, \quad V = \frac{d\psi}{dy}, \quad W = \frac{d\psi}{dz};$$

ϑ then becomes the Hessian of ψ, and the S of this (like every other invariant of ψ)‡ may be expressed, as is well known, as a rational integral function of

* I propose to append at the end of the next or some subsequent Section what ought to have been given in this or previous place, viz. the general differential equations for any concomitant to any congeries of forms, comprising amongst them any number of various distinct (that is unrelated) classes of systems of sets of variables, the relations between the sets belonging to any one system being supposed to be either simple or compound, and after the manner of either cogredience or contragredience; in fact, to do this only requires a slight extension of the formulæ given by me with that object in the fifth section of my paper in the *Philosophical Transactions* for the year 1853, Part III., which see. [Vol. I., p. 551.]

† *Vide* last number of this *Journal*, near the end of Author's paper therein. [Vol. I., p. 420.]

‡ I have given a perfectly rigid demonstration in the *Philosophical Magazine*, in the early part of 1853, that every invariant to a cubic function of three variables is a rational integral function of the two Aronholdian invariants S and T. [Vol. I., p. 599.]

the S and T of ψ. The relation between the S of the H and the S and T may readily be obtained from the canonical form

$$(\psi) = x^3 + y^3 + z^3 + 6mxyz.$$

The Hessian of this is

$$(1 + 2m^3)\, xyz - m^2 (x^3 + y^3 + z^3),$$

and making $-\dfrac{1 + 2m^3}{6m^2} = \mu$, the S of this Hessian will be

$$6^4 \,.\, m^8 \times (\mu - \mu^4),$$

which is $$(1 + 2m^3) \{(1 + 2m^3)^3 + 216m^6\}.$$

That is
$$\begin{aligned} &1 + 8m^3 + 240m^6 + 464m^9 + 16m^{12}, \\ &= (1 - 20m^3 - 8m^6)^2 + 48\,(m - m^4)^3, \\ &= (S)^3 + 48\,(T)^3, \end{aligned}$$

where (S) and (T) are respectively the S and T of (ψ). Hence we have in general

$$S \,.\, H \,.\, \psi = (S\psi)^3 + 48\,(T\psi)^3.$$

So that שׁ becomes $T^2 + 48S^3$, and ט evidently from Calculus of Forms [cf. Vol. I., p. 311] becomes

$$\tfrac{16}{8}(1 - 20m^3 - 8m^6),$$

that is $2T$, so that $$4\text{שׁ} - \tfrac{1}{4}\text{ט}^2 = 3T^2 + 192S^3;$$

so that equation (A) becomes

$$R = T^2 + 64S^3,$$

the Aronholdian representation of the Discriminant of ψ.

We see from this numerical calculation that it is not $\Sigma\Omega$ but $\frac{1}{2}\Sigma\Omega$, which ought to receive the appellation of ט, making which modification the general equation, written (A), becomes

$$\tfrac{1}{3}R = 4\text{שׁ} - \text{ט}^2.$$

The שׁ it will be observed is a compound combinant, being a biquadratic function of quantities all of which are invariants of the system U, V, W; the ט on the other hand is a simple combinant of the sixth degree.

The general dodecadic combinant שׁ may also in another manner be exhibited as a biquadratic function of cubic functions of the coefficients of the three given quadratics; but these cubic functions will no longer be invariants of the given quadratics. Thus, form the Jacobian of U, V, W, that is, the determinant

$$\begin{vmatrix} \dfrac{dU}{dx}, & \dfrac{dU}{dy}, & \dfrac{dU}{dz} \\ \dfrac{dV}{dx}, & \dfrac{dV}{dy}, & \dfrac{dV}{dz} \\ \dfrac{dW}{dx}, & \dfrac{dW}{dy}, & \dfrac{dW}{dz} \end{vmatrix},$$

which will be a cubic covariant to the system. The S of this will be another form of שׁ. So too, again, if we border the matrix to the Jacobian determinant above written vertically and horizontally with ξ, η, ζ, and call the determinant of the matrix thus formed I', I' will be quadratic in the system x, y, z, in the system ξ, η, ζ, and in the system formed by the coefficients of U, V, W, and the result of affecting this with the operator Σ will be the same as the result of the operation upon Ω with the same symbol; that is to say, $\frac{1}{24}E \,.\, I'$ will be equal to ט, this latter symbol being so taken (as last explained) in such a way that $3R$ shall equal 4שׁ $-$ ט^2, and each of the four lines in the operator Σ being supposed to go through their complete number (6) of permutations.

The terms sextic and dodecadic combinants will not be sufficient *per se* to characterize שׁ or ט (to a numerical factor *près*), supposing that there exist combinants of the 3rd and 9th degrees respectively in the coefficients, in which case the general sextic would contain two and the general dodecadic five arbitrary numerical parameters.

This makes so much the more remarkable and satisfactory the method above developed for finding שׁ and ט as undecompounded forms; the general dodecadic combinant at all events being rendered indeterminate by virtue of the existence of a sextic combinant above demonstrated.

It is interesting to evince the identity of the S of the Jacobian with that of the discriminant to the conjunctive of U, V, W, which latter has been called שׁ.

Starting with the canonical forms of the system U, V, W, and neglecting the ρ and σ, which cannot influence the result of the intended comparison, we have

$$J(U, V, W) = \begin{vmatrix} x, & -y, & 0 \\ 0, & -y, & z \\ gz + hy, & hx + y + fz, & gz + hy \end{vmatrix}$$
$$= fx(y^2 + z^2) + gy(x^2 + z^2) + hz(x^2 + y^2) + xyz.$$

And multiplying by 6 and adopting the same notation as before (from the *Higher Plane Curves*, p. 182), we have

$$\begin{aligned} b_1 &= 2f, & b_2 &= 0, & b_3 &= 2h, \\ a_1 &= 0, & a_2 &= 2g, & a_3 &= 2h, \\ c_1 &= 2f, & c_2 &= 2y, & c_3 &= 0, \\ & & d &= 1. \end{aligned}$$

And the expression for S in *Higher Plane Curves*, p. 184, becomes, omitting every term containing a_1, b_2, or c_3,

$$d^4 - 2d^2(b_1c_1 + c_2a_2 + a_3b_3) + 3d(a_2b_3c_1 + a_3b_1c_2)$$
$$- (b_3c_2a_2a_3 + a_3c_1b_1b_3 + a_2b_1c_1c_2) + (b_1^2c_1^2 + c_2^2a_2^2 + a_3^2b_3^2),$$

that is

$$1 - 8(f^2 + g^2 + h^2) + 48fgh - 16(h^2g^2 + g^2f^2 + f^2h^2) + 16(f^4 + g^4 + h^4),$$

so that

$$\underset{x,\,y,\,z}{S}\ J(U,\ V,\ W) = \text{שׁ} = \underset{\lambda,\,\mu,\,\nu}{S}\quad \underset{x,\,y,\,z}{\square}\,(\lambda U + \mu V + \nu W),$$

as was to be shown. As observed above, the form first found has the advantage over the one just obtained in disclosing the elements (cubic invariants to U, V, W) of which the שׁ is a biquadratic function. So, analogously, the resultant of two quadratic functions (P, Q) of x and y may be exhibited either under the form of the discriminant in respect to the coefficients of conjunction of the discriminant in respect to the original variables of the conjunctive of P, Q, or under the form of the discriminant of the Jacobian of P, Q. The former discloses the invariantive composition of the resultant which remains latent in the latter. As regards the ט, the proof of its being capable of the second mode of generation above indicated must, on account of the tediousness of the calculation, be for the present reserved; nor can I assert the fact with entire confidence until I have made a more complete investigation into the combinants of the system U, V, W, the remarks concerning which, in p. [416, Vol. I.], I wish to be considered as provisionally withdrawn.

The analogy between the invariants of a cubic form of three variables and a biquadratic of two has been frequently insisted upon in the foregoing pages; but we shall now see that this analogy has its foundation in the deeper-seated analogy which connects a ternary system of quadratics of three variables with a binary system of cubics of two variables.

We may suppose the two given functions so combined that the linear conjunctive $lP + mQ$ shall contain two equal roots, and so take the form x^2y; this may then be combined with either of the given functions so as to give a conjunctive of the form

$$ax^3 + 3xy^2 + dy^3,$$

and writing for x and y, $\dfrac{x}{\sqrt[3]{a}}$, $\dfrac{y}{\sqrt[3]{d}}$, respectively, and multiplying $lP + mQ$ by $\sqrt[3]{a^2d}$, we obtain for our standard form

$$P = 3x^2y,$$

$$Q = x^3 + 3exy^2 + y^3.$$

The resultant of this system rejecting an universally-irrelevant numerical factor is 1.

Again, write

$$\lambda P + \mu Q = 3\lambda x^2y + \mu x^3 + 3\mu exy^2 + \mu y^3,$$

and operate upon this with the commutator (say ω) [see Vol. I., p. 306],

$$\begin{vmatrix} \frac{d}{d\lambda}, & \frac{d}{d\mu} \\ \frac{d}{dx}, & \frac{d}{dy} \\ \frac{d}{dx}, & \frac{d}{dy} \\ \frac{d}{dx}, & \frac{d}{dy} \end{vmatrix}.$$

Keeping one of the lines (for example, the first) stationary, and, for greater brevity, writing δ_λ, δ_μ, δ_x, δ_y in place of $\frac{d}{d\lambda}, \frac{d}{d\mu}, \frac{d}{dx}, \frac{d}{dy}$, we obtain 8 positions, which, remembering that the order *in* the lines of these positions (and not the order *of* the lines) is the only thing to be attended to, are equivalent to

$$\begin{vmatrix} \delta_\lambda, & \delta_\mu \\ \delta_x, & \delta_y \\ \delta_x, & \delta_y \\ \delta_x, & \delta_y \end{vmatrix} - 3 \times \begin{vmatrix} \delta_\lambda, & \delta_\mu \\ \delta_x, & \delta_y \\ \delta_x, & \delta_y \\ \delta_y, & \delta_x \end{vmatrix} + 3 \times \begin{vmatrix} \delta_\lambda, & \delta_\mu \\ \delta_x, & \delta_y \\ \delta_y, & \delta_x \\ \delta_y, & \delta_x \end{vmatrix} - \begin{vmatrix} \delta_\lambda, & \delta_\mu \\ \delta_y, & \delta_x \\ \delta_y, & \delta_x \\ \delta_y, & \delta_x \end{vmatrix}.$$

Hence we have $\frac{1}{36}\omega(\lambda P + \mu Q) = -e$.

I need hardly observe, that in general for any two odd-degreed functions of the same degree in x, y, as

$$a_0x^m + ma_1x^{m-1}y + \tfrac{1}{2}m(m-1)\,a_2x^{m-2}y^2 + \ldots + m\,(a_1)\,xy^{m-1} + (a_0)\,y^m,$$
$$b_0x^m + mb_1x^{m-1}y + \tfrac{1}{2}m(m-1)\,b_2x^{m-2}y^2 + \ldots + m\,(b_1)\,xy^{m-1} + (b_0)\,y^m,$$

we may obtain, in an analogous manner, the combinant

$$a_0(b_0) - ma_1(b_1) + \tfrac{1}{2}m(m-1)\,a_2(b_2) + \&c.$$

Moreover it is easily shown that when m is an even integer the above expression will remain invariant, although of course it is no longer a combinant.

Again, the Hessian to $\lambda P + \mu Q$ will be

$$\begin{vmatrix} \mu x + \lambda y, & \lambda x + \mu e y \\ \lambda x + \mu e y, & \mu e x + \mu y \end{vmatrix},$$

which is equal to

$$e\mu^2x^2 + \mu^2xy - e^2\mu^2y^2 + \lambda\mu y^2 - e\lambda\mu xy - \lambda^2x^2,$$

which call $H.C$ (C meaning the conjunctive of P, Q). Let this be operated upon with the commutator

$$\begin{matrix} \delta_x^2, & \delta_x\delta_y, & \delta_y^2, \\ \delta_\lambda^2, & \delta_\lambda\delta_\mu, & \delta_\mu^2, \end{matrix}$$

which call Ω.

Since neither $y^2\lambda^2$ nor $xy\lambda^2$ enters in $H.C$, we have only to consider out of the full number 6 of positions the two effective positions

$$\begin{vmatrix} \delta_x^2, & \delta_x\delta_y, & \delta_y^2 \\ \delta_\lambda^2, & \delta_\lambda\delta_\mu, & \delta_\mu^2 \end{vmatrix} - \begin{vmatrix} \delta_x^2, & \delta_x\delta_y, & \delta_y^2 \\ \delta_\lambda^2, & \delta_\mu^2, & \delta_\lambda\delta_\mu \end{vmatrix}.$$

Hence $$\tfrac{1}{16}EHC(P, Q) = 1 - e^3.$$

So that $$\{-\tfrac{1}{36}\omega C(P, Q)\}^3 + \{\tfrac{1}{16}EHC(P, Q)\} = R(P, Q).$$

Thus R is expressed in terms of the cube of a simple quadratic combinant and a sextic compound combinant, which is made up of quadratic invariants. When P and Q become of the form $\frac{d\psi}{dx}, \frac{d\psi}{dy}$, respectively ($\psi$ being a quartic in x and y), these become respectively (to numerical factors *près*) the quadrinvariant of the given function and the cube invariant of its Hessian, which latter is a linear function of the cube of the quadrinvariant and the square of the cubinvariant of the given function, as we know *à priori* from the fact of the fundamental scale of the quartic consisting of the quadrinvariant and cubinvariant (for a rigid demonstration of which fact see the *Philosophical Magazine* in the early part of 1853 [Vol. I., p. 599]), and the expression for the resultant thus resolves itself into the known composite form of the sum of a square and cube.

The simple sextic combinant represented by $E.H.C(P, Q)$ may also, analogous to what has been observed concerning the ט, be expressed as a commutant (in fact the cubinvariant) of the Jacobian to P and Q, but then the form will no longer disclose its invariantive sub-composition. So too, if it were thought worth while to push the analogies to an extreme, the quadricombinant to P, Q might have been found, first by bordering the Hessian to the conjunctive to P, Q with ξ, η horizontally and vertically, and operating upon the result with the commutator

$$\begin{vmatrix} \frac{d}{dx}, & \frac{d}{dy} \\ \frac{d}{d\lambda}, & \frac{d}{d\mu} \\ \frac{d}{d\xi}, & \frac{d}{d\eta} \\ \frac{d}{d\xi}, & \frac{d}{d\eta} \end{vmatrix},$$

or by bordering the Jacobian to P, Q with ξ, η, as before, and then operating upon the result with the commutator

$$\begin{vmatrix} \frac{d}{dx}, & \frac{d}{dy} \\ \frac{d}{dx}, & \frac{d}{dy} \\ \frac{d}{d\xi}, & \frac{d}{d\eta} \\ \frac{d}{d\xi}, & \frac{d}{d\eta} \end{vmatrix}.$$

I propose hereafter to return to the consideration of the fundamental scale of combinants to the two systems, namely of three quadratics in x, y, z, and of two cubics in x, y, which have been treated of in this section.

SECTION VIII.

On the Reduction of a Sextic Function of Two Variables to its Canonical Form.

In the *London and Edinburgh Philosophical Magazine* for Nov. 1851, after giving a simple method for representing any function of two variables of an odd degree $(x, y)^{2m+1}$ under the form of

$$u_1^{2m+1} + u_2^{2m+1} + \ldots + u_{m+1}^{2m+1},$$

where $u_1, u_2 \ldots u_{m+1}$ are linear functions of x, y (which form, as appears from the method of obtaining it, is unique), I proceeded [Vol. I., p. 269] to show how by a certain method therein explained the biquadratic and octavic functions of $x, y, (x, y)^4, (x, y)^8$ could be thrown under the respective forms

$$u_1^4 + u_2^4 + mu_1^2u_2^2,$$

$$u_1^8 + u_2^8 + u_3^8 + u_4^8 + mu_1^2 u_2^2u_3^2u_4^2,$$

the number of values of m in the first form being three and in the second form five, the quantity m in the one case depending on the solution of the equation

$$\begin{vmatrix} a_0, & a_1, & a_2 + \lambda \\ a_1, & a_2 - \frac{1}{2}\lambda, & a_3 \\ a_2 + \lambda, & a_3, & a_4 \end{vmatrix} = 0,$$

where a_0, a_1, a_2, a_3, a_4 are the coefficients of $(x, y)^4$ multiplied respectively by $1, \frac{1}{4}, \frac{1}{6}, \frac{1}{4}, 1$; and in the other case, on the solution of the equation

$$\begin{vmatrix} a_0, & a_1, & a_2, & a_3, & a_4 + \lambda \\ a_1, & a_2, & a_3, & a_4 - \frac{1}{4}\lambda, & a_5 \\ a_2, & a_3, & a_4 + \frac{1}{6}\lambda, & a_5, & a_6 \\ a_3, & a_4 - \frac{1}{4}\lambda, & a_5, & a_6, & a_7 \\ a_4 + \lambda, & a_5, & a_6, & a_7, & a_8 \end{vmatrix} = 0,$$

$$a_0,\ a_1,\ a_2,\ a_3,\ a_4,\ a_5,\ a_6,\ a_7,\ a_8,$$

being the coefficients of $(x, y)^8$ multiplied respectively by

$$1,\ \tfrac{1}{8},\ \tfrac{1}{28},\ \tfrac{1}{56},\ \tfrac{1}{70},\ \tfrac{1}{56},\ \tfrac{1}{28},\ \tfrac{1}{8},\ 1.$$

Before proceeding to investigate the theory of these methods of reduction under any more general point of view, it will be convenient to seek to obtain the representation of $(x, y)^6$ under some analogous form.

It might at first be supposed that the corresponding form should be

$$u_1^6 + u_2^6 + u_3^6 + m u_1^2 u_2^2 u_3^2;$$

if, however, the method which succeeds for the quartic and octavic functions be attempted to be applied to this it will be found entirely to fail. Here, however, considerations of a purely morphological character step in to our aid and immediately lead to the true canonical representation of the sextic function. Algebraically speaking, the only connexion between two identical forms F and F is through the equation $F = \psi^{-1}\psi F$; but, morphologically considered, a *form* F may admit of being derived by a series of entirely heterogeneous operations from *itself*. In general, supposing

$$F(x, y) = ax^n + nbx^{n-1}y + \&\text{c.} \dots + n(b)\, xy^{n-1} + (a)\, y^n,$$

the form $$\xi^n \frac{d}{da} + \xi^{n-1}\eta \frac{d}{db} + \dots + \xi\eta^{n-1} \frac{d}{d(b)} + \eta^n \frac{d}{d(a)},$$

operating upon any concomitant to F will, we know (from the law of reciprocity in Section IV.), produce another concomitant. The operative form above written is termed the evector, and the result of operating therewith upon a concomitant is termed the evectant of the latter, which is said, when so operated upon, to be evected*. The polar reciprocal of the evector may be termed the contravector, and for two variables is of course of the form

$$y^n \frac{d}{da} - y^{n-1}x \frac{d}{db} \pm \&\text{c.}$$

If we suppose n to be even, $F(x, y)$ will have the well-known quadrinvariant

$$a(a) - nb(b) + \tfrac{1}{2}n(n-1)\, c(c) \mp \&\text{c.},$$

and if this be operated upon with the contravector, or if we like so to say, be contravected, we recover the original function F, so that any function of two variables of an even degree is the contravect of its quadrinvariant.

* These terms "evector, evectant, contravectant, to evect and contravect," will of course admit of an immediate extension to functions of any number of variables. Evection gives rise to contravariants, contravection to covariants; but on this account to interchange the meanings respectively attached to the terms evector and contravector, and their respective allied terms, would be a simplification too dearly purchased at the expense of contravening the principle that the *word for the base should be the base for the word.*

If now we return to the representation of $(x, y)^4$ under the form

$$u_1^4 + u_2^4 + m(u_1 u_2)^2,$$

and make

$$u_1 u_2 = F_2(x, y);$$

or to that of $(x, y)^8$ under the form

$$u_1^4 + u_2^4 + u_3^4 + u_4^4 + m(u_1 u_2 u_3 u_4)^2,$$

and make

$$u_1 u_2 u_3 u_4 = F_4(x, y),$$

the outstanding terms multiplied by the parameter m may be regarded in each of these two cases as the squared contravects of the quadrinvariants F_2 and F_4 respectively. Under this point of view we at once see a ground for the proved fact of $(x, y)^6$ not being capable of being thrown under the form

$$u_1^6 + u_2^6 + u_3^6 + m\{F_3(x, y)\}^2,$$

where

$$u_1 u_2 u_3 = F_3(x, y),$$

because there exists no quadrinvariant to $F_3(x, y)$, the only invariant which it possesses being the discriminant which is of the fourth degree; if however instead of $m\{F_3(x, y)\}^2$ we write $mF_3(x, y)\, G_3(x, y)$, where $G_3(x, y)$ is the contravect of the discriminant of F_3, we shall find that the method applied to the reduction of $(x, y)^4$ and to $(x, y)^8$ will perfectly well succeed for $(x, y)^6$, as I proceed to demonstrate.

Let this function be written under the form

$$a_0 x^6 + 6a_1 x^5 y + 15a_2 x^4 y^2 + 20a_3 x^3 y^3 + 15a_4 x^2 y^4 + 6a_5 x y^5 + a_6 y^6,$$

which suppose made equal to

$$(p_1 x + q_1 y)^6 + (p_2 x + q_2 y)^6 + (p_3 x + q_3 y)^6 \\ + (Ax^3 + 3Bx^2 y + 3Cxy^2 + Dy^3)(Lx^3 + Mx^2 y + Nxy^2 + Py^3);$$

where

$$(p_1 x + q_1 y)(p_2 x + q_2 y)(p_3 x + q_3 y) = Ax^3 + 3Bx^2 y + 3Cxy^2 + Dy^3;$$

the discriminant of this will be, as is well known,

$$A^2 D^2 + 4AC^3 + 4DB^3 - 3B^2 C^2 - 6ABCD,$$

and contravecting this with the operator

$$-y^3 \frac{d}{dA} + y^2 x \frac{d}{dB} - yx^2 \frac{d}{dC} + x^3 \frac{d}{dD},$$

and identifying the result with $Lx^3 + Mx^2 y + Nxy^2 + Py^3$, we have

$$\begin{aligned} L &= -6ABC + 2A^2 D + 4B^3, \\ M &= 6ABD - 12AC^2 + 6B^2 C, \\ N &= -6ACD + 12DB^2 - 6BC^2, \\ P &= 6BCD - 2AD^2 - 4C^3. \end{aligned}$$

A, B, C, D are known functions of p_1, p_2, p_3; q_1, q_2, q_3, and we shall have seven equations for determining these six unknown quantities and the unknown parameter m.

Let
$$q_1 = p_1\lambda_1, \quad q_2 = p_2\lambda_2^4, \quad q_3 = p_3\lambda_3,$$
$$\lambda_1 + \lambda_2 + \lambda_3 = 3s_1, \quad \lambda_2\lambda_3 + \lambda_3\lambda_1 + \lambda_1\lambda_2 = 3s_2, \quad \lambda_1\lambda_2\lambda_3 = s_3,$$
$$p_1 p_2 p_3 = m.$$
Then
$$A = m, \quad 3B = 3ms_1, \quad 3C = 3ms_2, \quad D = s_3.$$
$$L = m^3(4s_1^3 - 6s_1s_2 + 2s_3),$$
$$M = m^3(6s_1^2s_2 + 6s_1s_3 - 12s_2^2),$$
$$N = m^3(12s_1^2s_3 - 6s_1^2s_2 - 6s_2s_3),$$
$$P = m^3(+6s_1s_2s_3 - 4s_2^3 - 2s_3^2).$$
Let
$$(Ax^3 + 3Bx^2y + 3Cxy^2 + Dy^3)(Lx^3 + Mx^2y + Nxy^2 + Py^3)$$
$$= K_0x^6 + K_1x^5y + K_2x^4y^2 + K_3x^3y^3 + K_4x^2y^4 + K_5xy^5 + K_6y^6 = T.$$
Then, equating this term for term with
$$\lambda_1^6(x + p_1y)^6 + \lambda_2^6(x + p_2y)^6 + \lambda_3^6(x + p_3y)^6 + \mu T,$$
we obtain the seven equations following:
$$p_1^6 + p_2^6 + p_3^6 + \mu K_0 = a_0, \tag{1}$$
$$p_1^6\lambda_1 + p_2^6\lambda_2 + p_3^6\lambda_3 + \frac{\mu}{6}K_1 = a_1, \tag{2}$$
$$p_1^6\lambda_1^2 + p_2^6\lambda_2^2 + p_3^6\lambda_3^2 + \frac{\mu}{15}K_2 = a_2, \tag{3}$$
$$p_1^6\lambda_1^3 + p_2^6\lambda_2^3 + p_3^6\lambda_3^3 + \frac{\mu}{20}K_3 = a_3, \tag{4}$$
$$p_1^6\lambda_1^4 + p_2^6\lambda_2^4 + p_3^6\lambda_3^4 + \frac{\mu}{15}K_4 = a_4, \tag{5}$$
$$p_1^6\lambda_1^5 + p_2^6\lambda_2^5 + p_3^6\lambda_3^5 + \frac{\mu}{6}K_5 = a_5, \tag{6}$$
$$p_1^6\lambda_1^6 + p_2^6\lambda_2^6 + p_3^6\lambda_3^6 + \mu K_6 = a_6. \tag{7}$$
Eliminating linearly

p_1^6,	p_2^6,	p_3^6	between	equations	1, 2, 3, 4,
$\lambda_1 p_1^6$,	$\lambda_2 p_2^6$,	$\lambda_3 p_3^6$	"	"	2, 3, 4, 5,
$\lambda_1^2 p_1^6$,	$\lambda_2^2 p_2^6$,	$\lambda_3^2 p_3^6$	"	"	3, 4, 5, 6,
$\lambda_1^3 p_1^6$,	$\lambda_2^3 p_2^6$,	$\lambda_3^3 p_3^6$	"	"	4, 5, 6, 7,

we obtain the four equations following, namely,

$$a_0s_3 - 3a_1s_2 + 3a_2s_1 - a_3 = \mu\vartheta_0,$$
$$a_1s_3 - 3a_2s_2 + 3a_3s_1 - a_4 = \mu\vartheta_1,$$
$$a_2s_3 - 3a_3s_2 + 3a_4s_1 - a_5 = \mu\vartheta_2,$$
$$a_3s_3 - 3a_4s_2 + 3a_5s_1 - a_6 = \mu\vartheta_3,$$

where
$$\vartheta_0 = K_0s_3 - \tfrac{3}{6}K_1s_2 + \tfrac{3}{15}K_2s_1 - \tfrac{1}{20}K_3$$
$$= \tfrac{1}{60}(60K_0s_3 - 30K_1s_2 + 12K_2s_1 - 3K_3),$$
$$\vartheta_1 = \tfrac{1}{6}K_1s_3 - \tfrac{3}{15}K_2s_2 + \tfrac{3}{20}K_3s_1 - \tfrac{1}{15}K_4$$
$$= \tfrac{1}{60}(10K_1s_3 - 12K_2s_2 + 9K_3s_1 - 4K_4),$$
$$\vartheta_2 = \tfrac{1}{15}K_2s_3 - \tfrac{3}{20}K_3s_2 + \tfrac{3}{15}K_4s_1 - \tfrac{1}{6}K_5$$
$$= \tfrac{1}{60}(4K_2s_3 - 9K_3s_2 + 12K_4s_1 - 10K_5),$$
$$\vartheta_3 = \tfrac{1}{20}K_3s_3 - \tfrac{3}{15}K_4s_2 + \tfrac{3}{6}K_5s_1 - K_6$$
$$= \tfrac{1}{60}(3K_3s_3 - 12K_4s_2 + 30K_5s_1 - 60K_6),$$

and

$$\frac{1}{m^4}K_0 = \frac{A}{m}\cdot\frac{L}{m^3} = 4s_1^3 - 6s_1s_2 + 2s_3,$$

$$\frac{1}{m^4}K_1 = \frac{AM+3BL}{m^4} = (6s_1^2s_2 + 6s_1s_3 - 12s_2^2) + 3s_1(4s_1^3 - 6s_1s_2 + 2s_3)$$
$$= 12s_1^4 - 12s_1^2s_2 + 12s_1s_3 - 12s_2^2,$$

$$\frac{1}{m^4}K_2 = \frac{AN+3BM+3CL}{m^4} = (12s_1^2s_3 - 6s_1s_2^2 - 6s_2s_3)$$
$$+ (18s_1^3s_2 + 18s_1^2s_3 - 36s_1s_2^2)$$
$$+ (12s_1^3s_2 - 18s_1s_2^2 + 6s_2s_3)$$
$$= 30s_1^3s_2 + 30s_1^2s_3 - 60s_1s_2^2,$$

$$\frac{1}{m^4}K_3 = \frac{AP+3BN+3CM+DL}{m^4} = (6s_1s_2s_3 - 4s_2^3 - 2s_3^2)$$
$$+ (36s_1^3s_3 - 18s_1^2s_2^2 - 18s_1s_2s_3)$$
$$+ (18s_1^2s_2^2 + 18s_1s_2s_3 - 36s_2^3)$$
$$+ (4s_1^3s_3 - 6s_1s_2s_3 + 2s_3^2)$$
$$= 40s_1^3s_3 - 40s_2^3,$$

$$\frac{1}{m^4}K_4 = \frac{3BP+3CN+DM}{m^4} = 18s_1^2s_2s_3 - 12s_1s_2^3 - 6s_1s_3^2$$
$$+ 36s_1^2s_2s_3 - 18s_1s_2^3 - 18s_2^2s_3$$
$$+ 6s_1^2s_2s_3 + 6s_1s_3^2 - 12s_2^2s_3$$
$$= 60s_1^2s_2s_3 - 30s_1s_2^3 - 30s_2^2s_3,$$

$$\frac{1}{m^4}K_5 = \frac{3CP+DN}{m^4} = 18s_1s_2^2s_3^2 - 12s_2^4 - 6s_2s_3^2$$
$$+ 12s_1^2s_3^2 - 6s_1s_2^2s_3 - 6s_2s_3^2$$
$$= 12s_1^2s_3^2 + 12s_1s_2^2s_3 - 12s_2^4 - 12s_2s_3^2,$$

$$\frac{1}{m^4}K_6 = \frac{DP}{m^4} = 6s_1s_2s_3^2 - 4s_2^3s_3 - 2s_3^3;$$

therefore
$$\frac{60\vartheta_0}{\mu m^4} = 240s_1^3s_3 - 360s_1s_2s_3 + 120s_3^2$$
$$-360s_1^4s_2 + 360s_1^2s_2^2 - 360s_1s_2s_3 + 360s_2^3$$
$$+360s_1^4s_2 + 360s_1^3s_3 - 720s_1^2s_2^2$$
$$-120s_1^3s_3 + 120s_2^3$$
$$= 120(s_3^2 + 4s_2^3 + 4s_1^3s_3 - 3s_1^2s_2^2 - 6s_1s_2s_3),$$

that is
$$\vartheta_0 = 2\mu(A^2D^2 + 4AC^3 + 4DB^3 - 3B^2C^2 - 6ABCD).$$

Again,
$$\frac{\vartheta_1}{\mu m^4} = 120s_1^4s_3 - 120s_1^2s_2s_3 + 120s_1s_3^2 - 120s_2^2s_3$$
$$-360s_1^3s_2^2 - 360s_1^2s_2s_3 + 720s_1s_2^3$$
$$+360s_1^4s_3 - 360s_1s_2^3$$
$$-240s_1^2s_2s_3 + 120s_1s_2^3 + 120s_1s_2^2s_3$$
$$= 120(s_1s_3^2 + 4s_1s_2^3 + 4s_1^4s_3 - 3s_1^3s_2^2 - 6s_1^2s_2s_3);$$

therefore
$$\vartheta_1 = 2\mu(A^2D^2 + 4AC^3 + 4DB^3 - 3B^2C^2 - 6ABCD)s_1.$$

Again,
$$\frac{60\vartheta_2}{\mu m^4} = 120s_1^3s_2s_3 + 120s_1^2s_3^2 - 240s_1s_2^2s_3$$
$$-360s_1^3s_2s_3 + 360s_2^4$$
$$+720s_1^3s_2s_3 - 360s_1^2s_2^3 - 360s_1s_2^2s_3$$
$$-120s_1^2s_3^2 - 120s_1s_2^2s_3 + 120s_2^4 + 120s_2s_3^2$$
$$= 120(s_2s_3^2 + 4s_2^4 + 4s_1^3s_2s_3 - 3s_1^2s_2^3 - 6s_1s_2^2s_3),$$

therefore
$$\vartheta_2 = 2\mu(A^2D^2 + 4AC^3 + 4DB^3 - 3B^2C^2 - 6ABCD)s_2.$$

Finally,
$$\frac{60\vartheta_3}{\mu m^4} = 120s_1^3s_3^2 - 120s_2^3s_3$$
$$-720s_1^2s_2^2s_3 + 360s_1s_2^4 + 360s_2^3s_3$$
$$+360s_1^3s_3^2 + 360s_1^2s_2^2s_3 - 360s_1s_2^4 - 360s_1s_2s_3^2$$
$$-360s_1^2s_2s_3^2 + 240s_2^3s_3 + 120s_3^3$$
$$= 120(s_3^3 + 4s_2^3s_3 + 4s_1^3s_3^2 - 3s_1^2s_2^2s_3 - 6s_1s_2s_3^2),$$

therefore $\quad \vartheta_3 = 2\mu\,(A^2D^2 + 4AC^3 + 4DB^3 - 3B^2C^2 - 6ABCD)\,s_3.$

Hence, writing

$$2\mu\,(A^2D^2 + 4AC^3 + 4DB^3 - 3B^2C^2 - 6ABCD) = \rho,$$

the four equations connecting a_0, a_1, a_2, a_3 with ϑ_0, ϑ_1, ϑ_2, ϑ_3, take the form

$$\begin{aligned}
a_0s_3 - 3a_1s_2 + 3a_2s_1 - (a_3 + \rho) &= 0,\\
a_1s_3 - 3a_2s_2 + 3\,(a_3 - \tfrac{1}{3}\rho)\,s_1 - a_4 &= 0,\\
a_2s_3 - 3\,(a_3 + \tfrac{1}{3}\rho)\,s_2 + 3a_4s_1 - a_5 &= 0,\\
(a_3 - \rho) - 3a_4s_2 + 3a_5s_1 - a_6 &= 0.
\end{aligned}$$

Hence we derive the equation involving only the known coefficients of the given function for finding ρ, namely, the determinant

$$\begin{vmatrix} a_0, & a_1, & a_2, & a_3 + \rho \\ a_1, & a_2, & a_3 - \tfrac{1}{3}\rho, & a_4 \\ a_2, & a_3 + \tfrac{1}{3}\rho, & a_4, & a_5 \\ a_3 - \rho, & a_4, & a_5, & a_6 \end{vmatrix} = 0. \qquad \text{(R)}$$

If in this matrix ρ be changed into $-\rho$, the determinant evidently remains unaltered in value; hence the odd powers of ρ disappear from the equation, and ρ may be found by the solution of a double quadratic only. In fact the above equation for finding ρ, expanded out, becomes

$$\frac{\rho^4}{9} + \left(\frac{2}{3}\begin{vmatrix} a_2, & a_3 \\ a_3, & a_4 \end{vmatrix} - \frac{2}{3}\begin{vmatrix} a_1, & a_3 \\ a_3, & a_5 \end{vmatrix} + \begin{vmatrix} a_2, & a_3 \\ a_3, & a_4 \end{vmatrix} + \frac{1}{9}\begin{vmatrix} a_0, & a_3 \\ a_3, & a_6 \end{vmatrix}\right)\rho^2,$$

$$- \begin{vmatrix} a_0, & a_1, & a_2, & a_3 \\ a_1, & a_2, & a_3, & a_4 \\ a_2, & a_3, & a_4, & a_5 \\ a_3, & a_4, & a_5, & a_6 \end{vmatrix} = 0\,;$$

that is $\quad \rho^4 + (15a_2a_4 - 6a_1a_5 - 10a_3{}^2 + a_0a_6)\,\rho^2$

$$+ \begin{vmatrix} a_3, & a_2, & a_1, & a_0 \\ a_4, & a_3, & a_2, & a_1 \\ a_5, & a_4, & a_3, & a_2 \\ a_6, & a_5, & a_4, & a_3 \end{vmatrix} = 0\,;$$

the coefficient of ρ^2 being the well-known quadrinvariant, and the final term the meiocatalecticizant of the given function. There will consequently be four different values of ρ and four different systems of values of s_1, s_2, s_3, expressible for each system respectively in terms of ρ by means of any three out of the four equations (R), and consequently there will be four systems of

values of $\lambda_1, \lambda_2, \lambda_3$, each of which may be found separately by solving the cubic equation

$$\lambda^3 - 3s_1\lambda^2 + 3s_2\lambda - s_3 = 0;$$

also $K_0, K_1, K_2, K_3, K_4, K_5, K_6$ become known multiples of m^4, and finally, the values of any λ and K system being thus determined, we may then, by means of the identity

$$p_1^6(x+\lambda_1 y)^6 + p_2^6(x+\lambda_2 y)^6 + p_3^6(x+\lambda_3 y)^6$$
$$+ \mu m^4\left(\frac{K_0}{m^4}x^6 + \frac{K_1}{m^4}x^5y + \&\text{c.} + \frac{K_6}{m^4}y^6\right) = a_0x^6 + 6a_1x^5y + \&\text{c.} + a_6y^6,$$

write down at will any four equations out of the seven equations therefrom resulting, and these will serve to determine linearly the values of p_1^6, p_2^6, p_3^6, μm^4; and consequently, by means of the equations

$$q_1 = p_1\rho_1, \quad q_2 = p_2\rho_2, \quad q_3 = p_3\rho_3,$$

q_1, q_2, q_3 are known, and consequently every coefficient in

$$(p_1x+q_1y)^6 + (p_2x+q_2y)^6 + (p_3x+q_3y)^6 + \mu M$$

is completely determined. But we shall hereafter return to this theory, and seek for a direct method of finding the four values of the functions

$$(p_1x+q_1y), \quad (p_2x+q_2y), \quad (p_3x+q_3y).$$

It appears from the above investigation that there are four modes of throwing $(x, y)^6$ under the assumed form which possess the remarkable property of separating into two pairs of modes, as is obvious from the fact of the resolving equation in ρ having two pairs of roots, those of the same pair being equal but of contrary signs. As this form will be of extreme value in studying the invariants of $(x, y)^6$, it may be well to consider the simplest shape to which it admits of being reduced.

We may suppose $(p_1x+q_1y)(p_2x+q_2y)(p_3x+q_3y)$ thrown under the form of u^3+v^3, the contravectant of the discriminant to which in respect to u and v is v^3-u^3, so that we may use for the canonical form the expression

$$a(u+v)^6 + b(u+\rho v)^6 + c(u+\rho^2 v)^6 + \mu(u^6 - v^6),$$

where $\rho^3 = 1$; or if we please, more simply

$$a(u+v)^6 + b(u+\rho v)^6 + c(u+\rho^2 v)^6 + u^6 - v^6.$$

I may take this occasion to observe that there are generally two modes of a distinct kind for obtaining any simple concomitant; the difference (a most important practical one) consisting in the circumstance that in the one mode there are differentiations to be performed in respect to the coefficients, the consequence of which is that the whole of the operations must be gone through for obtaining the concomitant with the primitive in its

most general form, and no advantage can be taken in the course of these operations of the simplification resulting from the absence of any terms in the primitive or of any other speciality therein; whereas in the other mode of derivation, where all the differentiations have to be performed quâ the variables only, the partial form may be operated with throughout. Thus, for instance, to find the contravectant to the discriminant of a cubic function the general form of the cubic must be employed, and then the special values of the coefficient corresponding to a specific form of the cubic substituted at the close of the operations; but this same concomitant may also be obtained by taking the resultant of the first emanant of the given cubic and of the first emanant of its Hessian in respect to the variables of emanation, and consequently the specific form may, after this mode, be retained from the first. Thus, if we start with u^3+v^3, the Hessian is uv, and the two emanants in question will be $u^2u'+v^2v'$ and $vu'+uv'$, the resultant of which in respect to u' and v' is u^3-v^3; or, again, if we commence with uvw subject to the relation that $u+v+w=0$, the Hessian will be

$$\begin{vmatrix} 0, & w, & v, & 1 \\ w, & 0, & u, & 1 \\ v, & u, & 0, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix},$$

that is to say, $$u^2+v^2+w^2-2uv-2uw-2vw.$$

The two emanants will then be

$$vwu'+wuv'+uvw'$$

$$(u-v-w)\,u'+(v-w-u)\,v'+(w-u-v)\,w',$$

subject to the relation

$$u'+v'+w'=0\,;$$

and taking the resultant of these three equations, or, which is the same thing, of

$$\begin{aligned} &vwu'+wuv'+uvw', \\ &uu'+\ vv'\ +ww', \\ &u'+\ v'\ +w', \end{aligned}$$

we obtain the determinant

$$\begin{vmatrix} vw, & wu, & uv \\ u, & v, & w \\ 1, & 1, & 1 \end{vmatrix},$$

which is equal to

$$vw\,(v-w)+wu\,(w-u)+uv\,(u-v),$$

that is to say $$(u-v)\,(v-w)\,(w-u).$$

Hence another variety of the external shape to which the canonical form for the sextic function of x, y may be reduced will be

$$au^6 + bv^6 + cw^6 + \mu uvw\,(u - v)\,(v - w)\,(w - u).$$

I shall presently revert to the theory of the corresponding mode of reducing to their canonical forms the biquadratic and octavic functions of x, y, the number of solutions for which will be respectively three and five, and the discovery of which, as shown by me in the Number of the *Philosophical Magazine* before adverted to, depends upon the solution of equations of the third and fifth degrees in ρ expressed by means of determinants of the third and fifth orders formed in precise correspondence with that of the fourth order, upon which, as we have found above, the reduction of the sextic function to its canonical form depends.

5.

THÉORÈME SUR LES DÉTERMINANTS.

[*Nouvelles Annales de Mathématiques,* XIII. (1854), p. 305.]

SOIENT les déterminants

$$
\begin{vmatrix} \lambda \end{vmatrix}, \quad
\begin{vmatrix} \lambda, 1 \\ 1, \lambda \end{vmatrix}, \quad
\begin{vmatrix} \lambda, 1, 0 \\ 2, \lambda, 2 \\ 0, 1, \lambda \end{vmatrix}, \quad
\begin{vmatrix} \lambda, 1, 0, 0 \\ 3, \lambda, 2, 0 \\ 0, 2, \lambda, 3 \\ 0, 0, 1, \lambda \end{vmatrix}, \quad
\begin{vmatrix} \lambda, 1, 0, 0, 0 \\ 4, \lambda, 2, 0, 0 \\ 0, 3, \lambda, 3, 0 \\ 0, 0, 2, \lambda, 4 \\ 0, 0, 0, 1, \lambda \end{vmatrix}, \quad
\begin{vmatrix} \lambda, 1, 0, 0, 0, 0 \\ 5, \lambda, 2, 0, 0, 0 \\ 0, 4, \lambda, 3, 0, 0 \\ 0, 0, 3, \lambda, 4, 0 \\ 0, 0, 0, 2, \lambda, 5 \\ 0, 0, 0, 0, 1, \lambda \end{vmatrix};
$$

la loi de formation est évidente ; effectuant, on trouve

$$
\begin{gathered}
\lambda, \quad \lambda^2 - 1^2, \quad \lambda(\lambda^3 - 2^2), \quad (\lambda^2 - 1^2)(\lambda^2 - 3^2), \quad \lambda(\lambda^2 - 2^2)(\lambda^2 - 4^2), \\
(\lambda^2 - 1^2)(\lambda^2 - 3^2)(\lambda^2 - 5^2), \quad \lambda(\lambda^2 - 2^2)(\lambda^2 - 4^2)(\lambda^2 - 6^2),
\end{gathered}
$$

et ainsi de suite.

6.

NOTE ON A POINT OF NOTATION.

[*Philosophical Magazine*, VII. (1854), pp. 50—51.]

IT frequently becomes important in algebraical investigations, and in the representation of results, to have a means of expressing that the sign + or − is to be affixed to an algebraical expression, according as certain indices $\theta_1, \theta_2, \theta_3 \ldots \theta_n$ which occur therein, and which represent the natural numbers from 1 to n in some regular or irregular order, can be derived from the fundamental arrangement $1, 2, 3 \ldots n$ by an even or by an odd number of interchanges. An example of this occurred in my short paper in the last Number of the *Philosophical Magazine*, on the extension of Lagrange's Rule of Interpolation [Vol. I., p. 646], where I used to denote that such a choice of signs was to be made, the awkward and unsuggestive symbol "?" There exists, however, a very simple algebraical mode of denoting the presence of the factor $+1$ or -1, according to the order of the natural numbers in the scale $\theta_1, \theta_2, \theta_3 \ldots \theta_n$.

ζ has been always consecrated by me to the purpose of signifying that the product of the squared differences is to be taken of the elements with which it is in regimen; and in the paper adverted to I introduced the highly convenient new symbol $\zeta^{\frac{1}{2}}$ to denote that the product is to be taken of the simple differences obtained by subtracting from each element in regimen therewith every subsequent element in the arrangement of the elements as set down. By aid of this new symbol $\zeta^{\frac{1}{2}}$, the positive or negative character of any permutation, as $\theta_1, \theta_2 \ldots \theta_n$, can be completely expressed; for

$$\zeta^{\frac{1}{2}}(\theta_1, \theta_2, \theta_3 \ldots \theta_n) \div \zeta^{\frac{1}{2}}(1, 2, 3 \ldots n)$$

will be $+1$ or -1 according as $1, 2, 3 \ldots n$ and $\theta_1, \theta_2, \theta_3 \ldots \theta_n$ belong to the same group, or to opposite groups in the natural dichotomous separation of the permutations of the n symbols in question, and thereby the desired object of giving a functional representation of the ambiguous sign is perfectly attained.

7.

NOTE ON THE "ENUMERATION OF THE CONTACTS OF LINES AND SURFACES OF THE SECOND ORDER."

[*Philosophical Magazine*, VII. (1854), pp. 331—334.]

IN the month of February, 1851, I gave in this *Magazine* an *à priori* and exhaustive process, founded upon the method of determinants, for determining every different kind of simple or collective contact capable of happening between lines and surfaces, and in general between all loci (whether intraspatial or extraspatial) of the second order [Vol. I., p. 219]. The question was shown to resolve itself into that of determining the number of singular relations capable of existing between two quadratic homogeneous functions of any given degree. My object in the paper referred to was actually to calculate the geometrical and analytical characters of these contents and singularities for intraspatial loci, that is loci representable by homogeneous quadratics of two, three, and four variables; but I incidentally appended [Vol. I., p. 239] a statement of the number of such for loci of five, six, seven, and eight variables, without, however, dwelling upon the means of representing the general law. This statement is, however, affected with certain inaccuracies of computation which will be presently pointed out.

It will be at once apparent, from an inspection of the principle of my method, that it remains equally applicable (*mutatis mutandis*) to the more general question of determining the relative singularities (in character and amount) of two functions, each linear in respect of two systems of variables $x_1, x_2 \ldots x_n$; $x_1', x_2' \ldots x_n'$; which species of functions degenerate into quadratic forms, when the two systems of variables become identical so as to coalesce into a single system. Some researches of Mr Cayley into the autometamorphic substitutions of quadratic forms (meaning thereby the linear substitutions which leave the form unaltered) required him to consider the nature of the singular relations capable of existing between two linear substitutions, which is precisely the question, differently stated, of the singular relations

connecting two lineo-linear functions above adverted to; accordingly, I am indebted to Mr Cayley for making an observation on the effect of my rule for finding such singularities, which leads to a most elegant formulization of the number of singularities in question, and which I proceed to introduce to the notice of my readers.

If U and V be two quadratic functions, each of n variables, and if we call D the discriminant of $U+\lambda V = D(\lambda)$, $D(\lambda)$ will be a function of λ of the nth degree. Now, first, I have observed that if any of these n roots be repeated any number of times, there will be a corresponding degree of singularity about one of the points of intersection of the loci represented by $U=0$, $V=0$; so that if the n roots of $D(\lambda)$ be made up of r_1 roots a_1, r_2 roots a_2, r_3 roots a_3, &c., there will be an *inclusive singularity* r_1 at one point, r_2 at another, r_3 at a third, and so on—by *inclusive singularity* meaning a number one unit greater than the index of singularity properly so termed; the inclusive-singularity at an *ordinary* intersection being called 1, at a point of simple singularity 2, of double singularity 3, and in general at a point of the $(r-1)$th degree of singularity r.

Hence the total-inclusive singularity (which is an unit greater than the total-singularity, properly so called) may be broken up into as many partial heaps of inclusive-singularity as there are modes of decomposing n into integers. We may now confine our attention exclusively to the different modes in which a given amount of inclusive-singularity at a single point admits of subdivision into distinct species of singularity, for which I have given in my paper referred to the following rule: The minor systems of determinants corresponding to the matrix of $U+\lambda V$ are to be considered in succession; and if a be any root of the complete determinant of the matrix occurring r times, every hypothesis is to be exhausted as regards the number of times in which $(\lambda-a)$ may be conceived to enter as a factor into each of the system of 1st minors, into each minor of the system of 2nd minors, into each minor of the system of 3rd minors, and so on; the number of such hypotheses being limited by the condition that, if *quoad* the root a, $(\lambda-a)^{k_1}$, $(\lambda-a)^{k_2}$, $(\lambda-a)^{k_3}$ be the greatest common factors respectively to three consecutive systems of minor determinants, k_1 must be not less than $2k_2-k_3$. Here steps in the beautiful observation of Mr Cayley, that the question of assigning the different species of singularities respondent to the factor a supposed to occur r times, is, by virtue of the above condition, tantamount precisely to that of assigning the total number of decreasing* series of positive integers, commencing with a given number r, subject to the condition that the second differences shall be all positive; which (he adds), calling the successive second differences δ, δ', δ'', &c., is tantamount to finding

* Such a series must, from its very nature, be *always* decreasing or increasing in the same direction.

the number of ways that the equation $r = \delta + 2\delta' + 3\delta'' +$ &c., admits of being solved by positive integers, which is obviously the same as the number of modes in which r admits of being decomposed into positive integer parts. Thus the idea of partition, which arises naturally in the first part of the process (that, namely, of the decomposition of the collective inclusive-singularity in every possible way into modes of distributive inclusive-singularity), reappears quite unexpectedly (it may almost be said miraculously), and as the result of an analytical transformation in the second part of the same.

It should be observed that the case of complete coincidence between U and V, which, supposing them to be functions of n variables, corresponds to the supposition of the same factor occurring respectively n times, $(n-1)$ times, $(n-2)$ times, &c., 2 times and 1 time in the complete determinant, the 1st minor system, the 2nd minor system, &c., the $(n-2)$th minor system and the $(n-1)$th minor system respectively, is here taken as the highest case of singularity; this and the case of non-singularity, which also adds a unit to the index of singularity, properly so called, will together make a difference of two units in the numbers given by me in the paper referred to, which numbers will accordingly be 3, 6, 14, &c., in lieu of 1, 2, 12*, &c. We are now enabled to give the following simple statement of the law for determining the total number of singularities which can exist between two quadratic forms of n variables (or if we like so to say, more generally between two linear substitution-systems of the nth order), namely the number of the singularities (including absolute unrelatedness and entire coincidence within the purview of the term) is the index of double decomposition into parts of the number n. To raise up in the mind a clear conception of the idea of double decomposition, we may proceed as follows: First. Suppose a state of things in which a body is supposed to be determined completely, provided that the number of molecules which it contains, and the different number of atoms in each molecule are given, the index of simple decomposition, that is of ordinary partitionment of the number of n, will be the number of different bodies which are capable of being formed out of n atoms. Now imagine that, for the complete determination of a body, another step in the hierarchy of aggregation is to be taken into account, and that we must know for this purpose not only the number of molecules in the body and the number of atoms in each molecule, but also the number of monads in each atom; the number of bodies (differing by definition) capable of being formed out of n monads will then represent what I mean by the index of double decomposition of (or if we like so to say), to the modulus, n. And it is obvious that this idea admits of indefinite extension, and that we may speak of the index of decomposition of any order of multiplicity (single, double, treble, &c.) of, or to the modulus, n.

* These numbers refer to quadratic homogeneous functions, containing respectively 2, 3, 4, &c. variables. For the case of functions containing but one variable there is no distinction between coincidence and unrelatedness, and the number of modes of relation is a single unit.

For single decomposition it is well known and immediately obvious, that the indices to the successive moduli given by the rational numbers in regular progression will be the coefficients of x, x^2, x^3, &c. in the continued product

$$(1-x)^{-1}(1-x^2)^{-1}(1-x^3)^{-1} \text{ \&c. } ad\ inf.;$$

calling these n_1, n_2, n_3, &c., it is of course obvious, as Mr Cayley has observed, that the indices of double decomposition to the same successive moduli will be the coefficients of the same arguments x, x^2, x^3, &c., in the continued product

$$(1-x)^{-n_1}(1-x^2)^{-n_2}(1-x^3)^{-n_3} \text{ \&c. } ad\ inf.;$$

and by aid of this formula he has calculated (with extreme facility) the indices in question up to the modulus 11, and found that they form the series 1, 3, 6, 14, 27, 58, 111, 223, 424, 817, 1527, which accordingly is the series representing the number of singularities capable of existing between quadratic loci commencing with 1 and ending with 11 variables.

The values of n_1, n_2, n_3, ... n_{11}, &c. themselves are given in Euler's introduction, and are respectively

$$1,\ 2,\ 3,\ 5,\ 7,\ 11,\ 15,\ 22,\ 30,\ 42,\ 56,\ \text{\&c.},$$

which numbers will accordingly represent to their respectively corresponding moduli the number of *classes* of singularity, whether these classes be defined with reference to the different modes of distribution of the total collective singularity about different points, or with reference to the degree of the lowest system of minor determinants of the matrix to the determinant to $U + \lambda V$ having one or more factors in common, which latter is the mode of forming the classes adopted by me in the "Enumeration."

Let me be permitted to express the satisfaction which I have felt in finding this theory, which appeared to be doomed to hopeless oblivion, thus unexpectedly, after three years of interment, coming back to life, and at once filling a desired place in analytical researches pursued with apparently a totally different aim.

8.

NOTE ON A FORMULA BY AID OF WHICH AND OF A TABLE OF SINGLE ENTRY THE CONTINUED PRODUCT OF ANY SET OF NUMBERS (OR AT LEAST A GIVEN CONSTANT MULTIPLE THEREOF) MAY BE EFFECTED BY ADDITIONS AND SUBTRACTIONS ONLY WITHOUT THE USE OF LOGARITHMS.

[*Philosophical Magazine*, VII. (1854), pp. 430—436.]

INTRODUCTION.

THE remark to which this note refers is not new; it has been well observed somewhere in Gergonne's *Annales* (Mr Cayley being my informant), that by aid of the formula $4ab = (a+b)^2 - (a-b)^2$ the question of finding the product of two numbers is virtually reduced to a process of addition and subtraction, and of finding the values of two squares out of a table of squares. If the two factors a and b are both even or both odd, the formula ought to be changed into

$$ab = \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2;$$

if one of them is odd and the other even, we may employ the formula

$$ab = \left(\frac{a+b-1}{2}\right)^2 - \left(\frac{a-b+1}{2}\right)^2 + a.$$

So, again, for the product of three numbers, there exists the analogous formula

$$24abc = (a+b+c)^3 - (a+b-c)^3 - (b+c-a)^3 - (c+a-b)^3.$$

OBJECT OF THE PAPER.

The object of this brief note is to exhibit and demonstrate the generalization of the above formulæ, that is, to express the product of any n quantities $a_1, a_2, a_3, \dots a_n$ under the form of the sum of powers of simple linear functions of $a_1, a_2, \dots a_n$. This may be done as follows:

General Formula.

Let $$\theta_1, \theta_2, \theta_3, \dots \theta_n$$
be disjunctively equal to $$1, 2, 3, \dots n,$$
then
$$\begin{aligned} &2 \,.\, 4 \,.\, 6 \dots (2n)\, a_1 a_2 \dots a_n \\ &= (a_{\theta_1} + a_{\theta_2} + a_{\theta_3} + \dots + a_{\theta_n})^n - \Sigma(-a_{\theta_1} + a_{\theta_2} + \dots + a_{\theta_n})^n \\ &+ \Sigma(-a_{\theta_1} - a_{\theta_2} + a_{\theta_3} + \dots + a_{\theta_n})^n + \&\text{c.} \\ &+ (-)^n (-a_{\theta_1} - a_{\theta_2} - \dots - a_{\theta_n})^n, \end{aligned}$$
which I call the principal equation.

Demonstration of the Principal Equation.

Let $$\phi_1, \phi_2, \phi_3, \dots \phi_{n-1}$$
be disjunctively equal to
$$1, 2, 3, \dots (n-1),$$
then it is easily seen that
$$\begin{aligned} (a_{\theta_1} + a_{\theta_2} + \dots + a_{\theta_n})^n &= (a_{\phi_1} + a_{\phi_2} + \dots + a_{\phi_{n-1}} + a_n)^n \\ \Sigma(-a_{\theta_1} + a_{\theta_2} + \dots + a_{\theta_n})^n &= (a_{\phi_1} + a_{\phi_2} + \dots + a_{\phi_{n-1}} - a_n)^n \\ &\quad + \Sigma(-a_{\phi_1} + a_{\phi_2} + \dots + a_{\phi_{n-1}} + a_n)^n \\ \Sigma(-a_{\theta_1} - a_{\theta_2} + \dots + a_{\theta_n})^n &= \Sigma(-a_{\phi_1} + a_{\phi_2} + \dots + a_{\phi_{n-1}} - a_n)^n \\ &\quad + \Sigma(-a_{\phi_1} - a_{\phi_2} + \dots + a_{\phi_{n-1}} + a_n)^n \\ \&\text{c.} \qquad &= \qquad \&\text{c.} \\ \Sigma(-a_{\theta_1} - a_{\theta_2} \dots - a_{\theta_{n-1}} + a_{\theta_n})^n &= \Sigma(-a_{\phi_1} - a_{\phi_2} \dots - a_{\phi_{n-2}} + a_{\phi_{n-1}} - a_n)^n \\ &\quad + (-a_{\phi_1} - a_{\phi_2} \dots - a_{\phi_{n-1}} + a_n)^n \\ (-a_{\theta_1} - a_{\theta_2} \dots - a_{\theta_{n-1}} - a_{\theta_n})^n &= (-a_{\phi_1} - a_{\phi_2} \dots - a_{\phi_{n-1}} - a_n)^n. \end{aligned}$$
Hence it is apparent that when $a_n = 0$, the right-hand side of the so-called principal equation spontaneously vanishes; it will therefore always contain a_n as a factor, and by parity of reasoning it will contain every one of the quantities $a_1, a_2, \dots a_n$ as a factor, and will consequently be equal to the product $a_1 a_2 \dots a_n$ multiplied by a numerical factor, which, by making $a_1, a_2, \dots a_n$ each equal to unity, is readily seen to be
$$2^n \times (1 \,.\, 2 \,.\, 3 \dots n),$$
or if we please so to say, $2 \,.\, 4 \,.\, 6 \dots (2n)$. Q. E. D.

CONCLUSION.

If n is odd and be called $2m+1$, we have

$$4 \,.\, 6 \,.\, 8 \ldots (2n)\, a_1 a_2 \ldots a_n$$
$$= (a_{\theta_1} + a_{\theta_2} + \ldots + a_{\theta_n})^n - \Sigma(-a_{\theta_1} + a_{\theta_2} + \ldots + a_{\theta_n})^n$$
$$+ \Sigma(-a_{\theta_1} - a_{\theta_2} + a_{\theta_3} + \ldots + a_{\theta_n})^n \mp \&c.$$
$$+ (-)^m (-a_{\theta_1} - a_{\theta_2} \ldots - a_{\theta_m} + a_{\theta_{m+1}} + a_{\theta_{m+2}} + \ldots + a_{\theta_n})^n;$$

and if n be even and be called $2m$, we have

$$4 \,.\, 6 \,.\, 8 \ldots (2n)\, a_1 a_2 \ldots a_n$$
$$= (a_{\theta_1} + a_{\theta_2} + \ldots + a_{\theta_n})^n - \Sigma(-a_{\theta_1} + a_{\theta_2} + \ldots + a_{\theta_n})^n$$
$$+ \Sigma(-a_{\theta_1} - a_{\theta_2} + a_{\theta_3} + \ldots + a_{\theta_n})^n \mp \&c.$$
$$+ \tfrac{1}{2}(-)^m \Sigma(-a_{\theta_1} - a_{\theta_2} \ldots - a_{\theta m} + a_{\theta_{m+1}} + a_{\theta_{m+2}} + \ldots + a_{\theta_n})^n;$$

where, it should be observed, that the last term is made up of integer parts, notwithstanding the presence of the factor $\frac{1}{2}$, which factor may be construed as only serving to denote that, of any pair of complementary linear functions of those which enter into this term, such as

$$-a_{q_1} - a_{q_2} \ldots - a_{q_m} + a_{q_{m+1}} + a_{q_{m+2}} + \ldots + a_{q_n}$$

and $$-a_{q_{m+1}} - a_{q_{m+2}} \ldots - a_{q_n} + a_{q_1} + a_{q_2} + \ldots + a_{q_m},$$

one only is to be retained. The entire term is of course made up exclusively of such pairs.

COROLLARY.

If $R(a_1, a_2, \ldots a_n)$ denote any symmetrical algebraic function whatever of $a_1, a_2, \ldots a_n$,

$$\Sigma_n^{\overset{0}{i}} \Sigma_{\nu_i}^{\overset{0}{\theta}} (-)^i R(-a_{\theta_1}, -a_{\theta_2}, \ldots -a_{\theta_i}, a_{\theta_{i+1}}, a_{\theta_{i+2}}, \ldots a_{\theta_n})$$

will contain $a_1 a_2 a_3 \ldots a_n$ as a factor. In this formula ν_i denotes the number of combinations of n things taken i together.

POSTSCRIPT.

In constructing a table of single entry for applying the formula

$$4ab = (a+b)^2 - (a-b)^2,$$

that is, $$ab = \frac{1}{4}(a+b)^2 - \frac{1}{4}(a-b)^2,$$

it is only necessary to retain the integer part of the quarters of the squares of all the numbers from 2 to the sum of the highest of the values of a and b to which the application of the table is proposed to be restricted, because the

fractional parts of $\left(\frac{a+b}{2}\right)^2$ and $\left(\frac{a-b}{2}\right)^2$ will always destroy one another. A table for the multiplication of a ternary set of factors by means of the formula

$$abc = \frac{1}{24}(a+b+c)^3 - \frac{1}{24}(a+b-c)^3 - \frac{1}{24}(a-b+c)^3 - \frac{1}{24}(-a+b+c)^3$$

will imply the registration of the values of the 24th parts of all numbers up to the highest value of $(a+b+c)$, and it becomes a question of some practical interest to determine in what way the fractional remainders of these 24th parts are to be dealt with.

The formula last written may give rise to either of the two subjoined cases, according as the numbers a, b, c correspond or not to the lengths of a possible triangle, namely:

$$(1) \qquad abc = \frac{1}{24}N_1^3 - \frac{1}{24}N_2^3 - \frac{1}{24}N_3^3 - \frac{1}{24}N_4^3,$$

or

$$(2) \qquad abc = \frac{1}{24}N_1^3 + \frac{1}{24}N_2^3 - \frac{1}{24}N_3^3 - \frac{1}{24}N_4^3,$$

the quantities N_1, N_2, N_3, N_4 being all supposed to represent positive integers.

A very little consideration will show, that if we neglect fractions in the table there may be entailed an error of 2, 1, 0, or -1. Whether the error is, on the one hand, an error of an even order (namely, 0 or 2), or, on the other hand, of an odd order (namely, 1 or -1), would be at once obvious by looking to see whether the formula, after neglecting the fractions, gave an odd result when the result ought to be odd, and an even result when the result ought to be even, or *vice versâ*. And the nature of the result as to whether it *ought* to be odd or even could be immediately inferred from observing whether a, b, c were or were not all of them odd numbers. But there would still remain an ambiguity in the correction to be applied in either case, arising from the doubt whether it should be zero or 2 in the one case, or whether it should be $+1$ or -1 in the other case.

This ambiguity might of course be removed by inserting in the table employed the first decimal place of $\frac{N^3}{24}$, and increasing the decimal part in the final result to unity, or lowering it to zero, according as its value might be greater or less than $\frac{1}{2}$; and it would be easy to ascertain the limits within which the decimal digit in the result must lie, and the range of values (of which 5 is one) from which it is excluded. The same end may, however, be gained much more elegantly and expeditiously, and by a method more closely analogous to that employed for the evolution of binary products, by the intervention of a very simple expedient.

The cubic residues in respect to the modulus 24 are easily verified to be as follows: 0, 1, 3, 5, 7, 8, 9, 11, 13, 15, 16, 17, 19, 21, 23. Let the tabular value of $\frac{N^3}{24}$ be made $\left[\frac{N^3}{24}\right]+K_N$, where $\left[\frac{N^3}{24}\right]$ means the integer part of the quantity within the brackets, and K_N may have any one of the three values 0, $\frac{1}{2}$, 1, namely:

$K_N=0$ when the remainder of N^3 to the divisor 24 is 0, 1, 3 or 5;

$K_N=\frac{1}{2}$ when the said remainder is 7, 8, 9, 11, 13, 15, 16 or 17;

and

$K_N=1$ when the remainder is 19, 21 or 23;

and let $\left[\frac{N^3}{24}\right]+K_N$ be called the cubic respondent to N, and be denoted by $R(N)$;

and let the exact value of $\frac{N^3}{24}$ be called $R'(N)$.

Let

$$R'(a+b+c)-R'(a+b-c)-R'(a-b+c)-R'(-a+b+c)$$
$$=R(a+b+c)-R(a+b-c)-R(a-b+c)-R(-a+b+c)+\Delta.$$

If in general we write $R'(n)-R(n)=E(n)$, Δ must be of one or the other of the two forms

$$E(n_1)-E(n_2)-E(n_3)-E(n_4),$$

or

$$E(n_1)+E(n_2)-E(n_3)-E(n_4),$$

where n_1, n_2, n_3, n_4 are supposed to be all positive integers. Now it is easily seen that $E(n)$ always lies within the limits $\pm\frac{5}{24}$; that is to say, it may reach up to $\frac{5}{24}$ or down to $-\frac{5}{24}$, but can never transgress these values in either direction. Hence it is obvious that Δ, which is made up of four terms, each of the form $E(n)$, can never be so great as $+1$ or so small as -1, and consequently Δ can only have one of the three values $+\frac{1}{2}$, 0, $-\frac{1}{2}$.

Hence, then, we may work with the tabular cubic respondents in lieu of the exact cubic respondents; if the result is an integer, it is good without any correction; if it is a fraction, $\frac{1}{2}$ must be added to, or taken away from it. And to ascertain which of these processes is to be applied, it is only necessary to consider whether the three factors to be multiplied are or are not all of them odd.

In practically constructing a table of cubic respondents, it would not be necessary actually to insert the fraction $\frac{1}{2}$ in any case; a dot over, or a stroke through the last integer, would serve to denote that this fraction was to be understood.

A table of quadratic respondents (that is, of the integer parts of the fourths of the square numbers) up to the base 20,000, has been actually constructed and published by a M. Antoine Voisin, under the title "Tables des Multiplications ou Logarithmes de Nombres entiers depuis 1 jusqu'à 20,000, au moyen desquelles on peut multiplier tous les nombres qui n'excèdent pas 20,000 par 20,000," &c. 12mo. à Paris, Firmin Didot, 1817. A copy of this is in Mr J. T. Graves's valuable mathematical library at Cheltenham.

By logarithms the author intends the same quantities as I term respondents, certainly a less objectionable and safer term to employ. There appears to be an error in the title in affirming that any two numbers, not separately exceeding 20,000, may be multiplied by aid of these tables, as the sum of the two factors ought not to exceed 20,000. Mr Peter Gray, so favourably known to an important section of the public as the author of many useful tables, has informed me that Major Shortredd, now in India, has computed a table of quadratic respondents extending to the argument 200,000, which he is taking measures to have published. Such tables would be very useful to computers, as they would serve for the multiplication of any two numbers whatever not containing more than five figures each. I should like to see a table of cubic respondents up to 30,000 appended to this work*.

* The best practical mode of using and arranging such a table I find, after much thought and consideration, would be as follows. It is easy to add two quantities and subtract their sum from a third by a single operation. If, then, a, b, c are the three numbers whose product it is required to find, they should be written under one another; and against (a) should be set the value of $a-b-c$; against (b), that of $b-a-c$; and against (c), that of $c-a-b$; under these three last results should be written the value of $a+b+c$; of the three former, two at least must be, all *may* be negative; their values arithmetically expressed will be of the form $K(10,000)+N$, where K is 0, 1 or 2. In order that the final process of combining the 4 cubes may be made purely additive, the tables should show the values of $(10,000)^3$ less the respondent to $K(10,000)+N$, when K is 1 or 2 for all values of N from 1 to 9999. These complements to the respondents of the simple or augmented complements of N may be termed respectively the simply and doubly affected respondents of N, but in using the tables no distinction need be drawn between the respondents and the affected respondents. The arrangement of the tables will be as follows. In each page there will be a column for the arguments, which will extend from 1 to 9999, and five other columns containing respondents and bearing respectively for their headings the numbers $\bar{2}$, $\bar{1}$, 0, 1, 2. The four quantities formed by addition, or by addition and subtraction, from a, b, c, will all be of the form $K\nu_1\nu_2\nu_3\nu_4$ (ν_1, ν_2, ν_3, ν_4 denoting respectively some one or other of the digits from 0 to 9), and K being one of the five symbols $\bar{2}$, $\bar{1}$, 0, 1, 2; the value corresponding to $\nu_1\nu_2\nu_3\nu_4$ will then be sought for in its proper column (according to the value of the guiding figure K), and the sum of the four values so found will be taken (the last figure to the left, which will be 2 or 3, being rejected). This result, affected, if necessary, with the proper correction of $\pm\frac{1}{2}$, will express the value of abc.

9.

ON SOME NEW THEOREMS IN ARITHMETIC.

[*Philosophical Magazine*, VIII. (1854), pp. 187—190.]

LET $S_i\ (a, b, c, \ldots k, l)$ denote, as is not unusual, the complete sum of the products of the elements (n in number) $a, b, c, \ldots k, l$, combined in every possible way i together. Let $\ddot{S}_i\ (a, b, c, \ldots k, l)$ denote the sum of the products of the same elements combined i together, but so that all combinations are excluded in which any two consecutive elements as a and b, or b and $c, \ldots$ or k and l, appear simultaneously. S_i may be termed a complete sum of ith products, and $\ddot{S}_i$ a sum of products of *anakolouthic* elements, or briefly an anakolouthic sum of ith products. If we expand the continued fraction

$$\frac{1}{\rho +}\,\frac{a}{\rho +}\,\frac{b}{\rho +}\cdots\frac{k}{\rho +}\,\frac{l}{\rho},$$

it will be easily found to take the form

$$\frac{\rho^{n-1} + \ddot{S}_1{}'\rho^{n-3} + \ddot{S}_2{}'\rho^{n-5} + \&c.}{\rho^n + \ddot{S}_1\rho^{n-2} + \ddot{S}_2\rho^{n-4} + \&c.};$$

where $\ddot{S}_i{}'$ is intended to denote the anakolouthic sum of the ith products of $b, c, \ldots l$, and $\ddot{S}_i$ the anakolouthic sum of the ith products of $a, b, c, \ldots l$.

It is this fact, and the close relation of reciprocity in which the generating continued fraction for anakolouthic sums stands to ordinary continued fractions (a reciprocity which becomes more apparent when ρ is made unity), which gives a peculiar importance to the theory of anakolouthic sums of the kind denoted by $\ddot{S}$; otherwise we might be tempted to embark upon a premature generalization, extending the force of the term anakolouthic so as to denote by $\dddot{S}$ a sum of products in which no *three* consecutive elements came together, $\ddddot{S}$ a sum of products in which no four consecutive elements came together, and so on; these more general forms of anakolouthic sums may hereafter merit and reward attention, but my present business will be exclusively with a statement of some remarkable properties which have accidentally fallen under my observation, of anakolouthic sums of the kind first mentioned, and referring to elements formed in a manner presently to be

explained, from the natural progression of numbers. In order to familiarize the reader with the construction of anakolouthic series, I subjoin the following examples:

$$\ddot{S}_1(abcde) = a + b + c + d + e,$$
$$\ddot{S}_2(abcde) = ac + ad + ae + bd + be + ce,$$
$$\ddot{S}_3(abcde) = ace,$$
$$\ddot{S}_4(abcde) = 0,$$
$$\ddot{S}_4(abcdef) = 0,$$
$$\ddot{S}_4(abcdefg) = aceg,$$
$$\ddot{S}_4(abcdefgh) = aceg + aceh + bdfh.$$

First Theorem. Let n be any odd number; form the $\frac{1}{2}(n-1)$ elements

$$n,\ 2(n-1),\ 3(n-2)\ldots\frac{n-1}{2}\cdot\frac{n+3}{2};$$

the anakolouthic sum of the ith products of these elements is equal to the ith power of negative unity into the complete sum of the $2i$th products of the elements n, $-(n-2)$, $(n-4)$, ... ± 1. Thus suppose $n = 7$, the elements for the anakolouthic sums will be

$$7,\ 12,\ 15;$$

and for the complete sums,

$$7,\ -5,\ 3,\ -1;$$

and we find

$$\ddot{S}_1(7, 12, 15) = 7 + 12 + 15 = 34,\quad S_2(7, -5, 3, -1) = -7.3 - 5.2 - 3 = -34,$$
$$\ddot{S}_2(7, 12, 15) = 7.15 = 105,\quad S_4(7, -5, 3, -1) = 1.3.5.7 = 105.$$

Or, again, if $n = 9$, the one set of elements will be

$$9,\ 16,\ 21,\ 24,$$

and the other set

$$9,\ -7,\ 5,\ -3,\ 1;$$

and we have

$$-(9 + 16 + 21 + 24) = -70 = 9\times(-4) + 7(-3) + 5(-2) + 3(-1),$$
$$9.21 + 9.24 + 16.24$$
$$= 789 = 9.7.5.3 + 9.7.3.1 - 9.7.5.1 - 9.5.3.1 + 7.5.3.1.$$

Second Theorem. Take away the last element belonging to the anakolouthic group above written, so as to reduce the elements to the following sequence:

$$n,\ 2(n-1),\ 3(n-2)\ldots\frac{n-3}{2}\cdot\frac{n+5}{2};$$

$\frac{1}{2}(n+1)$ times the anakolouthic sum of ith products of this sequence will be equal to $(-1)^i$ multiplied by the complete sum of the $(2i+1)$th products

of the series $n, -(n-2), (n-4), \dots \pm 1$. Thus if $n = 9$, the two series of elements are respectively

$$9,\ 16,\ 21; \qquad 9,\ -7,\ 5,\ -3,\ 1;$$

and we find

$$5 \,.\, 1 = 9 - 7 + 5 - 3 + 1,$$

$$5 \,.\, (9 + 16 + 21) = 230 = 9 \,.\, 7 \,.\, 5 - 9 \,.\, 7 \,.\, 3 + 9 \,.\, 7 \,.\, 1 + 9 \,.\, 5 \,.\, 3 - 9 \,.\, 5 \,.\, 1$$
$$+ 9 \,.\, 3 \,.\, 1 - 7 \,.\, 5 \,.\, 3 + 7 \,.\, 5 \,.\, 1 - 7 \,.\, 3 \,.\, 1 + 5 \,.\, 3 \,.\, 1,$$

$$5 \,.\, (9 \,.\, 21) = 9 \,.\, 7 \,.\, 5 \,.\, 3 \,.\, 1.$$

I now pass on to the cases where n is an even number.

Third Theorem. Let n be of the form $4m + k$, where k is zero or 2; construct the sequence

$$1 \,.\, n,\ \ 2\,(n-1),\ \ 3\,(n-2) \dots \left(\frac{n}{2} - 1\right)\left(\frac{n}{2} + 2\right);$$

the ith anakolouthic series of products formed out of these elements is equal to the ith complete series of products formed out of the elements $(n-2)^2, (n-6)^2, \dots (k+2)^2$.

Ex. Let $n = 10$, the two sequences will be

$$10,\ 18,\ 24,\ 28,$$
$$64,\ 16,$$

and we have

$$10 + 18 + 24 + 28 = 80 = 64 + 16,$$

$$10 \,.\, 24 + 10 \,.\, 28 + 18 \,.\, 28 = 1024 = 64 \,.\, 16.$$

So, if $n = 12$, the two sequences will be

$$12,\ 22,\ 30,\ 36,\ 40,$$
$$100,\ 36,\ 4;$$

and we have

$$12 + 22 + 30 + 36 + 40 = 140 = 100 + 36 + 4,$$

$$12 \,.\, (30 + 36 + 40) + 22 \,.\, (36 + 40) + 30 \,.\, 40 = 4144$$
$$= 100 \,.\, 36 + 100 \,.\, 4 + 36 \,.\, 4,$$

$$12 \,.\, 30 \,.\, 40 = 4 \,.\, 36 \,.\, 100.$$

Fourth Theorem. If n be any even number, and we form the three sequences

$$1 \,.\, n, \quad 2\,(n-1), \quad 3\,(n-2) \dots \frac{n}{2}\left(\frac{n}{2} + 1\right),$$

$$1 \,.\, (n+2), \quad 2\,(n+1), \quad 3\,(n) \dots \frac{n}{2}\left(\frac{n}{2} + 3\right),$$

$$1 \,.\, n, \quad 2\,(n-1), \quad 3\,(n-2) \dots \left(\frac{n}{2} - 2\right)\left(\frac{n}{2} + 3\right),$$

the ith anakolouthic sum in respect to the second sequence less the ith anakolouthic sum in respect to the first sequence is equal to $\frac{n}{2}\left(\frac{n}{2}+1\right)$ into the $(i-1)$th anakolouthic sum in respect to the third sequence.

Ex. Take the three sequences

$$\begin{array}{lllll} 1.10, & 2.9, & 3.8, & 4.7, & 5.6, \\ 1.12, & 2.11, & 3.10, & 4.9, & 5.8, \\ 1.10, & 2.9, & 3.8. & & \end{array}$$

These, written out with simple elements, are as follows:

$$\begin{array}{lllll} 10 & 18 & 24 & 28 & 30, \\ 12 & 22 & 30 & 36 & 40, \\ 10 & 18 & 24; & & \end{array}$$

and we have

$$(12+22+30+36+40)-(10+18+24+28+30)=30.1,$$

$$\{12.(30+36+40)+22.(36+40)+30.40\}$$
$$-\{10.(24+28+30)+18.(28+30)+24.30\}$$
$$=4144-2584=1560=30.(10+18+24),$$

$$12.30.40-10.24.30=14400-7200=7200=30.(10.24).$$

These four theorems are only particular cases of one much more general relating to a determinant, to which I was led by my method of integrating the system of two partial differential equations to the general invariant of a function or system of functions of two variables. In like manner the integration of the system of t partial differential equations to the general invariant of a function or system of functions of t variables conducts to a determinant*, of which *à priori* we know the constitution, and which will (save as to the periodic occurrence of a single factor λ) resolve itself into factors of the form $\lambda^t \pm m^t$, m being an integer; and thus promises to lay open a road to the discovery of a new genus of theorems relating to the powers of the natural progression of integer numbers, destined apparently to occupy a sort of neutral ground between the formal and quantitative arithmetics.

* The integration of this system of equations always depends essentially upon the integration of one homogeneous equation which is doubly linear, that is of the first degree in the variables, and also of the first degree in respect to the order of the differentiations; such an equation can always be integrated, and the integral will depend upon the solution of an algebraical equation expressed by equating a certain determinant to zero.

10.

NOTE ON BURMAN'S LAW FOR THE INVERSION OF THE INDEPENDENT VARIABLE.

[*Philosophical Magazine,* VIII. (1854), pp. 535—540.]

THIS Note refers to the development of the nth differential coefficient of u in respect to x in terms of the nth and lower differential coefficients of x in respect to u.

The late Mr Gregory, in his very valuable book of examples on the Calculus, in alluding to this development, speaks of it as "extremely complicated, and involving so much preliminary matter for its demonstration," that he contents himself "with referring to a memoir by Mr Murphy on the subject in the *Philosophical Transactions,* 1837, p. 210." The development there given is of course essentially no other than that included in Burman's general formula. I recently have had occasion (as a preliminary step to the investigation of the laws of inverse transformation between two systems of t variables each, instead of between two single variables only, an investigation in which I have already made such progress that I expect shortly to be in possession of the general formula for the purpose) to reconsider what I shall term Burman's law, and have been somewhat surprised to find that, so far from affording a complicated expression, it does, when properly stated, give rise to an expression of the very simplest form that could be conceived or desired, and one that admits of an easy and elementary proof.

To fix the ideas, let us take the case of $\dfrac{d^7u}{dx^7}$, where $x = \phi u$. For greater brevity write $\dfrac{d^rx}{du^r}$ as x_r. The most cursory consideration will suffice to show, irrespective of all calculation, that we should have the following form of expansion, namely,

$$\frac{d^7u}{dx^7} = - x_7 \div x_1^8$$
$$+ \{(2, 6)\, x_2x_6 + (3, 5)\, x_3x_5 + (4, 4)\, x_4x_4\} \div x_1^9$$
$$- \{(2, 2, 5)\, x_2x_2x_5 + (2, 3, 4)\, x_2x_3x_4 + (3, 3, 3)\, x_3x_3x_3\} \div x_1^{10}$$
$$+ \{(2, 2, 2, 4)\, x_2x_2x_2x_4 + (2, 2, 3, 3)\, x_2x_2x_3x_3\} \div x_1^{11}$$
$$- \{(2, 2, 2, 2, 3)\, x_2x_2x_2x_2x_3\} \div x_1^{12}$$
$$+ (2, 2, 2, 2, 2, 2) \div x_1^{13}.$$

In the first group of a single term, 7 is taken in one part, in the second group of 3 terms, 8 is taken in every possible way of partition in two parts, in the third group of 3 terms, 9 is taken in every possible way of partition in three parts, and so on, until finally 12, that is, the double of the number next inferior to the given index 7, is taken in the sole possible way in which it can be taken of six parts; I ought to add, that in the groups of indices, *unity* is always understood to be inadmissible.

The groups of indices in the parentheses indicate numerical coefficients to be determined, and the whole and sole real difficulty (if any) of the question consists in determining the value of these numerical symbols. Now the law which furnishes these values would be seen on the most perfunctory examination to be the very simplest law that could possibly be stated, namely, any such symbol as $(r, s, t, \ldots)$ is to be understood to denote the number of distinct ways in which a number of things equal to the sum of the indices r, s, t, &c. admit of being thrown into combination groups of r, s, t, &c.!

Thus, for example,

$$(2, 6) = \frac{8!}{2!\,6!} = 28, \qquad (3, 5) = \frac{8!}{3!\,5!} = 56, \qquad (4, 4) = \frac{1}{2}\frac{8!}{(4!)^2} = 35,$$

$$(2, 2, 5) = \frac{1}{2}\frac{9!}{(2!)^2\,5!}, \qquad (2, 3, 4) = \frac{9!}{2!\,3!\,4!}, \qquad (3, 3, 3) = \frac{1}{6}\frac{9!}{(3!)^3},$$

$$(2, 2, 2, 4) = \frac{1}{3!}\frac{10!}{(2!)^3\,4!}, \qquad (2, 2, 3, 3) = \frac{1}{(2!)^2}\frac{10!}{(2!)^2\,(3!)^2},$$

and so on. The general law is obvious; and to prove its applicability in general, we have only to show that if it be true for the case of $\frac{d^ru}{dx^r}$, it is true for $\frac{d^{r+1}u}{dx^{r+1}}$. The proof is as follows. Let in general $[l, m, n, \&c.]$ indicate the value of

$$\frac{1\,.\,2\,.\,3 \ldots\ldots (l + m + n + \&c.)}{1\,.\,2 \ldots l \times 1\,.\,2 \ldots m \times 1\,.\,2 \ldots n \times \&c.},$$

without reference to $l, m, n,$ &c. being equal or unequal *inter se*.

Lemma 1. It is very easily seen that

$$[l, m, n, \&c.] = [l-1, m, n, \&c.] + [l, m-1, n, \&c.] + [l, m, n-1, \&c.] + \&c.$$

If now we use the notation $[\rho^r, \sigma^s, \tau^t, \ldots]$ as an abbreviated form of the notation $[\rho, \rho, \rho \ldots$ to r terms, $\sigma, \sigma, \sigma \ldots$ to s terms, $\tau, \tau \ldots$ to t terms, &c.$]$, it is obvious that the equation last written becomes

$$[\rho^r, \sigma^s, \tau^t, \ldots] = r[\rho-1, \rho^{r-1}, \sigma^s, \tau^t, \ldots] + s[\rho^r, \sigma-1, \sigma^{s-1}, \tau^t, \ldots] + t[\rho^r, \sigma^s, \tau-1, \tau^{t-1}, \ldots] + \ldots.$$

Lemma 2. Let $C(\rho^r, \sigma^s, \tau^t, \ldots)$ denote the number of ways in which $r\rho + s\sigma + t\tau + \ldots$ can be taken in combinations of $\rho, \rho \ldots$ to r places, $\sigma, \sigma \ldots$ to s places, &c., then upon the supposition that ρ, σ, τ, &c., which are to be understood as arranged in an ascending order of magnitude, are all unequal, we shall have

$$C(\rho^r, \sigma^s, \tau^t, \ldots) = [\rho^r, \sigma^s, \tau^t, \ldots]/r!\, s!\, t! \ldots,$$

which by Lemma 1

$$= \frac{[\rho-1, \rho^{r-1}, \sigma^s, \tau^t, \ldots]}{(r-1)!\, s!\, t! \ldots} + \frac{[\rho^r, \sigma-1, \sigma^{s-1}, \tau^t, \ldots]}{r!\, (s-1)!\, t! \ldots} + \frac{[\rho^r, \sigma^s, \tau-1, \tau^{t-1}, \ldots]}{r!\, s!\, (t-1)! \ldots} + \ldots$$

$$= C(\rho-1, \rho^{r-1}, \sigma^s, \tau^t, \ldots) + \{1 + rF(\sigma-\rho)\}\, C(\rho^r, \sigma-1, \sigma^{s-1}, \tau^t, \ldots) + \{1 + sF(\tau-\sigma)\}\, C(\rho^r, \sigma^s, \tau-1, \tau^{t-1}, \ldots) + \ldots.$$

$F(\sigma-\rho)$, $F(\tau-\sigma)$, &c. meaning quantities which are respectively zero when $\sigma - 1 > \rho$, $\tau - 1 > \sigma$, &c., and respectively units when $(\sigma-1) = \rho$, $\tau - 1 = \sigma$, &c.; for it will be obvious that if $\sigma - 1 = \rho$, the quantity

$$[\rho^r, \sigma-1, \sigma^{s-1}, \tau^t, \ldots]$$

becomes

$$[\rho^{r+1}, \sigma^{s-1}, \tau^t, \ldots],$$

and consequently when divided by $r!\,(s-1)!\,t! \ldots$ does not give

$$C(\rho^{r+1}, \sigma^{s-1}, \tau^t, \ldots),$$

but

$$(r+1)\, C(\rho^{r+1}, \sigma^{s-1}, \tau^t, \ldots),$$

and so similarly for the cases of $\tau - 1 = \sigma$, &c.

Now let us suppose that we are considering any group ($\rho, \rho \ldots$ to r places, $\sigma, \sigma \ldots$ to s places, &c.), or more briefly $(\rho^r, \sigma^s, \tau^t, \ldots)$, the numerical coefficient of the term $x_\rho{}^r x_\sigma{}^s x_\tau{}^t \ldots$ in the inverse development of $\dfrac{d^\mu x}{du^\mu}$.

And first, suppose that ρ is not 2.

The coefficient in question will evidently be made up exclusively of the following parts, each, however, affected with the factor $(-)^{N-1}$, derived from

the expansion of $\frac{d^{\mu-1}x}{du^{\mu-1}}$, for which the law to be established is supposed to hold, namely,

$$\left.\begin{aligned} &C(\rho-1,\ \rho^{r-1},\ \sigma^s,\ \tau^t,\ \ldots) \\ +\{1+rF(\sigma-\rho)\}\ &C(\rho^r,\ \sigma-1,\ \sigma^{s-1},\ \tau^t,\ \ldots) \\ +\{1+sF(\tau-\sigma)\}\ &C(\rho^r,\ \sigma^s,\ \tau-1,\ \tau^{t-1},\ \ldots) \\ +\ \&\text{c.}\ & \end{aligned}\right\} \quad (\vartheta),$$

each part being affected with the factor $(-1)^{N-1}$, derived from the differentiations performed upon

$$x_{\rho-1}\, x_\rho^{\ r-1} x_\sigma^{\ s} x_\tau^{\ t} \ldots \div x_1^{\ N},$$

$$x_\rho^{\ r} x_{\sigma-1}\, x_\sigma^{\ s-1} x_\tau^{\ t} \ldots \div x_1^{\ N},$$

$$x_\rho^{\ r} x_\sigma^{\ s} x_{\tau-1}\, x_\tau^{\ t-1} \ldots \div x_1^{\ N},$$

&c.

Secondly, suppose ρ, the lowest index, is 2, then the term

$$x_{\rho-1}\, x_\rho^{\ r} x_\sigma^{\ s} x_\tau^{\ t} \ldots$$

must be rejected, because $x_{\rho-1}$ becomes x_1, which is excluded from appearing in any numerator. But then, *per contra*, in this case there will be a portion of the coefficient derivable from the differentiation of the denominator of the term

$$(-)^{N-2} \cdot \frac{(2^{r-1},\ \sigma^s,\ \tau^t \ldots)\ x_2^{\ r-1} x_\sigma^{\ s} x_\tau^{\ t} \ldots}{x_1^{\ N-1}},$$

where $$(N-1) = 1 + (r-1)\,2 + s\sigma + t\tau + \&\text{c.}$$

This portion will be

$$(-)^{N-1}(N-1)\, C(2^{r-1},\ \sigma^s,\ \tau^t,\ \ldots),$$

or, which is the same thing,

$$C(1,\ 2^{r-1},\ \sigma^s,\ \tau^t,\ \ldots),$$

and therefore the portion of the coefficient corresponding to $x_{\rho-1}\, x_\rho^{\ r} x_\sigma^{\ s} x_\tau^{\ t} \ldots$, &c. is supplied from another source, and the expression (ϑ) remains good for *all* values of ρ, σ, τ, &c., and consequently, by virtue of the second lemma, is equal to $C(\rho^r,\ \sigma^s,\ \tau^t,\ \ldots)$; and thus we see that if the law assumed is true for $\frac{d^r u}{dx^r}$ it remains true for $\frac{d^{r+1}u}{dx^{r+1}}$, as was to be shown. And as it is evidently true for $r = 1$, it is true generally.

POSTSCRIPT.

The formula expressing Burman's law may be exhibited as follows: x_r will still be understood to denote $\dfrac{d^r x}{du^r}$, and $C\{p, q, \ldots m\}$ will, as before, denote the number of distinct modes of combining $p+q+\ldots+m$ things in sets of $p, q, \ldots m$ at a time; so that, for example, $C\{2, 2, 4, 4, 4\}$ will denote

$$\frac{1\times 2\times 3\ldots\times 16}{(1.2)^2.(1.2.3.4)^3}\cdot\frac{1}{1.2}\cdot\frac{1}{1.2.3}.$$

Let now $n-1$ be broken up *without restriction in every possible way* into parts, and let $r, s, t \ldots l$ denote one such system of parts so that

$$r+s+t+\ldots+l=n-1,$$

r, s, &c. being all *actual* positive integers. Then is $\dfrac{d^n u}{dx^n}$ equal to

$$\Sigma C\{(1+r), (1+s), (1+t)\ldots(1+l)\}\cdot\frac{1}{x_1{}^n}\cdot\left\{\frac{-x_{1+r}}{x_1}\cdot\frac{-x_{1+s}}{x_1}\cdot\frac{-x_{1+t}}{x_1}\cdots\frac{-x_{1+l}}{x_1}\right\},$$

than which nothing more clear and simple can be desired or imagined. And so more generally, if we make, as before, $r+s+t+\ldots+l=n-g$, and give g in succession every different value from 1 to n, we shall have $\dfrac{d^n\vartheta}{dx^n}$ equal to

$$\Sigma\Sigma\left[[\{(1+r), (1+s), \ldots (1+l)\}, (g-1)]\,\frac{d^g\vartheta}{du^g}\cdot\frac{1}{x_1{}^n}\left(\frac{-x_{1+r}}{x_1}\cdot\frac{-x_{1+s}}{x_1}\cdots\frac{-x_{1+l}}{x_1}\right)\right],$$

where $[\{(1+r), (1+s), \ldots (1+l)\}, (g-1)]$ means the number of ways in which $(1+r)+(1+s)+\ldots+(1+l)+(g-1)$ elements can be partitioned off into groups of one kind containing respectively $(1+r), (1+s), \ldots (1+l)$ of the elements, and into a group of another kind containing the remainder $(g-1)$ of the elements. This distinction of the groups into two kinds has no effect upon the result except when $g-1$ is equal to any of the numbers $(1+r)$, $(1+s), \ldots (1+l)$. If we write, according to the notation above employed, $(1+r), (1+s), \ldots (1+l)$ under the form $(\alpha^a, \beta^b, \ldots \gamma^c)$, then

$$[\{(1+r), (1+s), \ldots (1+l), (g-1)\}]$$

will represent

$$\frac{(a\alpha+b\beta+\ldots+c\gamma+g-1)!}{a!\,(\alpha!)^a\,b!\,(\beta!)^b\ldots c!\,(\gamma!)^c\,(g-1)!}.$$

This more general theorem may of course be demonstrated by a similar method to that employed in the text for the case of $\vartheta = u$, for which all the terms in the expansion vanish except those in which $g = 1$.

I have, since this paper was sent to the press, obtained a new solution of the far more difficult and interesting question of the change from one *system* of independent variables to another system*. I say a new solution, because one has already been *virtually* effected, but under a form leaving much to be desired, by the great Jacobi in his Memoir *De Resolutione Æquationum per series infinitas*, Crelle, Vol. VI. 1830. In my solution, a remarkable species of quantities, to which I give the name of Arborescent Functions, make their appearance in analysis for the first time.

[* p. 65 below.]

11.

ON DIFFERENTIAL TRANSFORMATION AND THE REVERSION OF SERIESES*.

[*Proceedings of the Royal Society of London*, VII. (1856), pp. 219—223.]

[Also *Philosophical Magazine*, IX. (1855), pp. 391—394.]

WITH a view to its publication in the *Proceedings* of the Society, I take occasion to communicate the result of my investigations, as far as they have yet extended, into the general theory of differential transformations, containing a complete and general solution of the important problem of expanding a given partial differential coefficient of a function in respect of one system of independent variables in terms of the partial differential coefficients thereof, in respect to a second system of independent variables, each respectively given as explicit functions of the first set.

This question may be shown to be exactly coincident with that of the reversion of simultaneous serieses proposed by Jacobi, which may be thus stated: given $(n+1)$ quantities, each expressed by rational infinite serieses as functions of n others; required to express any one of the first set in a rational infinite series in terms of the other n of the same set. This question has only been resolved by Jacobi for a particular case; the result hereunder given for the transformation of differential coefficients contains the solution of the general question. My method of investigation is entirely different from that adopted by the great Jacobi, and I hope in a short time to be able to lay it in a complete form before the Society, and probably to add a solution of the still more general question comprising the reversion of serieses as a particular case, namely, the question of expressing any one of n quantities connected by m equations in terms of any $(n-m)$ others of the same.

Let there be any number of variables, say u, v, w, of which x, y, z, ϑ are given functions, it is required to expand

$$\left(\frac{d}{dx}\right)^f \left(\frac{d}{dy}\right)^g \left(\frac{d}{dz}\right)^h \vartheta$$

in terms of the partial differential coefficients of ϑ, x, y, z in respect of u, v, w.

[* See p. 65, below.]

Form the determinant

$$\begin{vmatrix} \frac{dx}{du}, & \frac{dx}{dv}, & \frac{dx}{dw} \\ \frac{dy}{du}, & \frac{dy}{dv}, & \frac{dy}{dw} \\ \frac{dz}{du}, & \frac{dz}{dv}, & \frac{dz}{dw} \end{vmatrix},$$

which call J.

The required expansion will contain in each term an integer numerical coefficient, a power of $\frac{1}{J}$, one factor of the form

$$\left(\frac{d}{du}\right)^p \left(\frac{d}{dv}\right)^q \left(\frac{d}{dw}\right)^r \vartheta,$$

and other factors of the form

$$\left(\frac{d}{du}\right)^l \left(\frac{d}{dv}\right)^m \left(\frac{d}{dw}\right)^n x,$$

$$\left(\frac{d}{du}\right)^{l'} \left(\frac{d}{dv}\right)^{m'} \left(\frac{d}{dw}\right)^{n'} y,$$

$$\left(\frac{d}{du}\right)^{l''} \left(\frac{d}{dv}\right)^{m''} \left(\frac{d}{dw}\right)^{n''} z.$$

Let the latter class of factors be distinguished into two sets, those where $l+m+n=1$,

$$\left(\begin{array}{lll} \quad l=1 & m=0 & n=0 \\ \text{or } l=0 & m=1 & n=0 \\ \text{or } l=0 & m=0 & n=1 \end{array}\right),$$

which I shall call uni-differential factors, and those in which $l+m+n>1$, which I shall call pluri-differential factors.

First, then, as to the form of the general term abstracting from the numerical coefficient and the uni-differential factors (except of course so far as they enter into J). This will be as follows:

$$\left(\frac{d}{du}\right)^{{}^1l_1} \left(\frac{d}{dv}\right)^{{}^1m_1} \left(\frac{d}{dw}\right)^{{}^1n_1} x \times \left(\frac{d}{du}\right)^{{}^2l_1} \left(\frac{d}{dv}\right)^{{}^2m_1} \left(\frac{d}{dw}\right)^{{}^2n_1} x \times \ldots \left(\frac{d}{du}\right)^{{}^{e_1}l_1} \left(\frac{d}{dv}\right)^{{}^{e_1}m_1} \left(\frac{d}{dw}\right)^{{}^{e_1}n_1} x$$

$$\times \left(\frac{d}{du}\right)^{{}^1l_2} \left(\frac{d}{dv}\right)^{{}^1m_2} \left(\frac{d}{dw}\right)^{{}^1n_2} y \times \ldots\ldots \times \left(\frac{d}{du}\right)^{{}^{e_2}l_2} \left(\frac{d}{dv}\right)^{{}^{e_2}m_2} \left(\frac{d}{dw}\right)^{{}^{e_2}n_2} y$$

$$\times \left(\frac{d}{du}\right)^{{}^1l_3} \left(\frac{d}{dv}\right)^{{}^1m_3} \left(\frac{d}{dw}\right)^{{}^1n_3} z \times \ldots\ldots \times \left(\frac{d}{du}\right)^{{}^{e_3}l_3} \left(\frac{d}{dv}\right)^{{}^{e_3}m_3} \left(\frac{d}{dw}\right)^{{}^{e_3}n_3} z$$

$$\times \left(\frac{d}{du}\right)^p \left(\frac{d}{dv}\right)^q \left(\frac{d}{dw}\right)^r \vartheta \times \frac{1}{J^\omega},$$

subject to the limitations about to be expressed.

Call $$^1l_1 + {}^2l_1 + \ldots + {}^{e_1}l_1 = L_1,$$
$$^1l_2 + {}^2l_2 + \ldots + {}^{e_2}l_2 = L_2,$$
$$^1l_3 + {}^2l_3 + \ldots + {}^{e_3}l_3 = L_3,$$

and form the analogous quantities M_1, M_2, M_3; N_1, N_2, N_3. Then we must have

$$L_1 + L_2 + L_3 + M_1 + M_2 + M_3 + N_1 + N_2 + N_3 + p + q + r = f + g + h + e_1 + e_2 + e_3;$$

and as the sum of any group of indices l, m, n must not be less than 2, we have

$$f + g + h + e_1 + e_2 + e_3 + p + q + r, \text{ not less than } 2e_1 + 2e_2 + 2e_3,$$

so that $e_1 + e_2 + e_3$ must not exceed $f + g + h + p + q + r$; furthermore, $p + q + r$ must not exceed $f + g + h$; and finally,

$$\omega = f + g + h + e_1 + e_2 + e_3.$$

1. We may first take $e_1 + e_2 + e_3 = E$, giving to E in succession all integer values from $f + g + h$ to $2f + 2g + 2h$, and find all possible solutions of this equation with permutations between the values of e_1, e_2, e_3.

2. We may then take $p + q + r = s$, giving s in succession all integer values from 1 to $f + g + h$, and find all possible solutions of this equation with permutations between f, g, h.

3. We may then take $L + M + N = f + g + h + E - s$, and find all the values of L, M, N, with permutations allowable between the values of L, M, N.

4. We may then take
$$L_1 + L_2 + L_3 = L,$$
$$M_1 + M_2 + M_3 = M,$$
$$N_1 + N_2 + N_3 = N,$$

and solve these several equations in every way possible, with permutations as before.

5. We must take

$$^1l_1 + {}^2l_1 + \ldots + {}^{e_1}l_1 = L_1, \quad {}^1m_1 + {}^2m_1 + \ldots + {}^{e_1}m_1 = M_1, \quad {}^1n_1 + {}^2n_1 + \ldots + {}^{e_1}n_1 = N_1,$$
$$^1l_2 + {}^2l_2 \quad \ldots \quad {}^{e_2}l_2 = L_2, \quad {}^1m_2 + {}^2m_2 \quad \ldots \quad {}^{e_2}m_2 = M_2, \quad {}^1n_2 + {}^2n_2 \quad \ldots \quad {}^{e_2}n_2 = N_2,$$
$$^1l_3 + {}^2l_3 \quad \ldots \quad {}^{e_3}l_3 = L_3, \quad {}^1m_3 + {}^2m_3 \quad \ldots \quad {}^{e_3}m_3 = M_3, \quad {}^1n_3 + {}^2n_3 \quad \ldots \quad {}^{e_3}n_3 = N_3,$$

and solve in every possible manner these equations, but without admitting permutations between the values of $^1l_1, {}^2l_1 \ldots {}^{e_1}l_1$, or between the values of the members of the other of the third sets taken each *per se*, and subject to the

condition that every such sum as ${}^r l_i + {}^r m_i + {}^r n_i$ must be greater than unity. Every possible system of values of these nine sets will furnish a corresponding pluri-differential part to the general term.

Next, as to the uni-differential part, we may form the quantity

$$\left(\frac{dy}{dv}\frac{dz}{dw}-\frac{dy}{dw}\frac{dz}{dv}\right)^{\lambda_1}\left(\frac{dy}{dw}\frac{dz}{du}-\frac{dy}{du}\frac{dz}{dw}\right)^{\mu_1}\left(\frac{dy}{du}\frac{dz}{dv}-\frac{dy}{dv}\frac{dz}{du}\right)^{\nu_1},$$

$$\left(\frac{dz}{dv}\frac{dx}{dw}-\frac{dz}{dw}\frac{dx}{dv}\right)^{\lambda_2}\left(\frac{dz}{dw}\frac{dx}{du}-\frac{dz}{du}\frac{dx}{dw}\right)^{\mu_2}\left(\frac{dz}{du}\frac{dx}{dv}-\frac{dz}{dv}\frac{dx}{du}\right)^{\nu_2},$$

$$\left(\frac{dx}{dv}\frac{dy}{dw}-\frac{dx}{dw}\frac{dy}{dv}\right)^{\lambda_3}\left(\frac{dx}{dw}\frac{dy}{du}-\frac{dx}{du}\frac{dy}{dw}\right)^{\mu_3}\left(\frac{dx}{du}\frac{dy}{dv}-\frac{dx}{dv}\frac{dy}{du}\right)^{\nu_3},$$

where

$$\lambda_1+\lambda_2+\lambda_3=L+p,$$
$$\mu_1+\mu_2+\mu_3=M+q,$$
$$\nu_1+\nu_2+\nu_3=N+r.$$

These equations are to be solved in every possible manner with permutations between the members of the λ set, the μ set, and the ν set. Finally, we have to consider the numerical coefficient. To give a perfect representation of this, we must ascertain what identities exist in the factors of the pluri-differential part. Let us suppose that one set of operators upon x is repeated θ_1 times, another θ_2 times, and so on, giving rise to the powers $\theta_1, \theta_2, \ldots\ldots \theta_\alpha$ in the x line. Similarly, form $\phi_1, \phi_2, \ldots \phi_\beta$ from the y line, and $\psi_1, \psi_2, \ldots \psi_\gamma$ from the z line. Then the numerical part of the general term will be

$$\frac{\Pi(\lambda_1+\mu_1+\nu_1)\,\Pi(\lambda_2+\mu_2+\nu_2)\,\Pi(\lambda_3+\mu_3+\nu_3)}{\Pi\lambda_1\Pi\mu_1\Pi\nu_1\Pi\lambda_2\Pi\mu_2\Pi\nu_2\Pi\lambda_3\Pi\mu_3\Pi\nu_3}$$
$$\times\frac{\Pi(L+p)\,\Pi(M+q)\,\Pi(N+r)}{\left\{\begin{matrix}\Pi\,{}^1l_1\,\Pi\,{}^1m_1\,\Pi\,{}^1n_1\,\Pi\,{}^2l_1\,\Pi\,{}^2m_1\,\Pi\,{}^2n_1\ldots\ldots\\ \Pi\,{}^1l_2\,\Pi\,{}^1m_2\,\Pi\,{}^1n_2\,\Pi\,{}^2l_2\,\Pi\,{}^2m_2\,\Pi\,{}^2n_2\ldots\ldots\\ \Pi\,{}^1l_3\,\Pi\,{}^1m_3\,\Pi\,{}^1n_3\,\Pi\,{}^2l_3\,\Pi\,{}^2m_3\,\Pi\,{}^2n_3\ldots\ldots\end{matrix}\right\}}$$
$$\times\frac{D}{\Pi\theta_1\Pi\theta_2\ldots\Pi\theta_\alpha\Pi\phi_1\Pi\phi_2\ldots\Pi\phi_\beta\Pi\psi_1\Pi\psi_2\ldots\Pi\psi_\gamma},$$

where in general Πm means $1.2.3\ldots m$: as regards D, it is the following determinant, namely,

$$\begin{vmatrix}\lambda_1+\mu_1+\nu_1 & \nu & \nu & L_3 & M_3 & N_3\\ \nu & \lambda_2+\mu_2+\nu_2 & \nu & L_2 & M_2 & N_2\\ \nu & \nu & \lambda_3+\mu_3+\nu_3 & L_1 & M_1 & N_1\\ \lambda_1 & \lambda_2 & \lambda_3 & L_1+L_2+L_3+p & \nu & \nu\\ \mu_1 & \mu_2 & \mu_3 & \nu & M_1+M_2+M_3+q & \nu\\ \nu_1 & \nu_2 & \nu_3 & \nu & \nu & N_1+N_2+N_3+r\end{vmatrix}.$$

The result, for greater brevity, has been set out in the above pages for the case of ϑ, a function of three variables, but the reader can have no difficulty in extending the statement to any number. In the case of a single variable, the formula can easily be identified with that given by Burman's law. It is noticeable that the determinant written is of the form

$$Apqr + Bpq + Cqr + Drp + Ep + Fq + Gr,$$

the part independent of p, q, r being easily seen to vanish. Moreover, the coefficients $A, B, C, \ldots$ are all essentially positive, so that the determinant can only vanish (except for $p=0$, $r=0$, $q=0$) by virtue of one condition at least more than the number of the variables.

12.

A TRIFLE ON PROJECTILES.

[*Philosophical Magazine*, XI. (1856), pp. 450—453.]

In teaching the subject of projectiles *in vacuo*, the following solution has presented itself to me of a question not wholly without practical interest, namely, of determining the angle of projection to give the best range in the most general case, namely, when a gun is fired upon a slope at a given vertical height above the slope. The solution is not wholly either without theoretical interest in point of method, as leading to a result of some little complexity in maxima and minima by very simple calculations, and without the aid of the differential calculus. Therefore I venture to submit it to the readers of the *Philosophical Magazine*. In the next number of the Magazine I hope to have leisure to lay before them a subject of much greater interest, also belonging to the theory of projectiles, showing how, by the oblique action of gravity combined with the earth's rotation, a pendulum suitably adjusted may be caused to advance in a westerly direction, and so the earth be made the means of impelling a light carriage without any visible motive force, or any influence of magnetism.

To this pendulum I give, for reasons which will be apparent when the matter is more clearly set forth, and in contradistinction to the ordinary fixed or circular pendulum on the one hand, and to Foucault's free or spherical pendulum on the other, the name of the *Cylindrical* or *Travelling* Pendulum. But to resume the business of this present communication: let us begin with determining the angle of projection to give the maximum range when a gun is fired from a point *in* a plane sloping at an angle i from the horizon.

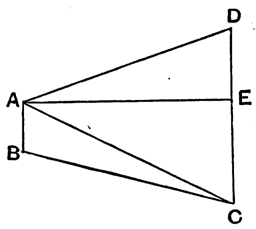
This question is most simply solved (the result itself is of course familiar to all who will read this paper) by resolving the velocity V, supposed to make an angle θ with the horizon, as also g, the accelerating force of gravity, each into two parts, V into $V\cos(\theta+i)$ and $V\sin(\theta+i)$, and g into $g\sin i$ and $g\cos i$, respectively parallel and perpendicular to the plane of the slope.

The time of flight is of course found by looking to the perpendicular part of the velocity and of gravity alone, and is evidently $2\dfrac{V\sin(\theta+i)}{g\cos i}$, which call τ; the range will evidently be

$$\frac{V\cos\theta\,.\,\tau}{\cos i},\ \text{that is,}\ \frac{V^2}{g\cos i}\{\sin(2\theta+i)+\sin i\}.$$

Hence the best angle of range for this case is found by making $2\theta+i=90°$, $\theta=\frac{1}{2}(90°-i)$.

Now let us proceed to apply this result to the general case, as in the figure below, where BC is the slope upon which the range is to be measured, A the point of projection, AD the direction which gives the maximum range upon the slope, and BC the actual extent of this range; then I say AD is the direction which would give also the best range upon the slope AC. Since if, with the given velocity of projection, any other direction than AD would give a better range upon AC, the path corresponding to such direction must evidently cut BC at a point beyond C in that line in order to strike a point beyond C in the line AC.



Hence if we draw the horizontal line AE, we know by the preceding case that the angle $DAE=\frac{1}{2}CAB$*.

Let $CAB=\phi$, which is to be found; also let $AB=h$, and the inclination of BC to $AE=i$, h and i being given; and let $t=$ time of flight, then

$$CAD=(90°-\phi)+\frac{\phi}{2}$$

$$=90°-\frac{\phi}{2}.$$

Hence also $$ADC=180°-\phi-\left(90°-\frac{\phi}{2}\right)=90°-\frac{\phi}{2}.$$

Hence $$\tfrac{1}{2}gt^2=CD=AC=h\frac{\sin ABC}{\sin ACB}$$

$$=\frac{h\cos i}{\cos(i+\phi)};$$

* This equation, and the isoscelism of the principal triangle of the figure to which it leads, would not readily present themselves to notice in the direct method of seeking the maximum range. It is for the sake of this pleasing geometrical relation, not unmixed perhaps with a desire of exhibiting the simple yet delicate turn of reasoning, the agreeable little point of method (a fly embalmed in amber) contained in the immediately preceding paragraph, that I have thought this trifle worth preserving in the pages of the Magazine.

and $$v \cos \frac{\phi}{2} t = AE = \frac{h \sin \phi \cos i}{\cos (i+\phi)}.$$

Hence eliminating t, we have

$$\frac{v^2}{gh \cos i} = \frac{(\sin \phi)^2}{1+\cos \phi} \cdot \frac{1}{\cos (i+\phi)} = \frac{1-\cos \phi}{\cos (i+\phi)}.$$

If $i=0$, that is, if the gun is fired from the top of a battery commanding a level plain, we have simply

$$\sec \phi = 1 + \frac{v^2}{gh},$$

which gives ϕ the double of the angle of elevation.

In other cases we may make $\phi + i = \psi$, we have then

$$\frac{1-\cos (\psi - i)}{\cos \psi} = \frac{1}{\cos \psi} - \frac{\sin \psi}{\cos \psi} \sin i - \cos i = \frac{v^2}{gh} \sec i.$$

Let $$\left(1 + \frac{v^2}{gh} \sec^2 i\right) \cot i = \cot \epsilon;$$

then $$\frac{\sin i}{\sin \epsilon} \cos (\psi - \epsilon) = 1,$$

$$\cos (\psi - \epsilon) = \frac{\sin \epsilon}{\sin i},$$

or $$\cos (\phi + i - \epsilon) = \frac{\sin \epsilon}{\sin i},$$

from which ϕ, the double of the angle of elevation, may be determined.

Calling $\frac{\sin \epsilon}{\sin i} = \cos \mu$, and taking ϕ_1, ϕ_2 as the two values of ϕ, we have

$$\phi_1 + i - \epsilon = \mu,$$

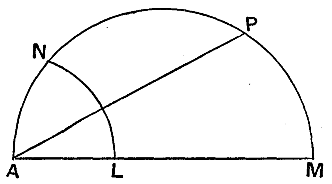
$$\phi_2 + i - \epsilon = 360° - \mu.$$

ϕ_1, ϕ_2 correspond to the angles of projection *down and up* the slope respectively, the one affording what in an *algebraical* sense is a maximum, and the other a minimum, but of course, *arithmetically* speaking, both giving maximum values of the range.

Thus when $h=0$, so that $\sin \epsilon = 0$, $\mu = 90°$, and $\frac{1}{2}(\phi_2 - \phi_1)$ is a right angle, as may easily be verified.

It may be worth while to exhibit the geometrical construction for the case of firing from a gun in position commanding a *horizontal plane.*

Let A be the position of the gun, LN a portion of a circle of radius AL which represents the height of the gun above the plain, LM twice the height due to the velocity of projection, ANM a semicircle on AM, P the point in it bisecting the arc MN, then (abstraction made of the resistance of the air) AP is the elevation at which the gun must be pointed to give the greatest range on the plain below, for sec $2PAM$ obviously $= 1 + \dfrac{(\text{velocity of ball})^2}{g \cdot AL}$.



Suppose a sea battery as much as 300 feet* above the water, and a cannon-ball projected at the low rate of 1200 feet per second (which is less than that of a common musket-ball), we should have twice the height due to the velocity of projection equal to 44720, and therefore

$$\sec 2\alpha = \frac{44720}{1200} + 1$$

$$= 38{,}2666,$$

and consequently $$2\alpha = 88^\circ\ 30'\ 9''$$

or $$\alpha = 44^\circ\ 15'\ 5'',$$

differing very little from 45°; showing that certainly in a non-resisting medium, and in all probability in air, the height of the point of fire above the plane which it commands will very little indeed influence, under any conceivable circumstances of practice, the angle of elevation which gives the best range.

[* The succeeding calculation uses 1200.]

13.

NOTE ON AN INTUITIVE PROOF OF THE EXISTENCE OF TWENTY-SEVEN CONICS OF CLOSEST CONTACT WITH A CURVE OF THE THIRD DEGREE.

[*Philosophical Magazine*, XI. (1856), pp. 463, 464.]

IN general a conic can only be made to have five coincident points with a curve, and if the curve be of the third degree, the conic will of course cut it in a remaining sixth point; but at certain points of the cubic all these six points may come together. How many of these are there, and where are they? This question, which originated with Steiner, who stated the number, and was subsequently treated by Plücker, who assigned the position of the points, may be resolved by very simple considerations and without calculation. For if we can succeed in putting the characteristic of the curve (I mean what is commonly, but not altogether commodiously, called "the-left-hand-side-of-the-equation-to-the-curve-when-the-right-hand-side-of-it-is-made-equal-to-zero") under the form $u^3 + v(uw + \omega^2)$, it is obvious that the conic $uw + \omega^2$ will intersect the cubic curve in the six coincident points $u^3 = 0$, $\omega^2 = 0$.

If now we take for our cubic the reduced form $x^3 + y^3 + z^3 - 6mxyz$, and make $x + y + 2mz = p$, $\rho x + \rho^2 y + 2mz = q$, $\rho^2 x + \rho y + 2mz = r$ [where ρ is an imaginary cube root of unity], it may be written under the form

$$(1 - 8m^3) z^3 + pqr, \text{ say } -\mu z^3 + pqr;$$

or, if we please, under the form

$$-\mu (z + kp)^3 + p(qr + \mu k^3 p^2 + 3\mu k^2 pz + 3\mu kz^2).$$

And if we assume k properly, $z + kp$ may be made to touch the multiplier of p, that is, the cubic may be made to take the form

$$-\mu (z + kp)^3 + p\{(z + kp) v + \omega^2\}.$$

From the symmetry which reigns between x and y, it is obvious *à priori* that any value of k which is rightly assumed for the object in view will make

ω (when z is eliminated from it by means of the equation $z+kp=0$) a multiple either of $x-y$ or $x+y$; the latter obviously cannot be true, since such values would make the given cubic a function of $x+y$ and z; the proper values of k will therefore make $x-y=0$, from which, combined with the equation $2x^3+z^3+6mx^2z=0$, the values of $x:y:z$ may be determined. These will be three in number; and as we may write, instead of x and y, ρx, $\rho^2 y$, or ρy, $\rho^2 x$, we obtain three sets of three points, corresponding to p being taken $x+y+2mz$; and consequently, by interchanging z with x and with y successively, we obtain altogether three systems of three sets of three points each; any such factor as $x+y+2mz$ is a tangent to a point of inflexion, and it is clear *à priori* that if the cubic is put under the form $u^3+v\,(uw+\omega^2)$, since $v=0$ makes $u^3=0$, v *can only be* a tangent at an inflexion. Hence the nine sets of three points just assigned are *all* that can be found enjoying the property in question, and it is readily seen that $x-y$ is the straight line containing the three points of intersection in which the second emanant,

$$\left(x'\frac{d}{dx}+y'\frac{d}{dy}+z'\frac{d}{dz}\right)^2(x^3+y^3+z^3-6mxyz),$$

at the point of inflexion ($x+y=0$, $z=0$) cuts the given cubic over and above the three coincident points $x+y=0$, $z=0$. In other words, each ternary group of the twenty-seven points in question consists of the three points in which the curve is met by the tangents drawn from a point of inflexion, which agrees with the geometrical construction given by Plücker in *Crelle's Journal*.

14.

LETTER ON PROFESSOR GALBRAITH'S CONSTRUCTION FOR THE RANGE OF PROJECTILES.

[*Philosophical Magazine*, XII. (1856), pp. 112—114.]

To the Editors of the Philosophical Magazine and Journal.

GENTLEMEN,

Professor Galbraith's geometrical construction for finding the elevations of a projectile corresponding to any given velocity and given range in a plane, horizontal or sloping, is truly elegant, and, if new, constitutes a real acquisition to the subject. It might be worth while for its accomplished author to see if some analogous construction can be found extending to the more general case where the field is a portion of a circle. I need hardly add that the *isoscelism* referred to is, except for some extreme suppositions (impossible to occur in practice), absolutely independent of the form of the field.

As well-constructed names are, in fact, condensed lessons, lending an aid to the memory and imagination, of which modern mathematicians are only beginning to appreciate the importance, I suggest the following designations.

The point of *projection* and point of *impact* speak for themselves; the point vertically over the point of impact in the direction of projection may be called the point of *aim*. The line joining the point of aim and the point of impact is the *drop* or *fall*; the line joining the point of projection and the point of impact may be called the *excursion*; and that joining the point of projection and the point of aim, the *length of aim*.

A vertical section of the ground (plane or curved) through the axis of the gun may be called the *field*. We may then say, that, for the maximum range, the fall is always equal to the excursion, whatever the form of the field; and that in general the locus of the point of aim, for a rectilinear field, when the point of the projection and the velocity are given, is a circle to which, in the

case of the angle of best elevation, the line of fall is of course a tangent. It would not be surprising if a good deal of elegant geometry (like ivy twining round an old wall) should hereafter associate itself with Mr Galbraith's "circle of aim": *à propos* of projectiles, it is not unworthy of observation, that the velocities at any two points P and Q of the parabolic path are as the lines PT, QT which the tangents at P and Q mutually cut off from one another, a remark which of course is easily seen to extend itself to the case of an elliptic orbit with the force in the centre.

Ever, Gentlemen,

Your faithful friend and reader,

J. J. SYLVESTER.

WOOLWICH COMMON,
July 3, 1856.

P.S. The value of Mr Galbraith's method consists simply in the *act of conception* of the locus of the point of aim; it was scarcely worth while (at this time of day) to append a synthetical proof of so simple a proposition, which may be got at immediately by calling the length of aim ρ, its inclination to the vertical, θ, and that of the field to the vertical, i; when by similar triangles (if H denote the quantity $\frac{2v^2}{g}$, and η the vertical distance of the point of projection from the field) we obtain the equation

$$\frac{\frac{\rho^2}{H} - \eta}{\rho} = \frac{\sin(i-\theta)}{\sin i},$$

or

$$\rho^2 - H\frac{\sin(i-\theta)}{\sin i}\rho - H\eta = 0;$$

which obviously corresponds to the circle of Professor Galbraith. I imagine this circle has been long known for the case of the point of projection being in the field, but it may have escaped notice for the more general case. The equality between the *fall* and the *excursion* for the angle of maximum range subsists, not merely for a rectilinear or curved section, but for the ground itself (whatever its form of surface) when the gun is supposed to admit of being laid to any angle, as well as at any elevation.

15.

RECHERCHES SUR LES SOLUTIONS EN NOMBRES ENTIERS POSITIFS OU NÉGATIFS DE L'ÉQUATION CUBIQUE HOMOGÈNE À TROIS VARIABLES.

[*Annali di Scienze Matematiche e Fisiche* (Tortolini), VII. (1856), pp. 398—400.]

J'AI l'honneur de vous envoyer pour être inséré dans votre journal estimable, si vous les jugéz dignes, les énoncés de quelques théorèmes que j'ai trouvé dans mes recherches sur les solutions en nombres entiers positifs ou négatifs de l'équation cubique homogène à trois variables.

On sait selon *Fermat* que l'équation

$$x^3 + y^3 + z^3 = 0$$

n'est pas résoluble en nombres entiers.

On peut ajouter la même chose pour les équations

$$x^3 + y^3 + 2z^3 = 0,$$

$$x^3 + y^3 + 3z^3 = 0\,;$$

j'ajoute que l'équation

$$x^3 + y^3 + z^3 + 6xyz = 0$$

est irrésoluble: aussi l'équation

$$2\,(x^3 + y^3 + z^3 + 6xyz) = 27nxyz,$$

quand $$27n^2 - 8n + 4$$

est un nombre premier, est irrésoluble: aussi l'équation

$$4\,(x^3 + y^3 + z^3 + 6xyz) = 27vxyz$$

est irrésoluble quand

$$27v^2 - 36v + 16$$

est un nombre premier.

De plus l'équation

$$x^3 + y^3 + Az^3 = Mxyz$$

est irrésoluble dans les circonstances suivantes.

Posons $$M^3 - 27A = \Delta^3\Delta',$$
où Δ' ne contient nul facteur cubique. Alors si Δ' est pair et ne contient nul nombre de la forme
$$f^2 + 3g^2,$$
et si A est un nombre premier, l'équation est irrésoluble, excepté dans les cas que $\sqrt{\frac{-M}{A}}$ soit un nombre entier, et dans ce cas-là on peut donner la solution général de l'équation.

La même chose a lieu quand, Δ' restant assujétie aux mêmes conditions qu'auparavant, A est une puissance d'un nombre premier de la forme $p^{3\omega\pm1}$.

La même chose a aussi lieu sans que Δ' soit pair, pourvu qu'il ne contient nul facteur $f^2 + 3g^2$, et que
$$A = 2^{3\omega\pm1}.$$
La même chose a lieu encore pourvu que Δ' ne contient nul nombre de la forme $f^2 + 3g^2$ avec les conditions suivantes:
$$\begin{cases} \dfrac{A}{2} = \text{un nombre premier de la forme } qi \pm 4, \\[2mm] \dfrac{M}{9} = \text{un nombre entier}, \end{cases}$$
ou
$$\begin{cases} \dfrac{A}{4} = \text{un nombre premier de la forme } qi \pm 2, \\[2mm] \dfrac{M}{18} = \text{un nombre entier}, \end{cases}$$
ou si A étant un nombre premier on a A, B respectivement de la forme
$$qn + 2,\ qn + 6,$$
ou bien de la forme $\quad qn - 2,\ qn - 6,$

ou bien de la forme $\quad qn + 4,\ qn + 3,$

ou de la forme $\quad qn - 4,\ qn - 3,$

ou de la forme $\quad qn \pm 3,\ qn.$

16.

ON THE CHANGE OF SYSTEMS OF INDEPENDENT VARIABLES.

[*Quarterly Journal of Mathematics,* I. (1857), pp. 42—56, 126—134.]

(1) THE theorem contained in the subjoined pages having been printed*, with many typographical and other errors†, in the *Proceedings of the Royal Society*, Vol. VII. No. 8, I think, on account of its importance to the direct march of the differential calculus, of which, as an instrument of expansion, it may be said to complete the processes, that the reissue of it in a more correct form may be acceptable and useful to the readers of this journal.

The purpose of the theorem is to effect for any number of variables the same end which has been accomplished by Burmann and others for a single variable; that is to say, ϑ being supposed to be a function of the variables, $x, y, \ldots z$, each of which is a given function of $u, v, \ldots w$, and $\alpha, \beta, \ldots \gamma$, being any positive integers, the theorem gives the complete development of $\left(\frac{d}{dx}\right)^\alpha \left(\frac{d}{dy}\right)^\beta \ldots \left(\frac{d}{dz}\right)^\gamma \vartheta$ in terms of $\frac{d}{du}, \frac{d}{dv}, \ldots \frac{d}{dw}, x, y, \ldots z, \vartheta$. Such, I say, is the primary form of the theorem; but it enables us, as will hereafter be shown in this paper, in fact, and as a consequence, to do much more than this, namely, to solve the question of differential transformation, under its most general aspect. The question so proposed may be stated as follows:

Given $\phi_1 = 0, \phi_2 = 0, \ldots \phi_n = 0$, where each ϕ is a function of $x_1, x_2, \ldots x_{n+i}$, it is required to pass from an expression in which the differentiations have respect to $x_1, x_2, \ldots x_i$ to an equivalent expression, in each of the terms of which the differentiations have respect to $x_{\theta_1}, x_{\theta_2}, \ldots x_{\theta_i}$, these last-written quantities being any i arbitrarily chosen terms out of the given set of $n+i$ variables, $x_1, x_2, \ldots x_{n+i}$. Through the medium of the reversion of series, the solution of this problem for the case contemplated in the theorem about to be enunciated (where $x_1, x_2, \ldots x_i$ are given *explicitly* in terms of

* Want of leisure prevented me then, and still prevents me, from producing the proof of the theorem, or the investigation by which I arrived at it. It must, however, be understood, that the theorem was not obtained tentatively, but that the proof of it is in my possession.

[† p. 50 above.]

$u_1, u_2, \ldots u_i$), enables us to write down the solution for the case where these two systems of variables are connected by equations in the more general manner just above supposed. It may then be asked whether it is meant to affirm that Burmann's law for passing from one independent variable x to another y, of which the first is a known function, conducts immediately to the law for effecting such change, when x and y are connected through the intervention of one equation between x and y, or several equations between x, y, and other connecting variables. The answer to this question is in the negative; for even if we take the simpler case where x and y are connected by a single equation, it will be found that to solve the problem for this case in the manner indicated, we shall need to know the solution of the problem, how to pass to *two* variables, u, v, from two others, x, y, given explicitly as functions of the former two; and so in general it is the fact, that the theorem applicable to the case of implicit connections between any number of variables, is always a corollary to the theorem applicable to the case of explicit connection between a *greater* number of variables. Thus it comes to pass, that Burmann's law for one variable explicitly connected with another, does not contain within itself the law for one variable implicitly connected with another; but the general law which I have discovered for a system of *any number* of variables explicitly connected with another such system, does contain within itself the general law for systems implicitly so connected*.

As the theorem is one of considerable complexity, it will be rendered most easily intelligible by taking separately and successively the cases of two and of three variables; the reader will then not experience any difficulty in seeing how it is to be extended to any greater number.

Problem for Two Variables.

(2) Let x, y be given functions of u, v, it is required to express $\left(\frac{d}{dx}\right)^f \left(\frac{d}{dy}\right)^g \vartheta$ in terms of the partial differential coefficients of x, y, ϑ in respect of u and v.

Solution.

Form the Jacobian determinant

$$\begin{vmatrix} \frac{dx}{du}, & \frac{dx}{dv} \\ \frac{dy}{du}, & \frac{dy}{dv} \end{vmatrix},$$

* Any linear function of infinity is still infinity, and all infinity is one, but not so of a finite integer; thus it is that the particular does not carry with it the particular, although the general does the general.

which call J; the required expression will be made up of terms, each of which will have for its components; 1°, a power of $(-)$; 2°, a positive integer numerical multiplier; 3°, a negative power of J; 4° and 5° (subject to a subsequent distinction into two sets), factors of the form

$$\left(\frac{d}{du}\right)^p \left(\frac{d}{dv}\right)^q x, \quad \left(\frac{d}{du}\right)^{p'} \left(\frac{d}{dv}\right)^{q'} y;$$

and 6°, a factor of the form $\left(\frac{d}{du}\right)^A \left(\frac{d}{dv}\right)^B \vartheta$.

The distinction of the factors under the combined headings 4 and 5 into two sets, referring to these headings separately taken, is dependent upon the values of $p, q; p', q'$. The 4th heading is intended to comprise the factors, for which $p=1$ and $q=0$ or $p=0$, $q=1$, and similarly for p' and q', that is, factors for which $p+q$ or $p'+q'$ is unity. The 5th heading comprises those factors in which $p+q$ or $p'+q'$ (as the case may be), exceeds unity. These two sets require to be carefully distinguished and considered apart: those values of $p, q; p', q'$ belonging to the second set will be distinguished by the letters $a, b; a', b'$, so that it is to be understood that $a+b>1$, $a'+b'>1$.

The general term may thus be put under the form

$$\begin{aligned}(-)^i N \frac{1}{J^\omega} \left(\frac{dy}{dv}\right)^\alpha \times \left(-\frac{dy}{du}\right)^\beta \times \left(-\frac{dx}{dv}\right)^{\alpha'} \times \left(\frac{dx}{du}\right)^{\beta'}& \\ \times \left\{\left(\frac{d}{du}\right)^a \left(\frac{d}{dv}\right)^b x\right\}^l \times \&\text{c.}& \\ \times \left\{\left(\frac{d}{du}\right)^{a'} \left(\frac{d}{dv}\right)^{b'} y\right\}^{l'} \times \&\text{c.}& \\ \times \left(\frac{d}{du}\right)^A \left(\frac{d}{dv}\right)^B \vartheta.&\end{aligned}$$

The negative signs are employed with $\frac{dy}{du}, \frac{dx}{dv}$ in the first line of factors, because, as will be seen when we pass to the case of more than two variables, it is the first minors of J which give rise to these factors, and these first minors are respectively

$$\frac{dy}{dv}; \quad -\frac{dy}{du}; \quad -\frac{dx}{dv}; \quad \frac{dx}{du}.$$

The &c. in the second line of factors refers to a, b, l becoming changed into a_1, b_1, l_1; a_2, b_2, l_2 &c.; and indicates that the product is to be taken of all the factors thus formed upon the type of

$$\left\{\left(\frac{d}{du}\right)^a \left(\frac{d}{dv}\right)^b x\right\}^l.$$

Similarly, the &c. in the third line of factors refers to a', b', l' becoming changed into a_1', b_1', l_1' &c.; and the product taken of all such factors so formed upon the type of

$$\left\{\left(\frac{d}{du}\right)^{a'}\left(\frac{d}{dv}\right)^{b'} y\right\}^{l'}.$$

[We may of course, if we please, write the first line under the form

$$(-)^{i'}\frac{N}{J^{\omega}}\left(\frac{dy}{dv}\right)^{\alpha}\left(\frac{dy}{du}\right)^{\beta}\left(\frac{dx}{dv}\right)^{\alpha'}\cdot\left(\frac{dx}{du}\right)^{\beta'}$$

by making $i' = i + \beta + \alpha'$.]

In the first place,

$$i = l + \&c. + l' + \&c.$$

In the second place,

$$\omega = \alpha + \beta + \alpha' + \beta'.$$

In the third place,

$$\alpha + \alpha' = la + \&c. + l'a' + \&c. + A, \qquad (1)$$

$$\beta + \beta' = lb + \&c. + l'b' + \&c. + B, \qquad (2)$$

and

$$\alpha + \beta = f + \Sigma l, \qquad (3)$$

$$\alpha' + \beta' = g + \Sigma l', \qquad (4)$$

which two systems of equations of course imply the existence of the equation

$$\Sigma l(a + b - 1) + \Sigma l'(a' + b' - 1) = (f + g) - (A + B). \qquad (5)$$

And finally:

$$N = D \times \frac{\Pi(\alpha+\beta-1)\,\Pi(\alpha'+\beta'-1)}{\Pi\alpha\,\Pi\beta\,\Pi\alpha'\,\Pi\beta'}$$

$$\times \frac{\Pi(\alpha+\alpha'-1)\,\Pi(\beta+\beta'-1)}{\{\Pi l\,(\Pi a\,\Pi b)^l\} \times \&c. \times \{\Pi l'\,(\Pi a'\,\Pi b')^{l'}\} \times \&c. \times \Pi A\;\Pi B},$$

Πn for any value of the integer n indicating the factorial $1.2.3 \ldots n$, and D denoting the determinant hereunder written, namely:

$$\begin{vmatrix} \alpha+\beta, & 0, & la + \&c., & lb + \&c. \\ 0, & \alpha'+\beta', & l'a' + \&c., & l'b' + \&c. \\ \alpha, & \alpha', & \alpha+\alpha', & 0 \\ \beta, & \beta', & 0, & \beta+\beta' \end{vmatrix},$$

which writing $la + \&c. = \Sigma la$, $l'a' + \&c. = \Sigma l'a'$, and substituting for $\alpha + \alpha'$, $\Sigma la + \Sigma l'a' + A$, and for $\beta + \beta'$, $\Sigma lb + \Sigma l'b' + B$, becomes when developed

$$\begin{aligned} &(\alpha + \beta)(\alpha' + \beta') AB \\ &+ \{\beta(\alpha' + \beta') \Sigma la + \beta'(\alpha + \beta) \Sigma l'a'\} B \\ &+ \{\alpha(\alpha' + \beta') \Sigma lb + \alpha'(\alpha + \beta) \Sigma l'b'\} A. \end{aligned}$$

D being essentially positive, N can only vanish when the following equations (or the analogues to them obtained by the interchange of a, α, A with b, β, B) are fulfilled, namely:

$$A = 0, \quad B = 0,$$

or $$A = 0, \quad \beta = 0, \quad \beta' = 0,$$

or $$A = 0, \quad \beta = 0, \quad \alpha = 0,$$

or $$A = 0, \quad \alpha' = 0, \quad \beta' = 0,$$

or $$A = 0, \quad \beta = 0, \quad \Sigma l'a' = 0,$$

or $$A = 0, \quad \beta' = 0, \quad \Sigma la = 0,$$

or $$A = 0, \quad \Sigma la = 0, \quad \Sigma l'a' = 0.$$

(3) By way of illustration, let us suppose $f = 2$, $g = 0$, so that the expression to be developed is $\frac{d^2\vartheta}{dx^2}$, which is to be expressed in terms of $\frac{d}{du}$, $\frac{d}{dv}$, x, y, ϑ. It will be the simpler mode of proceeding to find this development by actual expansion, and compare the result with that given by the theorem in the text.

We shall find without difficulty by the ordinary process

$$\begin{aligned} \frac{d^2\vartheta}{dx^2} = &\frac{1}{J^2}\left(\frac{dy}{dv}\right)^2 \frac{d^2\vartheta}{du^2} - \frac{2}{J^2}\frac{dy}{du}\frac{dy}{dv}\frac{d^2\vartheta}{du\,dv} + \frac{1}{J^2}\left(\frac{dy}{du}\right)^2\frac{d^2\vartheta}{dv^2} \\ &+ \frac{1}{J^3}\frac{dx}{dv}\left(\frac{dy}{dv}\right)^2\frac{d^2y}{du^2}\frac{d\vartheta}{du} - \frac{2}{J^3}\frac{dx}{dv}\frac{dy}{du}\frac{dy}{dv}\frac{d^2y}{du\,dv}\frac{d\vartheta}{du} \\ &+ \frac{1}{J^3}\frac{dx}{dv}\left(\frac{dy}{du}\right)^2\frac{d^2y}{dv^2}\frac{d\vartheta}{du} \\ &- \frac{1}{J^3}\frac{dy}{du}\left(\frac{dy}{dv}\right)^2\frac{d^2y}{du^2}\frac{d\vartheta}{du} + \frac{2}{J^3}\left(\frac{dy}{dv}\right)^2\frac{dy}{du}\frac{d^2x}{du\,dv}\frac{d\vartheta}{du} \\ &- \frac{1}{J^3}\frac{dy}{dv}\left(\frac{dy}{du}\right)^2\frac{d^2x}{dv^2}\frac{d\vartheta}{du} \\ &- \frac{1}{J^3}\frac{dx}{dv}\left(\frac{dy}{dv}\right)^2\frac{d^2y}{du^2}\frac{d\vartheta}{dv} + \frac{2}{J^3}\frac{dx}{du}\frac{dy}{du}\frac{dy}{dv}\frac{d^2y}{du\,dv}\frac{d\vartheta}{dv} \\ &- \frac{1}{J^3}\frac{dx}{dv}\left(\frac{dy}{du}\right)^2\frac{d^2y}{dv^2}\frac{d\vartheta}{dv} \\ &+ \frac{1}{J^3}\frac{dy}{du}\left(\frac{dy}{dv}\right)^2\frac{d^2x}{du^2}\frac{d\vartheta}{dv} - \frac{2}{J^3}\frac{dy}{du}\frac{dy}{du}\frac{dy}{dv}\frac{d^2x}{du\,dv}\frac{d\vartheta}{dv} \\ &+ \frac{1}{J^3}\frac{dx}{du}\left(\frac{dy}{du}\right)^2\frac{d^2y}{dv^2}\frac{d\vartheta}{dv}. \end{aligned}$$

(4) In the first term

$$\alpha = 2, \quad \beta = 0, \quad \alpha' = 0, \quad \beta' = 0,$$
$$a = 0, \quad b = 0, \quad \&c. = 0, \quad \Sigma l = 0,$$
$$a' = 0, \quad b' = 0, \quad \&c. = 0, \quad \Sigma l' = 0,$$
$$A = 2, \quad B = 0,$$

and we have, as indicated by the theorem,

$$i = \Sigma l + \Sigma l' = 0, \quad \omega = \alpha + \beta + \alpha' + \beta' = 2,$$
$$\alpha + \alpha' = \Sigma la + \Sigma l'a' + A = 2,$$
$$\beta + \beta' = \Sigma lb + \Sigma l'b' + B = 0,$$
$$\alpha + \beta = f = 2,$$
$$\alpha' + \beta' = g = 0.$$

N becomes

$$\frac{\Pi(\alpha+\beta-1)\,\Pi(\alpha'+\beta'-1)}{\Pi\alpha\,\Pi\beta\,\Pi\alpha'\,\Pi\beta'} \times \frac{\Pi(\alpha+\alpha'-1)\,\Pi(\beta+\beta'-1)}{\Pi A} \times (\alpha+\beta)(\alpha'+\beta')\,AB;$$

it is easily seen that

$$\Pi(\beta+\beta'-1) \times B$$
$$= \Pi(\beta+\beta'-1) \times (\beta+\beta')$$
$$= \Pi 0 = 1,$$
$$(\alpha'+\beta')\,\Pi(\alpha'+\beta'-1) = \Pi(\alpha'+\beta') = \Pi 0 = 1,$$

so that the value of the fraction above written is in fact

$$\frac{\Pi(\alpha+\beta)\,A}{\Pi\alpha\,\Pi A} = \frac{(\Pi 2)^2}{(\Pi 2)^2} = 1.$$

In the second term,

$$\alpha = 1, \quad \beta = 1, \quad \alpha' = 0, \quad \beta' = 0;$$

everything else remains as before, except that the numerical factor is $(-)^{\beta} N$, that is, $-N$, where $N = 2$.

(5) If we take the eighth term (the second one of the fourth line) we have

$$\alpha = 2, \quad \beta = 1, \quad \alpha' = 0, \quad \beta' = 0,$$
$$a = 1, \quad b = 1, \quad a' = 0, \quad b' = 0, \quad \Sigma l = l = 1,$$
$$A = 1, \quad B = 0,$$

and we have

$$i = l + \beta + \alpha' = \Sigma l + \beta + \alpha' = 2,$$
$$\omega = \alpha + \beta + \alpha' + \beta' = 3,$$
$$\alpha + \alpha' = la + A = 2,$$
$$\beta + \beta' = lb + B = 1,$$
$$\alpha + \beta = f + l = 3,$$
$$\alpha' + \beta' = g = 0,$$

and N becomes
$$\frac{\Pi(\alpha+\beta-1)\,\Pi(\alpha'+\beta'-1)}{\Pi\alpha\,\Pi\beta\,\Pi\alpha'\,\Pi\beta'}$$
$$\times\frac{\Pi(\alpha+\alpha'-1)\,\Pi(\beta+\beta'-1)}{\Pi a\,\Pi b\,\Pi A\,\Pi B}\times\{\alpha(\alpha'+\beta')\}\,bA,$$

which, since
$$\Pi(\alpha'+\beta'-1)\times(\alpha'+\beta')=\Pi 0=1,$$
$$=\frac{\Pi(\alpha+\beta-1)\,A}{\Pi A}=\frac{\Pi 2\,.\,2}{\Pi 2}=2.$$

(6) The above examples, although taken from the simplest terms, are in a certain sense exceptional cases, inasmuch as N for these cases involves one or more fractions of the form $\frac{0}{0}$; but this is a mere accident, resulting from the peculiar form of representation which I choose to employ, as being in general the most convenient to operate with.

If we take the fifth term (that is, the second term of the second line), this exception does not apply. We have for this term
$$\alpha=1,\quad \beta=1,\quad \alpha'=1,\quad \beta'=0,$$
$$a=0,\quad b=0,\quad a'=1,\quad b'=1,\quad \Sigma l'=l'=1,$$
$$A=1,\quad B=0,$$

and we find
$$N=\frac{\Pi 1\times\Pi 0}{\Pi 1\times\Pi 1}\times\frac{\Pi 1\,\Pi 0}{\Pi 1\times\Pi 1\times\Pi 1}\times D=D$$
$$=\left\{\begin{array}{l} \quad 2\times 0\times 1\times 0 \\ +(1\times 1+0+0\times 1\times 1)\,0 \\ +\;1\times 1\times 0+1\times 2\times 1 \end{array}\right\}=2.$$

(7) In general, to form all the terms in $\left(\frac{d}{dx}\right)^f\left(\frac{d}{dy}\right)^g\vartheta$, that is, to find all the systems of indices, we may begin by taking $A+B=\mu$, and giving to μ in succession, every value from 1 to $f+g$, and calling $f+g=n$, and writing
$$\Sigma l\,(a+b-1)=L,$$
$$\Sigma l'\,(a'+b'-1)=L',$$
we have to combine each solution of the equation $A+B=\mu$ with each of the equation $L+L'=n-\mu$, that is, we may assume for A in succession each value from 1 to μ, and for L, from 1 to $n-\mu$.

It will be convenient to denote in general an integer which may be anything from 1 to p by $[p]$. We have then
$$\mu=[n],$$
$$A=\Big[[n]\Big],\quad B=[n]-\Big[n-[n]\Big],$$
$$L=\Big[n-[n]\Big],\quad L'=n-[n]-\Big[n-[n]\Big].$$

We have then to break up L in every possible way into parts which will give by combining equal parts into groups all the values of l, $(a+b-1)$. In like manner, the partitionment of L' will give all the values of l', $(a'+b'-1)$.

Any of the values of $a+b-1$ and of $a'+b'-1$ respectively being called c and c', we have

$$a=[c+1],\quad b=c+1-[c+1],\quad a'=[c'+1],\quad b'=c'+1-[c'+1].$$

Hence every system of $\quad l_1,\ a_1,\ b_1;\ \ l_2,\ a_2,\ b_2;\ \dots$

and of $\quad l_1',\ a_1',\ b_1';\ \ l_2',\ a_2',\ b_2';\ \dots$

satisfying the equations of condition may be found. To find the corresponding values of α, β; α', β' we must observe that one combination of the equations (1), (2), (3), (4), having been employed to obtain the quantities already found, only three of these equations are independent; we shall accordingly have

$$\alpha=[f+\Sigma l],\quad \beta=f+\Sigma l-[f+\Sigma l],$$
$$\alpha'=\Sigma la+\Sigma l'a'+A-\alpha,$$
$$\beta'=\Sigma lb+\Sigma l'b'+B-\beta,$$

and the problem is completely resolved.

(8) If now we pass to the case of three variables x, x', x'', given explicitly as functions of u, u', u'', we must take

$$J=\begin{vmatrix} \dfrac{dx}{du}, & \dfrac{dx}{du'}, & \dfrac{dx}{du''} \\ \dfrac{dx'}{du}, & \dfrac{dx'}{du'}, & \dfrac{dx'}{du''} \\ \dfrac{dx''}{du}, & \dfrac{dx''}{du'}, & \dfrac{dx''}{du''} \end{vmatrix},$$

which, for greater brevity, using $\bar{u}, \bar{u}', \bar{u}''$, to denote $\dfrac{d}{du}, \dfrac{d}{du'}, \dfrac{d}{du''}$, may be written

$$\begin{vmatrix} \bar{u}x, & \bar{u}'x, & \bar{u}''x \\ \bar{u}x', & \bar{u}'x', & \bar{u}''x' \\ \bar{u}x'', & \bar{u}'x'', & \bar{u}''x'' \end{vmatrix}.$$

The nine first minor determinants may then be expressed under the respective forms

$$\begin{matrix} \dfrac{dJ}{d\bar{u}x}, & \dfrac{dJ}{d\bar{u}'x}, & \dfrac{dJ}{d\bar{u}''x}, \\ \dfrac{dJ}{d\bar{u}x'}, & \dfrac{dJ}{d\bar{u}'x'}, & \dfrac{dJ}{d\bar{u}''x'}, \\ \dfrac{dJ}{d\bar{u}x''}, & \dfrac{dJ}{d\bar{u}'x''}, & \dfrac{dJ}{d\bar{u}''x''}. \end{matrix}$$

The general term in $\left(\frac{d}{dx}\right)^f \left(\frac{d}{dx'}\right)^{f'} \left(\frac{d}{dx''}\right)^{f''} \vartheta$ will then be

$$
\begin{aligned}
&(-)^i \frac{N}{J^\omega} \left(\frac{dJ}{d\bar{u}x}\right)^\alpha \left(\frac{dJ}{d\bar{u}'x}\right)^\beta \left(\frac{dJ}{d\bar{u}''x}\right)^\gamma \\
&\times \left(\frac{dJ}{d\bar{u}x'}\right)^{\alpha'} \left(\frac{dJ}{d\bar{u}'x'}\right)^{\beta'} \left(\frac{dJ}{d\bar{u}''x'}\right)^{\gamma'} \\
&\times \left(\frac{dJ}{d\bar{u}x''}\right)^{\alpha''} \left(\frac{dJ}{d\bar{u}'x''}\right)^{\beta''} \left(\frac{dJ}{d\bar{u}''x''}\right)^{\gamma''} \\
&\times \left\{\left(\frac{d}{du}\right)^a \left(\frac{d}{du'}\right)^b \left(\frac{d}{du''}\right)^c x\right\}^l \times \&c. \\
&\times \left\{\left(\frac{d}{du}\right)^{a'} \left(\frac{d}{du'}\right)^{b'} \left(\frac{d}{du''}\right)^{c'} x'\right\}^{l'} \times \&c. \\
&\times \left\{\left(\frac{d}{du}\right)^{a''} \left(\frac{d}{du'}\right)^{b''} \left(\frac{d}{du''}\right)^{c''} x''\right\}^{l''} \times \&c. \\
&\times \left(\frac{d}{du}\right)^A \left(\frac{d}{du'}\right)^B \left(\frac{d}{du''}\right)^C \vartheta;
\end{aligned}
$$

and similarly to the last case

$$
\begin{aligned}
i &= \Sigma l + \Sigma l' + \Sigma l'', \\
\omega &= \alpha + \beta + \gamma \\
&\quad + \alpha' + \beta' + \gamma' \\
&\quad + \alpha'' + \beta'' + \gamma'', \\
\alpha + \alpha' + \alpha'' &= \Sigma la + \Sigma l'a' + \Sigma l''a'' + A \\
\beta + \beta' + \beta'' &= \Sigma lb + \Sigma l'b' + \Sigma l''b'' + B \\
\gamma + \gamma' + \gamma'' &= \Sigma lc + \Sigma l'c' + \Sigma l''c'' + C, \\
\alpha + \beta + \gamma &= f + \Sigma l \\
\alpha' + \beta' + \gamma' &= f' + \Sigma l' \\
\alpha'' + \beta'' + \gamma'' &= f'' + \Sigma l'',
\end{aligned}
$$

from which six equations we may deduce

$$
\Sigma l\,(a+b+c-1) + \Sigma l'\,(a'+b'+c'-1) + \Sigma l''\,(a''+b''+c''-1) = f+g+h-(A+B+C).
$$

(9) And the six equations first written may be solved in a manner analogous to the four equations in the preceding case.

We have finally

$$
N = \frac{\Pi(\alpha+\beta+\gamma-1)\,\Pi(\alpha'+\beta'+\gamma'-1)\,\Pi(\alpha''+\beta''+\gamma''-1)}{\Pi\alpha\,\Pi\beta\,\Pi\gamma\,\Pi\alpha'\,\Pi\beta'\,\Pi\gamma'\,\Pi\alpha''\,\Pi\beta''\,\Pi\gamma''}
$$

$$
\times \frac{\Pi(\alpha+\alpha'+\alpha''-1)\,\Pi(\beta+\beta'+\beta''-1)\,\Pi(\gamma+\gamma'+\gamma''-1)}{\Pi l\,(\Pi a\,\Pi b\,\Pi c)^l \times \&c. \times \Pi l'\,(\Pi a'\,\Pi b'\,\Pi c')^{l'} \times \&c. \times \Pi l''\,(\Pi a''\,\Pi b''\,\Pi c'')^{l''} \times \&c.}
$$

$$
\times D \div (\Pi A\,\Pi B\,\Pi C).
$$

where D = the determinant following, namely,

$$\begin{vmatrix} \alpha+\beta+\gamma, & 0, & 0, & \Sigma la, & \Sigma lb, & \Sigma lc \\ 0, & \alpha'+\beta'+\gamma', & 0, & \Sigma l'a', & \Sigma l'b', & \Sigma l'c' \\ 0, & 0, & \alpha''+\beta''+\gamma'', & \Sigma l''a'', & \Sigma l''b'', & \Sigma l''c'' \\ \alpha, & \alpha', & \alpha'', & \alpha+\alpha'+\alpha'', & 0, & 0 \\ \beta, & \beta', & \beta'', & 0, & \beta+\beta'+\beta'', & 0 \\ \gamma, & \gamma', & \gamma'', & 0, & 0, & \gamma+\gamma'+\gamma'' \end{vmatrix},$$

which, employing the equations

$$\alpha + \alpha' + \alpha'' = \Sigma la + \Sigma l'a' + \Sigma l''a'' + A$$
$$\beta + \beta' + \beta'' = \Sigma lb + \Sigma l'b' + \Sigma l''b'' + B$$
$$\gamma + \gamma' + \gamma'' = \Sigma lc + \Sigma l'c' + \Sigma l''c'' + C,$$

may be expressed under the forms

$$\lambda ABC + \mu BC + \mu' CA + \mu'' AB + \nu A + \nu' B + \nu'' C,$$

where all the coefficients λ, μ, ν, are essentially positive functions of α, β, γ, &c., Σla, Σlb, Σlc, &c.

The general form of D is apparent, as is also the reason why there is no term in which one of the indices, A, B, C ... does not appear, namely, that the sum of the lines in the lower half of the square, minus the sum of the lines in its upper half, gives rise to the line of terms following, which may be substituted in place of any one of the existing lines

$$0,\ 0,\ 0\ \ldots\ A,\ B,\ C\ \ldots$$

so that one of the letters A, B, C ... must appear in every actual term of the development.

(10) Let us return for a moment to show what the theorem becomes for the case of a single variable x, from which the transition is to be made to u.

For this case $$J = \frac{dx}{du},$$

and the 1st minor which is a determinant of zero places, as is well known to those conversant with determinants, must be taken $+1$. The formula then becomes

$$(-)^i (1)^\alpha \frac{N}{J^\omega} \left\{\left(\frac{d}{du}\right)^{a_1} x\right\}^{l_1} \left\{\left(\frac{d}{du}\right)^{a_2} x\right\}^{l_2} \cdots \left\{\left(\frac{d}{du}\right)^{a_e} x\right\}^{l_e} \cdot \left(\frac{d}{du}\right)^{A} \vartheta,$$

where $i = l_1 + l_2 + \ldots + l_e$, $\omega = \alpha = l_1 a_1 + l_2 a_2 + \ldots + l_e a_e + A$,

and $$N = \frac{\Pi(\alpha-1)}{\Pi\alpha} \frac{\Pi(\alpha-1)}{\Pi l_1 (\Pi a_1)^{l_1} \Pi l_2 (\Pi a_2)^{l_2} \ldots \Pi l_e (\Pi a_e)^{l_e} \Pi A} \cdot D,$$

where
$$D = \begin{vmatrix} \alpha, & \alpha - A \\ \alpha, & \alpha \end{vmatrix} = \alpha A.$$

Hence
$$N = \frac{\Pi(\alpha - 1)}{\Pi l_1 (\Pi a_1)^{l_1} \times \text{\&c.} \times \Pi l_e (\Pi a_e)^{l_e} \times \Pi(A-1)},$$

agreeing, as it ought, with Burmann's Law.

(11) For particular classes of terms N admits of a reduction to a simpler form.

Thus, in the case of three variables, suppose that the matrix

$$\begin{matrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{matrix} \quad \text{assumes the form} \quad \begin{matrix} \alpha, & 0, & 0, \\ 0, & \beta', & 0, \\ 0, & 0, & \gamma'', \end{matrix}$$

by which I mean that
$$\beta = 0, \quad \gamma = 0,$$
$$\alpha' = 0, \quad \gamma' = 0,$$
$$\alpha'' = 0, \quad \beta'' = 0.$$

Then by substituting for the 4th, 5th, and 6th lines in D the differences between the 4th and 1st, the 5th and 2nd, the 6th and 3rd, respectively, D assumes the form

$$\begin{vmatrix} \alpha, & 0, & 0, & \Sigma la, & \Sigma lb, & \Sigma lc \\ 0, & \beta', & 0, & \Sigma l'a', & \Sigma l'b', & \Sigma l'c' \\ 0, & 0, & \gamma'', & \Sigma l''a'', & \Sigma l''b'', & \Sigma l''c'' \\ 0, & 0, & 0, & \begin{matrix}\Sigma l'a' \\ +\Sigma l''a'', \\ +A\end{matrix} & -\Sigma lb, & -\Sigma lc \\ 0, & 0, & 0, & -\Sigma l'a', & \begin{matrix}\Sigma lb \\ +\Sigma l''b'', \\ +B,\end{matrix} & -\Sigma l'c' \\ 0, & 0, & 0, & -\Sigma l''a'', & -\Sigma l''b'', & \begin{matrix}\Sigma lc \\ +\Sigma l'c' \\ +C\end{matrix} \end{vmatrix},$$

which
$$= \alpha\beta'\gamma'' \times \begin{vmatrix} \Sigma l'a' + \Sigma l''a'' + A, & -\Sigma l'a', & -\Sigma l''a'' \\ -\Sigma lb, & \Sigma lb + \Sigma l''b'' + B, & -\Sigma l''b'' \\ -\Sigma lc, & -\Sigma l'c', & \Sigma lc + \Sigma l'c' + C \end{vmatrix},$$

which we may call $\alpha\beta'\gamma''D'$.

The entire value of N is consequently

$$\alpha\beta'\gamma'' \frac{\Pi(\alpha-1)\,\Pi(\beta'-1)\,\Pi(\gamma''-1)}{\Pi\alpha\,\Pi\beta'\,\Pi\gamma''}$$

$$\times \frac{\Pi(\alpha-1)\,\Pi(\beta'-1)\,\Pi(\gamma''-1)}{\Pi l\,(\Pi a\,\Pi b\,\Pi c)^l \times \&c. \times \Pi l'\,(\Pi a'\,\Pi b'\,\Pi c')^{l'} \times \&c. \times \Pi l''\,(\Pi a''\,\Pi b''\,\Pi c'')^{l''} \times \&c.}$$

$$\times \frac{D}{\Pi A\,\Pi B\,\Pi C}$$

$$= \frac{\Pi\alpha\,\Pi\beta'\,\Pi\gamma''}{\Pi l\,(\Pi a\,\Pi b\,\Pi c)^l \times \&c. \times \Pi l'\,(\Pi a'\,\Pi b'\,\Pi c')^{l'} \times \&c. \times \Pi l''\,(\Pi a''\,\Pi b''\,\Pi c'')^{l''} \times \&c.}$$

$$\times \frac{D'}{\Pi A\,\Pi B\,\Pi C}.$$

(12) The form of D' is deserving of consideration on its own account. Call

$$\Sigma l'a' = A_b, \quad \Sigma l''a'' = A_c,$$
$$\Sigma lb = B_a, \quad \Sigma l''b'' = B_c,$$
$$\Sigma lc = C_a, \quad \Sigma l'c' = C_b.$$

Then
$$\begin{aligned} D' = ABC &+ (A_b + A_c)\,BC + (B_c + B_a)\,CA + (C_a + C_b)\,AB \\ &+ (B_cC_a + B_aC_b + B_aC_a)\,A + (C_aA_b + C_bA_c + C_bA_b)\,B \\ &+ (A_bB_c + A_cB_a + A_cB_c)\,C. \end{aligned}$$

The entire number of terms is 16. In general, for m variables the corresponding number will be $(m+1)^{m-1}$, as may easily be shown*.

* The number of terms in D', since each of them has positive unity for its numerical coefficient, is evidently the value of a determinant, which, for three variables, is

$$\begin{vmatrix} 3, & -1, & -1 \\ -1, & 3, & -1 \\ -1, & -1, & 3 \end{vmatrix}.$$

To find in general the value of such a determinant in its more general form

$$\begin{vmatrix} a, & -1, & -1 \\ -1, & a, & -1 \\ -1, & -1, & a \end{vmatrix},$$

which is the discriminant of $a(x^2+y^2+z^2) - 2yz - 2zx - 2xy$, we may observe that this latter formula becomes a perfect square, that is, loses two orders when $a = -1$. Hence $(a+1)^2$ is a factor of the determinant. Again, when $a=2$ the sum of all the terms in each column is zero. Hence $(a-2)$ is also contained in it as a factor; the complete value of the determinant is therefore $(a-2)(a+1)^2$, that is, 4^2, when $a=3$; and so for a determinant of the mth order we obtain $\{a-(m-1)\}(a+1)^{m-1}$, which becomes $(m+1)^{m-1}$ when $a=m$.

The same result may also be obtained directly by the integration of a linear equation of differences of the second order of the form given in the example at the foot of page 14, in Mr Cohen's paper in this *Journal*.

If we take D, which also, like D', consists exclusively of positive terms, only with unit coefficients, the number of these terms for the case of 1, 2, 3 variables I find to be 1, 12, 432; and for the general case of m variables I presume that the law is $m^m(m+1)^{m-1}$.

The terms themselves may be found without calculation by means of a simple rule.

Suppose that there are four variables, we may then find D' for the case corresponding to the one just treated of for three variables by taking the product of

$$\begin{gathered} A_b + A_c + A_d + A, \\ B_a + B_c + B_d + B, \\ C_a + C_b + C_d + C, \\ D_a + D_b + D_c + D, \end{gathered}$$

and *rejecting* every term in such product in which any group of the letters forms a cycle.

Thus, for example, every term in which $A_b \times B_a$ enters must be rejected, because AB, BA is a cycle.

So, again, every term in which $A_b \times B_c \times C_a$ enters must be rejected, because AB, BC, CA forms a cycle.

We might take the product of $A_a + A_b + A_c + A_d + A$, and the quantities similarly formed, and proceed as above; for since AA is a cycle, as is also BB, CC, DD, therefore $A_a B_b C_c D_d$ will not appear in the final result.

Applying the method of rejection, we find without difficulty D', which represents the determinant

$$\begin{vmatrix} A + A_b + A_c + A_d, & -A_b, & -A_c, & -A_d \\ -B_a, & B + B_a + B_c + B_d, & -B_c, & -B_d \\ -C_a, & -C_b, & C + C_a + C_b + C_d, & -C_d \\ -D_a, & -D_b, & -D_c, & D + D_a + D_b + D_c \end{vmatrix},$$

$$\begin{aligned} &= ABCD + \Sigma (A_b + A_c + A_d) BCD + \Sigma (A_b B_c + A_b B_d + A_c B_a \\ &\qquad + A_c B_c + A_c B_d + A_d B_a + A_d B_c + A_d B_d) CD \\ &\quad + \Sigma \begin{pmatrix} A_d B_d C_d + (A_b + A_c) B_d C_d + (B_c + B_a) C_d A_d + (C_a + C_b) A_d B_d \\ \{B_a (C_a + C_b) + B_c C_a\} C_d + \{C_b (A_b + A_c) + C_a A_b\} A_d \\ + \{A_c (B_c + B_a) + A_b B_c\} B_d \end{pmatrix} D. \end{aligned}$$

The total number of terms being

$$1 + 4 \times 3 + 8 \times 6 + 4 \times 16 = 125 = 5^3,$$

as it ought to be.

Other cases of simplification will readily suggest themselves; and, of course, when $\gamma = 0$, $\gamma' = 0$, $\gamma'' = 0$, which equations imply also $\Sigma lc = 0$, $\Sigma l'c' = 0$, $\Sigma l''c'' = 0$, and $C = 0$, the value of N will reduce as it ought to the form corresponding to the case of only two variables, and so in general (the value of the coefficient of any term in the development of the transformed

value of any differential coefficient of a function of several variables depending only upon such of them as appear in the term itself, and in no way upon the other variables not so appearing).

(13) To indicate the method of passing from the theory of transformation of systems explicitly to that of systems of variables implicitly connected, let us suppose $\phi(x, y) = 0$ and that $\dfrac{d^f\vartheta}{dx^f}$ is to be expressed in terms of $\dfrac{d}{dy}$, ϕ, ϑ.

We may make this transformation depend upon our being able to solve the following question in the reversion of series, namely:

Given
$$\xi = a\rho + b\sigma + \frac{1}{1\,.\,2}(c\rho^2 + 2d\rho\sigma + e\sigma^2) + \&c.,$$
$$\eta = a'\rho + b'\sigma + \frac{1}{1\,.\,2}(c'\rho^2 + 2d'\rho\sigma + e'\sigma^2) + \&c.,$$

to express $\rho^h\sigma^k$ *in terms of* ξ, η. The solution of this question, when $b = 0$, $a' = 0$, has been given by Jacobi, *Crelle*, t. VI. 1830; and as is obvious and pointed out by Jacobi, the general case, by either of two methods, namely, combination of the equations or linear transformations effected in the variables ρ, σ contained in them, may be made to depend on the particular case for which $b = 0$, $a' = 0$; but Jacobi has not followed out the effects of these processes, and apparently was not aware of the results being (as we may now see is the case) capable of an explicit representation, which mode of representation is essential for the purpose we have in view.

Let x, y be functions of u, v; and suppose x, y, ϑ to become $x + \xi$, $y + \eta$, $\vartheta + \tau$, when u and v become $u + h$ and $v + k$ respectively; then we shall have

$$\xi = \frac{dx}{du}h + \frac{dx}{dv}k + \&c. + \left\{\&c. + \frac{1}{\Pi A\,\Pi B}\frac{d^{A+B}x}{du^A dv^B}h^A k^B + \&c.\right\} + \&c.,$$
$$\eta = \frac{dy}{du}h + \frac{dy}{dv}k + \&c. + \left\{\&c. + \frac{1}{\Pi A\,\Pi B}\frac{d^{A+B}y}{du^A dv^B}h^A k^B + \&c.\right\} + \&c.,$$
$$\tau = \frac{d\vartheta}{du}h + \frac{d\vartheta}{dv}k + \&c. + \left\{\&c. + \frac{1}{\Pi A\,\Pi B}\frac{d^{A+B}\vartheta}{du^A dv^B}h^A k^B + \&c.\right\} + \&c.;$$

but treating τ as a function of ξ, η, we have also

$$\tau = \&c. + \left\{\&c. + \frac{1}{\Pi f\,\Pi g}\cdot\frac{d^{f+g}\vartheta}{dx^f dy^g}\xi^f\eta^g + \&c.\right\} + \&c.$$

Hence $\dfrac{1}{\Pi f\,\Pi g}\cdot\dfrac{d^{f+g}\vartheta}{dx^f dy^g}$ being expanded by means of our theorem in terms of $\dfrac{d}{du}$, $\dfrac{d}{dv}$, x, y, ϑ, the coefficient in such expansion of $\dfrac{d^{A+B}\vartheta}{du^A dv^B}$ will exhibit the value of the coefficient of $\xi^f\eta^g$ in the expansion of $\dfrac{1}{\Pi A\,\Pi B}h^A k^B$ in terms of ξ and η.

(14) As there are no quantitative relations between the coefficients in the equations above written which express ξ and η, we are therefore now able to express the value of $h^A k^B$ in terms of ξ and η when ξ and η are respectively expressed as rational integral functions of (and vanishing with) h and k. Thus, let us write in general

$$\xi = \Sigma p_{r,s} h^r k^s,$$

$$\eta = \Sigma q_{r,s} h^r k^s,$$

where $p_{0,0}$ and $q_{0,0}$ are each zero, but all the other values of p and q absolutely arbitrary. We have now $p_{r,s}$, $q_{r,s}$ respectively replacing

$$\frac{1}{\Pi r \, \Pi s} \frac{d^{r+s} x}{du^r dv^s}, \quad \frac{1}{\Pi r \, \Pi s} \frac{d^{r+s} y}{du^r dv^s}$$

and consequently the general term in the expansion of $h^A k^B$ as a function of ξ and η will be $I_{f,g} \xi^f \eta^g$, where

$$I_{f,g} = \frac{\Pi A \, \Pi B}{\Pi f \, \Pi g} \Sigma (-)^{i+\beta+\alpha'} \frac{N}{J^{\alpha+\beta+\alpha'+\beta'}} q^{\alpha}{}_{0,1} \, p^{\alpha'}{}_{0,1} \, q^{\beta}{}_{1,0} \, p^{\beta'}{}_{1,0}$$

$$\times (\Pi a \, \Pi b \, p_{a,b})^l \times \&\text{c.} \times (\Pi a' \, \Pi b' \, q_{a',b'})^{l'} \times \&\text{c.},$$

where

$$N = \frac{\Pi(\alpha+\beta-1) \, \Pi(\alpha'+\beta'-1)}{\Pi\alpha \, \Pi\beta \, \Pi\alpha' \, \Pi\beta'}$$

$$\times \frac{\Pi(\alpha+\alpha'-1) \, \Pi(\beta+\beta'-1)}{\Pi l \, (\Pi a \, \Pi b)^l \times \&\text{c.} \times \Pi l' \, (\Pi a' \, \Pi b')^{l'} \times \&\text{c.}} \times \frac{D}{\Pi A \, \Pi B},$$

and

$$J = \begin{vmatrix} p_{1,0}, & p_{0,1} \\ q_{1,0}, & q_{0,1} \end{vmatrix}.$$

Hence

$$I_{f,g} = \Sigma \left\{ (-)^{\Sigma l + \Sigma l' + \beta + \alpha'} \frac{\Pi(\alpha+\beta-1) \, \Pi(\alpha'+\beta'-1)}{\Pi\alpha \, \Pi\beta \, \Pi\alpha' \, \Pi\beta'} \right.$$

$$\times \frac{\Pi(\alpha+\alpha'-1) \, \Pi(\beta+\beta'-1)}{\Pi f \, \Pi g} \times \frac{q^{\alpha}{}_{0,1} \, q^{\beta}{}_{1,0} \, p^{\alpha'}{}_{0,1} \, p^{\beta'}{}_{1,0}}{(p_{1,0} \, q_{0,1} - p_{0,1} \, q_{1,0})^{\alpha+\beta+\alpha'+\beta'}}$$

$$\times \begin{vmatrix} \alpha+\beta, & & \Sigma la, & \Sigma lb \\ & \alpha'+\beta', & \Sigma l'a', & \Sigma l'b' \\ \alpha, & \alpha', & \alpha+\alpha', & \\ \beta, & \beta', & & \beta+\beta' \end{vmatrix}$$

$$\left. \times \left(\frac{p^l{}_{a,b}}{\Pi l} \times \&\text{c.} \times \frac{q^{l'}{}_{a',b'}}{\Pi l'} \times \&\text{c.} \right) \right\},$$

α, β; α', β'; l, l', &c., being any system of positive integers which are capable of satisfying the equations

$$\alpha+\beta=\Sigma l+f,$$
$$\alpha'+\beta'=\Sigma l'+f',$$
$$\alpha+\alpha'=\Sigma la+\Sigma l'a'+A,$$
$$\beta+\beta'=\Sigma lb+\Sigma l'b'+B.$$

Hence the value of $h^A k^B$, which $=\Sigma I_{f,g}\xi^f\eta^g$, is completely determined as an explicit function of ξ, η, and the coefficients p, q, of the equations by which ξ, η, are given in terms of h and k.

(15) So for three variables, supposing

$$\xi=\Sigma m_{r,s,t}\,h^r k^s l^t,$$
$$\eta=\Sigma n_{r,s,t}\,h^r k^s l^t,$$
$$\zeta=\Sigma p_{r,s,t}\,h^r k^s l^t,$$

where $m_{0,0,0}$, $n_{0,0,0}$ and $p_{0,0,0}$, are each zero, but all other values of m, n, p absolutely arbitrary, making

$$\begin{vmatrix} m_{1,0,0}, & m_{0,1,0}, & m_{0,0,1} \\ n_{1,0,0}, & n_{0,1,0}, & n_{0,0,1} \\ p_{1,0,0}, & p_{0,1,0}, & p_{0,0,1} \end{vmatrix}=J,$$

and writing in general

$$\frac{d\log J}{dm_{i,i',i''}}=\mu_{i,i',i''},$$
$$\frac{d\log J}{dn_{i,i',i''}}=\nu_{i,i',i''},$$
$$\frac{d\log J}{dp_{i,i',i''}}=\phi_{i,i',i''},$$

we shall find
$$h^A k^B l^C=\Sigma I_{f,g,h}\xi^f\eta^g\zeta^h,$$
where

$$I_{f,g,h}=\Sigma\left\{\left((-)^{\Sigma l+\Sigma l'+\Sigma l''}\frac{\Pi(\alpha+\beta+\gamma-1)\,\Pi(\alpha'+\beta'+\gamma'-1)\,\Pi(\alpha''+\beta''+\gamma''-1)}{\Pi\alpha\,\Pi\beta\,\Pi\gamma\,\Pi\alpha'\,\Pi\beta'\,\Pi\gamma'\,\Pi\alpha''\,\Pi\beta''\,\Pi\gamma''}\right.\right.$$
$$\left.\frac{\Pi(\alpha+\alpha'+\alpha''-1)\,\Pi(\beta+\beta'+\beta''-1)\,\Pi(\gamma+\gamma'+\gamma''-1)}{\Pi f\,\Pi g\,\Pi h}\right)$$
$$\times D\times\mu^{\alpha}{}_{1,0,0}\,\mu^{\beta}{}_{0,1,0}\,\mu^{\gamma}{}_{0,0,1}\,\nu^{\alpha'}{}_{1,0,0}\,\nu^{\beta'}{}_{0,1,0}\,\nu^{\gamma'}{}_{0,0,1}\,\phi^{\alpha''}{}_{1,0,0}\,\phi^{\beta''}{}_{0,1,0}\,\phi^{\gamma''}{}_{0,0,1}$$
$$\left.\times\frac{m^{l}{}_{a,b,c}}{\Pi l}\times\text{\&c.}\times\frac{n^{l'}{}_{a',b',c'}}{\Pi l'}\times\text{\&c.}\times\frac{p^{l''}{}_{a'',b'',c''}}{\Pi l''}\times\text{\&c.}\right\},$$

where D is a known determinant of the sixth order expressible in terms of

α, β, γ; α', β', γ'; $\alpha'', \beta'', \gamma''$; $\Sigma la, \Sigma lb, \Sigma lc$; $\Sigma l'a', \Sigma l'b', \Sigma l'c'$; $\Sigma l''a'', \Sigma l''b''$, $\Sigma l''c''$, and where

$$\begin{aligned}
\alpha+\beta+\gamma &= \Sigma l + f,\\
\alpha'+\beta'+\gamma' &= \Sigma l' + g,\\
\alpha''+\beta''+\gamma'' &= \Sigma l'' + h,\\
\alpha+\alpha'+\alpha'' &= \Sigma la + \Sigma l'a' + \Sigma l''a'' + A,\\
\beta+\beta'+\beta'' &= \Sigma lb + \Sigma l'b' + \Sigma l''b'' + B,\\
\gamma+\gamma'+\gamma'' &= \Sigma lc + \Sigma l'c' + \Sigma l''c'' + C.
\end{aligned}$$

(16) Suppose now that we wish from the equation $0 = \Sigma p_{r,s} h^r k^s$ to deduce the value of k^β in terms of h.

We may put
$$\xi = \Sigma p_{r,s} h^r k^s,$$
$$\eta = h,$$

and then apply the formula of reversion for finding k^β in terms of ξ and η; but since $\xi = 0$, we may reject all the terms out of $\Sigma I_{f,g} \xi^f \eta^g$, except those in which $f = 0$; moreover, in adapting the formula applicable to this case, we must put $q_{r,s} = 0$ for all values of the system r, s, except 1, 0, and for that system $q_{1,0} = 1$; we have, therefore, to retain such terms only in $I_{0,g}$ for which $\alpha = 0$, $\Sigma l'a' = 0$, $\Sigma l'b' = 0$;

D consequently becomes $\begin{vmatrix} \beta, & 0, & \Sigma la, & \Sigma lb \\ 0, & \alpha'+\beta', & 0, & 0 \\ 0, & \alpha', & \alpha', & 0 \\ \beta, & \beta', & 0, & \beta+\beta' \end{vmatrix}$

$$= \alpha'(\alpha'+\beta') \begin{vmatrix} \beta, & \Sigma lb \\ \beta, & \beta+\beta' \end{vmatrix} = \alpha'(\alpha'+\beta')\beta \begin{vmatrix} 1, & \beta+\beta'-B \\ 1, & \beta+\beta' \end{vmatrix}$$
$$= \alpha'(\alpha'+\beta')\beta B;$$

hence

$$I_{0,g} = \Sigma \left\{ (-)^{\Sigma l+\beta+\alpha'} \frac{\Pi\beta\,\Pi(\alpha'+\beta')}{\Pi\beta\,\Pi\alpha'\,\Pi\beta'} \times \frac{\Pi\alpha'\,\Pi(\beta+\beta'-1)B}{\Pi g} \times \frac{p^{\alpha'}{}_{0,1}\, p^{\beta'}{}_{1,0}}{p_{0,1}{}^{\beta+\alpha'+\beta'}} \times \frac{p^l{}_{a,b}}{\Pi l} \times \&\text{c.} \right\},$$

with the conditions

$$\left.\begin{matrix} \alpha' = \Sigma la, & \beta = \Sigma l \\ \beta+\beta' = \Sigma lb + B, & \alpha'+\beta' = g \end{matrix}\right\}; \qquad (\omega)$$

we have, therefore, finally

$$k^B = B\Sigma\left(\left\{\Sigma\,(-)^{\Sigma la}\,\frac{\Pi\,\{\Sigma\,(lb)+B-1\}}{\Pi\,\Sigma\,\{l\,(b-1)+B\}}\cdot\frac{{p_{1,0}}^{\Sigma\,(l\,(b-1))+B}}{{p_{0,1}}^{\Sigma\,(lb)+B}}\cdot\frac{(p^l{}_{a_1 b_1})}{\Pi l}\cdots\frac{(p^{l_e}{}_{a_e, b_e})}{\Pi l_e}\right\}h\right)$$

with the sole condition deduced from the system (ω),

$$l_1\,(a_1+b_1-1)+l_2\,(a_2+b_2-1)+\ldots+l_e\,(a_e+b_e-1)=g-B.$$

Suppose, now, that $\phi\,(x, y)=0$, and that we wish to express $\dfrac{d^g\vartheta}{dx^g}$ (where, for greater simplicity, I consider ϑ a function only of y) in terms of x, y, without solving the equation $\phi=0$; we know that if we write

$$\frac{d\phi}{dx}\,h+\frac{d\phi}{dy}\,k+\&\text{c.}+\left\{\&\text{c.}+\frac{\left(\dfrac{d}{dx}\right)^{\epsilon}\left(\dfrac{d}{dy}\right)^{\omega}}{\Pi\epsilon\,\Pi\omega}\,h^{\epsilon}k^{\omega}+\&\text{c.}\right\}+\&\text{c.},$$

then $\dfrac{1}{\Pi g}\,\dfrac{d^g\vartheta}{dx^g}$ will be the coefficient of h^g in the expansion of

$$\frac{d\vartheta}{dy}\,k+\frac{d^2\vartheta}{dy^2}\,\frac{k^2}{1\,.\,2}+\frac{d^3\vartheta}{dy^3}\,\frac{k^3}{1\,.\,2\,.\,3}\ \&\text{c. in terms of } h.$$

Consequently, if we make

$$\frac{d^g\vartheta}{dx^g}=\Sigma E_B\,\frac{d^B\vartheta}{dy^B},$$

$$E_B=\frac{\Pi g}{\Pi\,(B-1)}\,\Sigma\,(-)^{\Sigma la}\,\frac{\Pi\,\{\Sigma\,(lb)+B-1\}}{\{\Pi\Sigma\,(lb)+B-\Sigma l\}}$$

$$\times\frac{\left(\dfrac{d\phi}{dx}\right)^{\Sigma lb+B-\Sigma l}}{\left(\dfrac{d\phi}{dy}\right)^{\Sigma lb+B}}\,\frac{\left\{\left(\dfrac{d}{dx}\right)^{a}\left(\dfrac{d}{dy}\right)^{b}\phi\right\}^{l}}{\Pi l\,(\Pi a\,\Pi b)^{l}}\cdots\frac{\left\{\left(\dfrac{d}{dx}\right)^{a_e}\left(\dfrac{d}{dy}\right)^{b_e}\phi\right\}^{l_e}}{\Pi l_e\,(\Pi a_e\,\Pi b_e)^{l_e}},$$

where, as before, writing in general $a_k+b_k-1=c_k$,

$$l_1c_1+l_2c_2+\ldots+l_ec_e=g-B,$$

g being now given, and B variable and subject to assume in succession every value from 1 up to B.

(17) By way of verifying the above formula, and as a protection against accidental errors of calculation, suppose $\phi=-x+\psi\,(y)$,

so that $$\frac{d\phi}{dx}=-1,\quad\frac{d\phi}{dy}=\frac{d\psi}{dy};$$

the only terms to be retained are those in which no (a) index appears.

We have, therefore, for this case $\Sigma l(b-1)=g-B$,

that is, $$\Sigma lb+B-\Sigma l=g,$$

and $$E_B=\frac{\Pi\{\Sigma(lb)+B-1\}}{\Pi l(\Pi b_1)^l\ldots\Pi l_e(\Pi b_e)^{l_e}}\frac{(-)^g}{\left(\frac{d\psi}{dy}\right)^{g+\Sigma l}}\left\{\left(\frac{d}{dy}\right)^b\psi\right\}^l\ldots\left\{\left(\frac{d}{dy}\right)^{b_e}\psi\right\}^{l_e}$$

agreeing, as required, with Burmann's law.

(18) As another example, in illustration of the fact that our general theorem embraces the whole theory of reversion, suppose we have the equation $\Sigma m_{r,s,t}q^r k^s l^t=0$, and that it is required from this equation to deduce l^C as a function of h and k.

We may write
$$\xi=\Sigma m_{r,s,t}q^r k^s l^t,$$
$$\eta=q,$$
$$\zeta=k.$$

We have then $$l^C=\Sigma I_{0,g,h}q^g k^h,$$

and in assigning the value of $I_{0,g,h}$, we need, moreover, to retain in $I_{0,g,h}$ only those terms in which the a', b', c' and a'', b'', c'' systems of indices are wanting; for
$$\Sigma l'a'=0,\quad \Sigma l'b'=0,\quad \Sigma l'c'=0,$$
$$\Sigma l''a''=0,\quad \Sigma l''b''=0,\quad \Sigma l''c''=0.$$

Moreover α, β, γ; α', β', γ'; $\alpha'', \beta'', \gamma''$; being the indices respectively of the minor determinants of the matrix

$$\begin{matrix} m_{1,0,0}; & m_{0,1,0}; & m_{0,0,1}; \\ 1\ ; & 0\ ; & 0\ ; \\ 0\ ; & 1\ ; & 0\ ; \end{matrix}$$

we may consider $\alpha=0$, $\beta=0$, $\beta'=0$, $\alpha''=0$, since the minor determinants which have these indices are all zero.

Hence, for the actual terms in $I_{0,g,h}$, D becomes

$$\begin{vmatrix} \gamma, & \ldots, & \ldots, & \Sigma la, & \Sigma lb, & \Sigma lc \\ \ldots, & \alpha'+\gamma', & \ldots, & \ldots, & \ldots, & \ldots \\ \ldots, & \ldots, & \beta''+\gamma'', & \ldots, & \ldots, & \ldots \\ \ldots, & \alpha', & \ldots, & \alpha', & \ldots, & \ldots \\ \ldots, & \ldots, & \beta'', & \ldots, & \beta'', & \ldots \\ \gamma, & \gamma', & \gamma'', & \ldots, & \ldots, & \gamma+\gamma'+\gamma'' \end{vmatrix}$$

which obviously reduces to the form

$$\alpha'\beta''(\alpha'+\gamma')(\beta''+\gamma'')\begin{vmatrix}\gamma, & \Sigma lc\\ \gamma, & \Sigma lc + C\end{vmatrix}$$
$$=\alpha'\beta''(\alpha'+\gamma')(\beta''+\gamma'')\gamma C;$$

also the equations of condition between the indices become

$$\gamma=\Sigma l, \qquad \alpha'=\Sigma la,$$
$$\alpha'+\gamma'=g, \qquad \beta''=\Sigma lb,$$
$$\beta''+\gamma''=h, \quad \gamma+\gamma'+\gamma''=\Sigma lc+C,$$

in addition to the special equations

$$\alpha=0, \quad \beta=0, \quad \alpha''=0, \quad \beta'=0.$$

Hence $l^C=\Sigma I_{g,h}q^g k^h$, where $I_{g,h}$ represents

$$(-)^{\Sigma l}\frac{\Pi g\,\Pi h}{\Pi\gamma\,\Pi\alpha'\,\Pi\gamma'\,\Pi\beta''\,\Pi\gamma''}\cdot\frac{\Pi\alpha'\,\Pi\beta''\,\Pi(\gamma+\gamma'+\gamma''-1)\,C}{\Pi g\,\Pi h}\,.\,\&\text{c.}$$
$$=(-)^{\Sigma l}\,C\,\frac{\Pi(\gamma+\gamma'+\gamma''-1)}{\Pi\gamma\,\Pi\gamma'\,\Pi\gamma''}\cdot\frac{m_{0,0,1}{}^{\alpha'+\beta''}(-m_{0,1,0})^{\gamma''}(-m_{1,0,0})^{\gamma'}}{m_{0,0,1}{}^{\gamma+\gamma'+\gamma''+\alpha'+\beta''}}\times\&\text{c.}$$
$$=\Sigma(-)^{\gamma+\gamma'+\gamma''}\,C\,\frac{\Pi(\gamma+\gamma'+\gamma''-1)}{\Pi\gamma\,\Pi\gamma'\,\Pi\gamma''}\,\frac{m^{\gamma''}{}_{0,1,0}\,m^{\gamma'}{}_{1,0,0}}{m_{0,0,1}{}^{\gamma+\gamma'+\gamma''}}$$
$$\times\frac{m^{l}{}_{abc}}{\Pi l}\times\frac{m^{l_2}{}_{a_2b_2c_2}}{\Pi l_2}\dots\times\frac{m^{l_e}{}_{a_eb_ec_e}}{\Pi l_e},$$

where $l_1(a_1+b_1+c_1-1)+\dots+l_e(a_e+b_e+c_e-1)=g+h-C$, g and h being assumed of any values respectively, such that their sum is not less than C: the partitionment of $g+h-C$, gives every possible system

$$l_1\dots l_e;\quad (a+b+c-1)\dots(a_e+b_e+c_e-1);$$

and to every such system correspond known systems of values of $a, b, c; \dots; a_e, b_e, c_e$. We have then $\gamma=\Sigma l$, $\gamma'+\gamma''=\Sigma l(c-1)+C$, which latter equation, for each value of c, gives $\Sigma l(c-1)+C+1$ systems of values of γ' and γ''. Thus we have the complete solution of the equation $\Sigma m_{r,s,t}q^r k^s l^t=0$.

In like manner, if we suppose i variables $q_1, q_2, \dots q_i$, and for greater simplicity, in addition to the condition always supposed of the constant term being zero, likewise conceive that the coefficient shall be unity in each linear term of the equation

$$\Sigma m_{r_1 r_2\dots r_i}q_1{}^{r_1}q_2{}^{r_2}\dots q_i{}^{r_i}=0,$$

we shall find

$$q_i{}^{A_i}=\Sigma(-)^{\gamma_1+\gamma_2+\dots+\gamma_i}A_i\frac{\Pi(\gamma_1+\gamma_2+\dots+\gamma_i-1)}{\Pi\gamma_1\,\Pi\gamma_2\dots\Pi\gamma_i}$$
$$\times\frac{(m^{l}{}_{1_a2_a\dots i_a})^{l}}{\Pi l}\times\&\text{c.}\times\frac{(m_{1_{ae}2_{ae}\dots i_{ae}})^{l_e}}{\Pi l_e}\,q_1{}^{f_1}q_2{}^{f_2}\dots q_{i-1}{}^{f_{i-1}},$$

with the conditions following for finding the (γ) and ${}^1a$, ${}^2a \ldots {}^ia \ldots {}^1a_e$, ${}^2a_e \ldots {}^ia_e$ systems, namely,

$$\Sigma l\,({}^1a + {}^2a + \ldots + {}^ia - 1) = f_1 + f_2 + \ldots + f_{i-1} - A_i,$$

$$\gamma_1 = \Sigma l, \quad \gamma_2 + \gamma_3 + \ldots + \gamma_i = \Sigma l_i\,(a_i - 1) + A_i + 1.$$

(19) In like manner we may without difficulty assign the general law for solving with like generality any number of simultaneous equations between any greater number of variables, the functions equivalent to zero being all supposed to be without a constant term, and to be expressed as rational integral functions of the variables; and we can consequently pass from one system of independent variables to any new system in whatever way, whether explicitly or implicitly, through any number of equations and any number of connecting variables, the two systems may be supposed to be related.

17.

ON A DISCOVERY IN THE PARTITION OF NUMBERS*.

[*Quarterly Journal of Mathematics*, I. (1857), pp. 81—84.]

LET $a_1, a_2, \ldots a_r$ be any given system of integer elements; I call the number of ways of composing the number n with these elements the *quotity*† of n in respect to the given elements. Let the least common multiple of $a_1, a_2, a_3, \ldots a_r$ be called p and let the roots of $\dfrac{x^p-1}{x-1}=0$ be called ρ, then we may express the quotity in question under the form

$$A + U,$$

* From the last foot-note at p. 87, it follows that the non-periodical part of the analytical expression for the number of ways in which n can be composed of the r elements $a, b, c \ldots l$, is the coefficient of $\frac{1}{t}$ in the expansion, in a series of ascending powers of t, of the fraction $\dfrac{e^{nt}}{(1-e^{-at})(1-e^{-bt})\ldots(1-e^{-lt})}$. Moreover, if we suppose $\dfrac{1}{(1-x^a)(1-x^b)\ldots(1-x^l)}=\dfrac{P}{(1-x)^r}+$ fractions not containing $(1-x)$ in the denominator, it further follows that, for values of n not less than r, the coefficient of x^n in P will be the coefficient of $\frac{1}{t}$ in the expression

$$\frac{e^{(n-r)t}\,.\,(e^t-1)^r}{(1-e^{-at})(1-e^{-bt})\ldots(1-e^{-lt})},$$

which is evidently zero, as it ought to be.

† Thus the quotity of n in respect to a and 1 is the integer next greater than $\frac{n}{a}$; the complete expression for this quantity, it may be mentioned, is

$$\frac{1}{a}(n+\tfrac{1}{2})-\frac{1}{a^2}\Sigma\,\{(a-1)+(a-2)\rho+\ldots+\rho^{a-2}\}\,\rho^{n+1},$$

where ρ is a prime root of $\dfrac{\rho^a-1}{\rho-1}=0$.

The quotity of n in respect to the consecutive elements $1, 2, 3 \ldots r$ is equal to the number of ways of partitioning $n+r$ into r parts.

where A is an algebraical function of n and the elements, which is clear of all exponential expressions, and U is of the form

$$\Sigma (A_0 + A_1\rho + A_2\rho^2 + \&c. + A_{p-1}\rho^{p-1})\, \rho^n *.$$

I call U the quot-undulant†; A the quot-additant‡. I shall say nothing at present about the former, although I can express its value completely for any system of elements which are prime each to each, or of which the relations of identity existing between the prime factors are given: my theorem, for present purposes, is confined to the quot-additant, which may be written under the form following, namely,

$$\frac{1}{a_1 a_2 \ldots a_r}\left\{B_1 + B_2 n + B_3 \frac{n^2}{1\,.\,2} + \&c. \ldots + B_{r-1}\frac{n^{r-2}}{1\,.\,2 \ldots (r-2)} + B_r \frac{n^{r-1}}{1\,.\,2 \ldots (r-1)}\right\},$$

where $$B_\omega = \Sigma C_{\omega_{\theta_1}} C_{\omega_{\theta_2}} \ldots C_{\omega_{\theta_r}} S\,(a_1^{\omega_{\theta_1}} a_2^{\omega_{\theta_2}} \ldots a_r^{\omega_{\theta_r}}),$$

S denoting as usual that a symmetrical function is to be formed, of which the quantity which follows it is the type of the general term, and the symbol Σ referring to a summation to be effected in respect to all the distinct systems of integer values (zeros included in the number) of $\omega_{\theta_1}, \omega_{\theta_2}, \ldots \omega_{\theta_r}$, whose sum is $r - \omega$, and where, in general, C_m denotes the coefficient of t^m in

$$\frac{te^t}{e^t - 1}\,\S.$$

* The coefficients A_0, A_1, &c., are, in general, algebraical functions of n and of the elements whose degree in n is one unit inferior to the greatest number of the elements having the same common measure.

† The quot-undulant, although for present purposes presented as a single whole, is in fact a collective quantity made up, and most simply and naturally expressed by means of the sum of a series of analogous periodic or periodico-progressive functions, whose number is the same as that of the distinct elements, and whose periods are respectively measured by the number of units in each such element; it may be compared with a great wave, composed of a number of wavelets, whose lengths are either the same as or submultiples of its own. This is the view first taken by Mr Cayley, who, in his researches, has followed in the footsteps of Euler, but to which, also, I have been independently and unavoidably conducted by the method of investigation peculiar to myself. Sir John Herschel and Mr Kirkman have not taken this view, and accordingly there is an unnecessary complexity in their statements of results.

‡ If we suppose the fraction $\dfrac{1}{(1-x^{a_1})(1-x^{a_2})\ldots(1-x^{a_r})}$ thrown under the form

$$\frac{P}{(1-x)^r} + \frac{Q}{(1-x^{a_1})(1-x^{a_2})\ldots(1-x^{a_r}) \div (1-x)^r},$$

the quot-additant of n is the coefficient of x^n in $\dfrac{P}{(1-x)^r}$ which gives the means of expressing P, and consequently also Q. Compare Note iv. in M. Serret's excellent *Cours d'Algèbre supérieure*, 2nd edition.

§ The theorem may also be stated as follows. Let A_0, A_1, A_2, &c., denote the successive coefficients in the expansion in a series of ascending powers of x of the reciprocal of the product of $1-e^{-ax}$, $1-e^{-bx}$, ... $1-e^{-lx}$, then will the quot-additant of n in respect of the r elements $a, b, c \ldots l$, be expressed by $A_{r-1} + A_{r-2}\,.\,n + A_{r-3}\,.\,\dfrac{n^2}{1\,.\,2} + \ldots + A_0\,.\,\dfrac{n^{r-1}}{1\,.\,2\,.\,3 \ldots (r-1)}$. [See note * of p. 86.]

Examples. The quot-additants of n, in respect to the systems a; a, b; a, b, c; a, b, c, d, &c., respectively, are as follows:

$$\frac{1}{a},\ \frac{1}{ab}\left(\frac{a+b}{2}+n\right),$$

$$\frac{1}{abc}\left(\frac{a^2+b^2+c^2+3ab+3ac+3bc}{12}+\frac{a+b+c}{2}n+\frac{n^2}{1.2}\right),$$

$$\frac{1}{abcd}\left(\frac{\Sigma a^2b+3\Sigma abc}{24}+\frac{(\Sigma a)^2+3\Sigma ab}{12}n+\frac{\Sigma a}{4}n^2+\frac{n^3}{1.2.3}\right)^*.$$

So, again, the constant term in the quot-additant to the system of elements a, b, c, d, e will be

$$\frac{1}{abcde}\Sigma\left\{-\frac{a^4}{720}+\frac{a^2b^2}{144}+\frac{a^2bc}{48}+\frac{abcd}{16}\right\},$$

and to the system of six elements it will be

$$\frac{1}{abcdef}\Sigma\left\{-\frac{a^4b}{1440}+\frac{a^2b^2c}{288}+\frac{a^2bcd}{96}+\frac{abcde}{32}\right\};$$

it will be seen that the latter quantity under the sign of summation is obtained term for term from the one above by introducing a new element with the index unity in the numerator and doubling each denominator; this law is general, and is an immediate consequence of the fact that for a coefficient of i dimensions in the elements the only partitionments of i which appear in the groups of indices are those which are made up of the elements 1, 2, 4, 6, &c., all the odd elements except 1 from the nature of Bernouilli's numbers giving rise to the coefficient zero, so that, consequently, the partitionments of $2\iota+1$ which enter into the expression in question, are all derived from those of 2ι by the addition of a single distinct unit.

The series of fractions $\frac{1}{2}, \frac{1}{12}, 0, \frac{1}{120}, 0$, &c., arise in my method as the results of substituting $\dfrac{1}{\omega+1}$ in place of $\dfrac{\delta^\omega\phi}{\phi}$ in the expansion of the successive variations of $\log\phi$.

Thus, $\delta\log\phi=\dfrac{\phi'}{\phi}=\frac{1}{2}$, $\delta^2\log\phi=\dfrac{\phi''}{\phi}-\left(\dfrac{\phi'}{\phi}\right)^2=\frac{1}{3}-\frac{1}{4}=\frac{1}{12}$,

$$\delta^3\log\phi=\frac{\phi'''}{\phi}-3\frac{\phi''\phi'}{\phi^2}+2\frac{\phi'^3}{\phi^3}=\tfrac{1}{4}-\tfrac{3}{6}+\tfrac{2}{8}=0,\ \text{\&c. \&c.}$$

* When the elements $a, b, c \dots l$ are prime each to each, the quot-undulant will not contain n, that is, will be strictly periodic. For this case, therefore, the difference between the quot-additant of n and that of $n-(a.b.c \dots l)$ will represent the difference between the entire quotity of n and that of $n-(a.b.c \dots l)$ in respect to the system supposed. We have consequently an easy method of verifying, by actual decompositions of numbers, the general expression for the additant part without knowing the value of the undulant part in the complete expression for the quotity. For the case in question the quot-additant may also be defined and calculated as the algebraical expression whose mean value is the same as the mean value of the quotity when the partible number passes through a period of $a.b.c \dots l$ consecutive integer values.

Hence it may easily be collected, that if we write

$$\frac{te^t}{e^t-1} = 1 + K_1 t - 2K_2 t^2 + 3K_3 t^3 \mp \text{\&c.},$$

we ought to have

$$K_i = \frac{1}{i} E_0 - \frac{1}{i-1} E_1 + \frac{1}{i-2} E_2 \text{ \&c. } \pm E_{i-1},$$

where E_ω denotes, in general, the coefficient of h^ω in

$$\left(\frac{1}{2} + \frac{h}{2\,.\,3} + \frac{h^2}{2\,.\,3\,.\,4} + \text{\&c.}\right)^{i-\omega},$$

or, which is the same thing, K_i ought to be equal to the coefficient of t^i in $\log \dfrac{e^t-1}{t}$ as is easily demonstrable to be the case by Maclaurin's Theorem.

In general, if the quot-additant of n, in respect to the roots of

$$x^r + p_1 x^{r-1} + p_2 x^{r-2} + \text{\&c.} + p_r,$$

be expressed as a function of $p_1, p_2, \ldots p_r$ and n, and be called $\dfrac{1}{p_r} Q_r$, we have the following equations existing, namely,

$$Q_r = \int dn\, Q_{r-1} \text{ and } \left(\frac{d}{dp_1} + p_1 \frac{d}{dp_2} + \ldots + p_{r-1} \frac{d}{dp_r}\right) Q_r = -\frac{1}{2} Q_{r-1}.$$

Observation. My method which has led me to the preceding theorem reposes upon the axiom, which I believe is quite new, that the mean value of the $a\,.\,b\,.\,c \ldots l$ sums of homogeneous powers and products (all affected with the coefficient unity) of n dimensions in $\alpha, \beta, \gamma, \ldots, \lambda$, where $\alpha^a = 1$, $\beta^b = 1, \ldots \lambda^l = 1$, is equal to the quotity of n in respect to $a, b, c, \ldots l$.

18.

ON THE PARTITION OF NUMBERS.

[*Quarterly Journal of Mathematics*, I. (1857), pp. 141—152. *Also printed, Tortolini's Annali di Matematiche*, VIII. (1857), pp. 12—21.]

I MUST reluctantly content myself for the present (unexpected events, which have robbed me of the leisure and calm of mind necessary for composition, and the due evolution and embodiment of ideas on any extensive scale, forbid me to do more) with a brief statement of the general solution of this important question, which (as known to my thrice-distinguished friend, Mr Cayley) I succeeded in completing almost immediately after the appearance of the last number of the Journal.

It must be clearly understood that the methods of Euler, De Morgan, Herschel, Kirkman, and Cayley (the last a great advance upon all that went before) have only afforded the means (with more or less generality) of determining the *quotity* of a number in respect of given elements in any particular case; the existence of a universal algebraical representation of this quotity seems not even to have been suspected. Moreover, it will be found that the general formula, which I am about to give, possesses an immense practical advantage in point of facility of computation over the methods previously employed. Thus, for example, I have been able to compute by it, in a moderate space of time, the number of ways of partitioning n into nine parts; the enormous complexity of the calculations required by the methods of Herschel and Cayley had induced those distinguished authors to rest satisfied with stopping short at the formula for only five parts.

My result has been erected upon a completely independent basis, and deduced by an equally original method, namely, the axiom contained in the observation at the end of my former paper combined with a simple theorem for expressing, by means of partial fractions, the sum of the homogeneous

powers and products of any number of quantities, not merely for the *special* case of these quantities being all unlike, but for the *general* case of their being made up of any sets of equals. MM. Cayley and Terquem have both suggested, what is no doubt true, the possibility of obtaining my result otherwise, and perhaps a little more simply, by aid of M. Cauchy's Theory of Residues.

I now proceed to enunciate the theorem.

$a_1, a_2, \ldots a_r$ (all positive integers) are supposed to be the elements, n the partible number, and the object in view is the expression of the quotity of n quâ the elements $a_1, a_2, \ldots a_r$, that is, of the number of solutions of the equation in integers $a_1x_1 + a_2x_2 + \&c. + a_rx_r = n$, in which equation it may be observed no further condition is imposed upon the coefficients $a_1, a_2, \ldots a_r$ than that of their being positive integers. There is no restriction upon their being equal in any manner *inter se*. Call Q the quotity in question: then we may consider Q as made up of an infinite number of waves, of which, however (as it will immediately be seen), only a finite number have an *actual* existence, the rest will be *abortive*.

Let $\frac{p}{q}$ be any rational numerical fraction whatever, not exceeding unity, in its lowest terms, and use $w_{\frac{p}{q}}$ to denote the coefficient of $\frac{1}{t}$ in the development of the expression

$$e^{nw}(1-e^{a_1u})^{-1}(1-e^{a_2u})^{-1}\ldots(1-e^{a_ru})^{-1},$$

where $$w = \frac{2\pi ip}{q} + t, \quad u = \frac{2\pi ip}{q} - t, \quad i = (-1)^{\frac{1}{2}};$$

then $$Q = \Sigma w_{\frac{p_i}{q}}.$$

If $p_1, p_2, \ldots p_i$ be all the numbers (unity included) less than q, and prime to it, and if we write

$$w_{\frac{p_1}{q}} + w_{\frac{p_2}{q}} + \ldots + w_{\frac{p_i}{q}} = W_q,$$

we shall have more simply

$$Q = \Sigma W_q.$$

W_q again may be expressed* under a more easily intelligible form as the coefficient of $\frac{1}{t}$ in the development in ascending powers of t of

$$\Sigma \frac{\rho^n e^{nt}}{(1-\rho^{a_1}e^{-a_1t})(1-\rho^{a_2}e^{-a_2t})\ldots(1-\rho^{a_r}e^{-a_rt})},$$

where ρ is in succession each of the roots of the *prime factor* of $\rho^q - 1$; and

[* Cf. p. 157 below.]

consequently since this development will contain no term where $\frac{1}{t}$ enters, unless some one at least of the quantities $\rho^{a_1}, \rho^{a_2}, \ldots \rho^{a_r}$ is unity, it follows that $W_q = 0$, except for those values of q which are contained in some one or more of the elements $a_1, a_2, \ldots a_r$. The number of *actual* waves expressing a given quotity is consequently the number of distinct integers, unity included, which enter into the composition of the elements to which the quotity has reference.

It will readily be seen that on making $q = 1$ we shall obtain the expression for the so-called quot-additant (a name only adopted for provisional purposes, and which I now discard) given [p. 87 above] in the preceding number of the Journal. The generating function for this part of the quotity becomes, from the general formula,

$$\frac{e^{nt}}{(1-e^{-a_1 t})(1-e^{-a_2 t})\ldots(1-e^{-a_r t})}.$$

The coefficient of $\frac{1}{t}$ in this expression will give the formulæ contained in the body of the paper referred to; but far more expeditious formulæ of computation may be substituted in lieu of these. For we may write

$$W_1 = \text{coefficient of } \frac{1}{t} \text{ in}$$

$$e^{nt - \{\log(1-e^{-a_1 t}) + \log(1-e^{-a_2 t}) + \ldots + \log(1-e^{-a_r t})\}}.$$

But in general

$$\log(1-e^{-t}) = \log t - \frac{t}{2} + \frac{t^2}{24} + \&\text{c.}$$

$$= \log t - \frac{t}{2} + \frac{B_1}{1.2^2} t^2 - \frac{B_2}{1.2.3.4^2} t^4 + \&\text{c.},$$

B_1, B_2, &c., denoting Bernouilli's numbers, namely,

$$\frac{1}{6}, \frac{1}{30}, \frac{1}{42}, \frac{1}{30}, \&\text{c.}$$

Hence

$$W_1 = \frac{1}{a_1 a_2 \ldots a_r} \times \text{coefficient of } t^{r-1} \text{ in } e^{(n+\frac{1}{2}s_1)t - \frac{B_1 s_2}{1.2^2} t^2 + \frac{B_2 s_4}{1.2.3.4^2} t^4 \,\&\text{c.}},$$

where s_ω in general denotes the sum of the ωth powers of the elements $a_1, a_2, \ldots a_r$.

Hence, writing $n + \frac{1}{2}s_1 = \nu$, we have

$$W_1 = \text{coefficient of } t^{r-1} \text{ in}$$

$$(a_1 a_2 \ldots a_r)^{-1}\left(1 + \nu t + \nu^2 \frac{t^2}{1.2} + \nu^3 \frac{t^3}{1.2.3} + \&c.\right)$$
$$\times\left(1 - \frac{1}{24} s_2 t^2 + \frac{1}{1152} s_2{}^2 t^4 + \&c.\right)$$
$$\times\left(1 + \frac{1}{2880} s_4 t^4 + \frac{1}{165888} s_4{}^2 t^8 + \&c.\right)$$
$$\times\left(1 - \frac{1}{181440} s_6 t^6 + \&c.\right)$$
$$\times \&c.*$$

The wave W_2 is also deserving of particular notice, on account of it also being free from the sign of summation, and involving only the Bernouillian numbers. To find this wave we have to take ρ, the root of the prime factor of $\rho^2 - 1$, that is, we have simply $\rho = -1$.

And if we distinguish the elements $a_1, a_2 \ldots a_r$ into two groups, say $\alpha_1, \alpha_2 \ldots \alpha_l$, all even, and $\beta_1, \beta_2 \ldots \beta_m$ all odd, we have

$$W_2 = \text{coefficient of } \frac{1}{t} \text{ in the generator}$$

$$e^{nt}(-)^n \frac{1}{(1-e^{-\alpha_1 t})(1-e^{-\alpha_2 t})\ldots(1-e^{-\alpha_l t})} \frac{1}{(1+e^{-\beta_1 t})(1+e^{-\beta_2 t})\ldots(1+e^{-\beta_m t})},$$

which
$$= (-)^n e^{nt-R},$$
where
$$R = \Sigma \log(1 - e^{-\alpha_1 t}) + \Sigma \log(1 + e^{-\beta_1 t}).$$

But $\log(1 - e^{-t})$ has been already expressed, and

$$\log(1 + e^{-t}) = \log 2 - \frac{t}{2} + \frac{1}{8} t^2 + \&c.$$
$$= \log 2 - \frac{t}{2} + \frac{3}{4} B_1 t^2 - \frac{15}{16} \frac{B_2}{1.2.3} t^4 + \&c.$$

* To save circumlocution, I have not expressed in the text the general value of the coefficient of t_i, but of course there is not the slightest difficulty in so doing; let i be thrown in every possible way under the form $K_1 + 2K_2 + 4K_4 + 6K_6 + \&c.$ (that is to say, all the partitions of i quâ the elements 1, 2, 4, 6, &c., are to be written down), then the coefficient in question is

$$\Sigma\phi(K_1, K_2, K_4, \&c.)\, \nu^{K_1} . s_2{}^{K_2} . s_4{}^{K_4} . s_6{}^{K_6} \&c.,$$

where
$$\phi(K_1, K_2, K_4, \&c.) = \frac{\pm 1}{\Pi K_1 \, \Pi K_2 \, \Pi K_4 \, \&c.} \left(\frac{B_1}{1.2^2}\right)^{K_2} \left(\frac{B_2}{1.2.3.4^2}\right)^{K_4} \left(\frac{B_3}{1.2.3.4.5.6^2}\right)^{K_6} \&c.$$

It is deeply interesting to observe how, in the very formula for expressing partitions, a class of partitions reappears,—in fact, partitions constitute the sphere in which analysis lives, moves, and has its being; and no power of language can exaggerate or paint too forcibly the importance of this till recently almost neglected, but vast, subtle, and universally permeating element of algebraical thought and expression.

Happy ought I to feel in the reflection of having been the appointed instrument to make so great an advance in a doctrine which contains a large part of the future of pure analysis, and to have impressed upon it a form which must inevitably give rise to an illimitable host of the most important applications and consequences.

Hence, using $s_1, s_2 \ldots$ to denote the sums of the 1st, 2nd ... powers of $\alpha_1, \alpha_2 \ldots$ and $\sigma_1, \sigma_2 \ldots$ to denote the sums of the 1st, 2nd ... powers of $\beta_1, \beta_2 \ldots$ and writing $n + \frac{1}{2}(s_1 + \sigma_1) = \nu$, we have

$$nt - R = -\log(2^m \alpha_1\alpha_2 \ldots \alpha_l) + l \log t + \nu t - \frac{B_1}{1.2^2}(s_2 + 3\sigma_2)t^2$$

$$+ \frac{B_2}{1.2.3.4^2}(s_4 + 15\sigma_4)t^4 + \&c.$$

and consequently, $W_2 = \dfrac{1}{2^r\left(\frac{\alpha_1}{2}\cdot\frac{\alpha_2}{2}\cdots\frac{\alpha_l}{2}\right)} \times$ coefficient of t^{l-1} in

$$\left(1 + \nu t + \nu^2 \frac{t^2}{1.2} + \nu^3 \frac{t^3}{1.2.3} + \&c.\right)$$
$$\times \left(1 - \frac{1}{24}(s_2 + 3\sigma_2)t^2 + \frac{1}{1152}(s_2 + 3\sigma_2)^2 t^4 + \&c.\right)$$
$$\times \left(1 + \frac{1}{2880} s_4 t^4 + \&c.\right)$$
$$\times \&c.$$

So in general if we wish to find the wave W_q, we must distinguish the elements into two groups,

$\alpha_1, \alpha_2 \ldots \alpha_l$, all exactly divisible by q,

and $\beta_1, \beta_2 \ldots \beta_m$ not so divisible;

W_q will then be the coefficient of $\dfrac{1}{t}$ in

$$\Sigma \frac{\rho^n e^{nt}}{(1-\rho^{\alpha_1}e^{-\alpha_1 t})(1-\rho^{\alpha_2}e^{-\alpha_2 t})\ldots(1-\rho^{\alpha_l}e^{-\alpha_l t}) \times (1-\rho^{\beta_1}e^{-\beta_1 t})(1-\rho^{\beta_2}e^{-\beta_2 t})\ldots(1-\rho^{\beta_m}e^{-\beta_m t})}$$

where the sign of summation indicates that all the values are to be taken in succession of the prime roots of $\rho^q - 1 = 0$; this, again, may be expressed as the coefficient of t^{l-1} in a quantity of the form ρ^{nt-R} where R may be expressed by means of the prime qth roots of unity and the known numerical coefficients which enter into the expansion in ascending powers of t of the quantity $\dfrac{1}{1 - c\rho^{-t}}$; but I do not propose here to enter into the details of the method. It will be enough for present purposes to illustrate it by an example. Suppose, then, that we take the elements 1, 2, 3, 4, 5, 6; in other words, that we propose to express algebraically the number of ways in which n can be divided into six or a smaller number of parts.

The expression will here consist of six parts, which I shall reckon in inverse order, beginning with W_6.

W_6 will be the coefficient of $\frac{1}{t}$ in

$$\Sigma \frac{e^{nt}\rho^n}{(1-e^{-6t})\times(1-\rho e^{-t})(1-\rho^2e^{-2t})(1-\rho^3e^{-3t})(1-\rho^4e^{-4t})(1-\rho^5e^{-5t})},$$

where ρ is either root of $\qquad \rho^2-\rho+1=0;$

this is evidently equal to

$$\frac{1}{6}\times\Sigma\rho^n\frac{1}{(1-\rho)(1-\rho^2)(1-\rho^3)(1-\rho^4)(1-\rho^5)}$$

$$=\Sigma\frac{1}{6}\rho^n\frac{1}{(1-\rho)(1-\rho^2)\,2\,(1+\rho)(1+\rho^2)}$$

$$=\Sigma\frac{\rho^n}{12}\frac{1}{(1-\rho^2)(1-\rho^4)}$$

$$=\Sigma\frac{\rho^n}{12}\frac{1}{(2-\rho)(1+\rho)}=\Sigma\frac{\rho^n}{36}.$$

In like manner,

$$W_5=\frac{1}{5}\,.\,\Sigma\frac{\rho^n}{(1-\rho)(1-\rho^2)(1-\rho^3)(1-\rho^4)(1-\rho)},$$

where ρ is any root of

$$\rho^4+\rho^3+\rho^2+\rho+1=0.$$

Hence $\qquad W_5=\frac{1}{25}\Sigma\frac{\rho^n}{1-\rho}$

$$=\frac{1}{125}\Sigma\rho^n(4+3\rho+2\rho^2+\rho^3)$$

$$=\frac{1}{125}\Sigma\rho^n(2+\rho-\rho^3-2\rho^4).$$

Again,

$$W_4=\frac{1}{4}\Sigma\frac{\rho^n}{(1-\rho)(1-\rho^2)(1-\rho^3)(1-\rho)(1-\rho^2)},$$

where ρ is either root of $\rho^2+1=0$.

Hence $\qquad W_4=\frac{1}{16}\Sigma\frac{\rho^n}{(1-\rho)^2(1+\rho)}$

$$=\frac{1}{64}\Sigma\frac{\rho^n(1-\rho)}{-\rho}$$

$$=\frac{1}{64}\Sigma(\rho^n-\rho^{n-1}).$$

Again, $W_3=$ coefficient of $\frac{1}{t}$ in

$$\Sigma\frac{\rho^n e^{nt}}{(1-e^{-3t})(1-e^{-6t})\times\&c.},$$

where $$\rho^2+\rho+1=0,$$

$$=\frac{1}{18}\Sigma\frac{\rho^n\nu}{(1-\rho)(1-\rho^2)(1-\rho^4)(1-\rho^5)}=\frac{1}{18}\frac{\Sigma\rho^n\nu}{\{(1-\rho)(1-\rho^2)\}^2}$$

$$=\frac{\Sigma\rho^n\nu}{162},$$

where $$\nu=n+\frac{3}{2}+\frac{6}{2}+\frac{\rho}{\rho-1}+\frac{2\rho^2}{\rho^2-1}+\frac{4\rho^4}{\rho^4-1}+\frac{5\rho^5}{\rho^5-1}$$

$$=n+\frac{9}{2}+\frac{\rho}{\rho-1}+\frac{2\rho^2}{\rho^2-1}+\frac{4\rho}{\rho-1}+\frac{5\rho^2}{\rho^2-1}$$

$$=n+\frac{9}{2}-\frac{1}{3}\{5(2+\rho)+7(2+\rho^2)\}$$

$$=n+\frac{9}{2}-\frac{1}{3}(24+5\rho+7\rho^2)$$

$$=n-\frac{1}{6}(21+5\rho+7\rho^2)=n-\frac{1}{3}(7-\rho).$$

$$W_2=\frac{(-)^n}{2^6(1.2.3)}\times\text{coefficient of } t^2 \text{ in } \left(1+\nu t+\frac{\nu^2t^2}{2}\right)\left(1-\frac{1}{24}(s_2+3\sigma_2)t^2\right),$$

where $$\nu=n+\frac{1}{2}(1+2+3+4+5+6)$$

$$=n+\frac{21}{2},$$

$$s_2=2^2+4^2+6^2=56;\quad \sigma_2=1^2+3^2+5^2=35;\quad 3\sigma_2=105.$$

Hence $$W_2=(-)^n\frac{1}{384}\times\left(\frac{\nu^2}{2}-\frac{161}{24}\right)$$

$$=(-)^n\left(\frac{\nu^2}{768}-\frac{161}{9216}\right).$$

Finally $W_1=\dfrac{1}{1.2.3.4.5.6}\times$ coefficient of t^5 in

$$\left(1+\nu t+\nu^2\frac{t^2}{2}+\nu^3\frac{t^3}{6}+\nu^4\frac{t^4}{24}+\nu^5\frac{t^5}{120}\right)$$

$$\times\left(1-\frac{1}{24}s_2t^2+\frac{1}{1152}s_2^{\,2}t^4\right)$$

$$\times\left(1+\frac{1}{2880}s_4t^4\right)$$

$$=\frac{1}{720}\left\{\frac{\nu^5}{120}-\frac{\nu^3}{144}s_2+\nu\left(\frac{s_2^{\,2}}{1152}+\frac{s_4}{2880}\right)\right\},$$

where $$\nu = n + \frac{21}{2}$$

$$s_2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 = 91$$

$$s_4 = 1 \ + 2^4 + 3^4 + 4^4 + 5^4 + 6^4 = 2275$$

$$s_2^{\,2} = \qquad 8281.$$

Hence $$\frac{s_2^{\,2}}{1152} + \frac{s_4}{2880} = \frac{1}{1152}(8281 + 910)$$

$$= \frac{1}{1152}(9191),$$

and $$W_1 = \frac{\nu}{720}\left(\frac{\nu^4}{120} - \frac{91\nu^2}{144} + \frac{9191}{1152}\right).$$

As no useful object would be attained by substituting for ν its value $n + \frac{21}{2}$, I leave the expressions for W_1, W_2 in their present form as explicit functions of ν.

It is well worthy of observation that the exponent of the generating function of W_1, namely, $nt - R$, when the elements are taken the consecutive numbers 1, 2, 3 ... r, consists exclusively of Bernouillian numbers and sums of powers of 1, 2, 3 ... r; but as these latter are themselves expressible by Euler's theorem, in terms of powers of r and the Bernouillians, R is for this case a quadratic function of the numbers of Bernouilli.

If we express the quantities of the form $\Sigma\rho^w$, which occur in the different modes in terms of Herschel's circulating functions, then our expression assumes the very same form in which Mr Cayley had observed it was the most advantageous to express the quotity, and to which he has given the name of "prime radical circulators*."

* Thus what with Mr Cayley was an invention, with me becomes a theorem. Mr Cayley was led to the use of prime circulators from a perception of their affording the best analytical means of giving determinateness to the representation of the results; in my method they offer themselves spontaneously, and cannot be rejected.

Supposing that Mr Cayley could claim a right to the exclusive use of these forms, we should have an instructive instance of one of the mischiefs ascribed to the general system of patent law, namely, of blocking up the necessary march of invention. For the benefit of foreign readers of the *Journal*, I should add that r_n is used by Herschel to denote a quantity which is unity when n contains r as a factor, and is otherwise zero; and that any function of r_n, r_{n-1}, $r_{n-2} \ldots r_{n-r+1}$ is called a circulating function, and may, of course, be expressed as a linear function of the above quantities.

Suppose ρ to be any factor whatever of r, i to be less than ρ, and $r = \rho\sigma$. Then if for all admissible values of ρ and i

$$A_{n-i} + A_{n-i-\rho} + A_{n-i-2\rho} + \&c. + A_{n-i-(\sigma-1)\rho} = 0,$$

$A_n . r_n + A_{n-1} . r_{n-1} + \&c. + A_{n-r+1}$ (where A_n, $A_{n-1} \ldots$ are ordinary constants) is, according to Cayley, a prime radical circulator (prime circulator would be quite as specific and more convenient).

I shall conclude this very brief notice of my theory by converting the W waves in the example above treated into the form of these prime circulators.

For W_6 ρ is any root of $\rho^2-\rho+1=0$.

Hence

$$\Sigma\rho^0=2;\ \Sigma\rho=1;\ \Sigma\rho^2=-1;\ \Sigma\rho^3=-2;\ \Sigma\rho^4=-1;\ \Sigma\rho^5=1.$$

Hence in the notation of Herschel

$$W_6=\frac{6_n}{18}+\frac{6_{n-1}}{36}-\frac{6_{n-2}}{36}-\frac{6_{n-3}}{18}-\frac{6_{n-4}}{36}+\frac{6_{n-5}}{36}.$$

For W_5 ρ is any root of $\rho^4+\rho^3+\rho^2+\rho+1=0$.

Hence

$$\Sigma\rho^0=4;\ \Sigma\rho=-1;\ \Sigma\rho^2=-1;\ \Sigma\rho^3=-1;\ \Sigma\rho^4=-1.$$

Hence

$$\begin{aligned}W_5=\frac{1}{125}\{(8-1+1+2)\,5_n&+(-2-1+1-8)\,5_{n-1}\\ &+(-2-1-4+2)\,5_{n-2}+(-2-1+1+2)\,5_{n-3}\\ &+(-2+4+1+2)\,5_{n-4}\}\\ &=\frac{2}{25}5_n-\frac{2}{25}5_{n-1}-\frac{1}{25}5_{n-2}+\frac{1}{25}5_{n-3}.\end{aligned}$$

In W_4 ρ is either root of $\rho^2+1=0$.

Hence $\Sigma\rho^0=2;\ \Sigma\rho=0;\ \Sigma\rho^2=-2;\ \Sigma\rho^3=0,$

and hence

$$\begin{aligned}W_4&=\frac{1}{16}(2\,.\,4_n+2\,.\,4_{n-1}-2\,.\,4_{n-2}-2\,.\,4_{n-3})\\ &=\frac{4_n}{8}+\frac{4_{n-1}}{8}-\frac{4_{n-2}}{8}-\frac{4_{n-3}}{8}.\end{aligned}$$

In W_3 $\rho^2+\rho+1=0$.

Hence $\Sigma\rho^0=2;\ \Sigma\rho=-1;\ \Sigma\rho^2=-1.$

Hence

$$\begin{aligned}W_3&=\left(\frac{1}{81}3_n-\frac{1}{162}3_{n-1}-\frac{1}{162}3_{n-2}\right)\left(n-\frac{7}{3}\right)\\ &\quad-\frac{1}{486}3_n-\frac{1}{486}3_{n-1}+\frac{1}{243}3_{n-2}\\ &=\left(\frac{1}{81}3_n-\frac{1}{162}3_{n-1}-\frac{1}{162}3_{n-2}\right)n\\ &\quad-\left(\frac{5}{162}3_n-\frac{1}{81}3_{n-1}-\frac{3}{162}3_{n-2}\right).\end{aligned}$$

Finally

$$W_2=\left(\frac{\nu^2}{768}-\frac{161}{9216}\right)(2_n-2_{n-1}).$$

W_1 has already been expressed in its simplest terms; and the solution of the question of the partition of n into six parts is now complete.

The same causes which have interposed to prevent my setting forth at length the method above sketched out, have also interposed to preclude me from extending the exposition which I had intended of the application of the principles contained in my paper on Differential Transformations to the general question of the solution of equations, or systems of equations, containing any number of variables, thereby entirely superseding the necessity of all special considerations whatever in obtaining Lagrange's and Laplace's Theorems, either in their developed or ordinary form; and theorems to many degrees of infinity more general than these; for their method being, by the aid of this most desiderated discovery, now capable of being substituted for artifice, which, in general, may be said to stand in the same relation to method as what instinct is to reason, or the craft of the savage to the wisdom of the civilised man.

19.

NOTE ON A FORMAL PROPERTY OF A LATENT INTEGER.

[*Quarterly Journal of Mathematics*, I. (1857), p. 185.]

THE following was proposed some years ago by an author, whose name I do not recollect, among the mathematical questions in the *Educational Times**.

"Required to prove that the integer part of $(1+\sqrt{3})^{2m+1}$ contains 2^{m+1} as a factor."

The proof probably ran, or at all events might have run, as follows:

$(1-\sqrt{3})^{2m+1}$ being a negative fraction less than unity, the integer part of $(1+\sqrt{3})^{2m+1}$ is evidently

$$(1+\sqrt{3})^{2m+1}+(1-\sqrt{3})^{2m+1},$$

or is the sum of the $(2m+1)$th powers of the roots of the equation

$$x^2-2x-2=0,$$

from which the truth of the proposition is manifest.

We may add the remark that it may easily be shown in like manner that the integer next *above* the fraction $(1+\sqrt{3})^{2m}$, will also contain 2^{m+1} as a factor; and more generally, if we suppose that a is that integer congruent *quâ* the modulus 2 with n, which is next above or next below $\sqrt{n}$, then in the former case the two integers next above $(\sqrt{n}+a)^{2m+1}$ and $(\sqrt{n}+a)^{2m}$ respectively, and in the latter case the two integers next below the first and next above the second respectively, will each of them contain the factor 2^{m+1}.

The student is invited to ascertain whether any analogous theorem exists for latent integers expressed by means of higher surd forms.

* Questions of a similar nature, I am informed by Mr Ferrers, appeared in the Cambridge Senate House problems for the years 1847 and 1848.

20.

NOTE ON A PRINCIPLE IN THE THEORY OF NUMBERS AND THE RESOLUBILITY OF ANY NUMBER INTO THE SUM OF FOUR SQUARES.

[*Quarterly Journal of Mathematics*, I. (1857), pp. 196, 197.]

PROBABLY no one who has had any experience in the properties of numbers, could be found seriously doubting the truth of the proposition that any function of a variable integer, not algebraically decomposable into factors, must, among the infinite number of positive integers which it represents, be capable of affording prime as well as composite numbers, except in the case of its being of such a form as to admit of a constant divisor for all values of the variable. To prove this generally is probably a task reserved for remote generations, and for a more advanced development of the cerebral organization, but it is to my mind and conviction, and probably to that of most others, no less certain than the equally undemonstrable theorem which lies at the basis of the ordinary empirical geometry, that two parallels to the same line cannot be drawn through the same point. It would certainly be interesting to be able to deduce a connected body of doctrine as consequences flowing from the assumption of this principle, nor could the rigour of mathematical demonstration be in any degree prejudiced by its use, provided that every such consequence were stated only as a contingent truth, until either the principle itself had been as far as necessary apodictically established, or some other mode of demonstration substituted in its place. If this plan were followed out, it is not unlikely that the path would ultimately be discovered leading back to the demonstration of the fundamental principle, and, in the meanwhile, the *à priori* probability of its truth (if supposed to be inferior to moral certainty) would be confirmed by each additional experience of the correctness of the results to which it might be found to conduct.

Under this point of view it may not be uninteresting to show how the principle in question affords an almost instantaneous demonstration of the celebrated theorem of the resolubility of every integer into the sum of four squares.

Lemma. If M be any integer, and $3M = p^2 + q^2 + r^2 + s^2$, M may be expressed under the form $p'^2 + q'^2 + r'^2 + s'^2$.

For it may be observed, that of the four quantities p, q, r, s, either all are divisible by 3, or else one will be so divisible, and each of the others not; in either case, let p be divisible by 3, and give to the absolute values of $\sqrt{q^2}$, $\sqrt{r^2}$, $\sqrt{s^2}$ respectively such signs (if they do not all contain 3) as will make them congruent to one another *quâ* the modulus 3, then

$$M = \frac{p^2 + q^2 + r^2 + s^2}{3} = p'^2 + q'^2 + r'^2 + s'^2,$$

where
$$p' = \frac{\sqrt{q^2} + \sqrt{r^2} + \sqrt{s^2}}{3},$$

$$q' = \frac{\sqrt{r^2} - \sqrt{s^2} + p}{3},$$

$$r' = \frac{\sqrt{s^2} - \sqrt{q^2} + p}{3},$$

$$s' = \frac{\sqrt{q^2} - \sqrt{r^2} + p}{3};$$

consequently p', q', r', s' in either case, are all of them integers.

Suppose N to be odd, and of the form $4\mu + 1$; take the expression $3^{2x+1}N - 2$, and let T be one of the primes which, by virtue of our *principle* (since obviously in this case there is no constant factor), we assume it must contain. Then T is a prime number of the form $4\mu + 1$, that is, the sum of two squares, and consequently $T + 2$, that is, $3^{2x+1}N$ is the sum of four squares, whence, by the Lemma, it follows that N is the same.

In like manner if N is of the form $4\mu + 3$, we take T, one of the primes contained in the form $3^{2x}N - 2$, and as before $T + 2$, that is, $3^{2x}N$, and consequently N will be the sum of four squares.

If N be even, we may obviously consider it to be of the form $4\mu + 2$ (since the theorem, if true for N, will be so for $4N$), and then if T be a prime contained in the form $3^x N - 1$, $T + 1$ will be the sum of three squares, or which is the same thing of four squares, of which zero is one, and the reasoning is the same as before. Hence, in all cases, N is the sum of four squares; and the same result might be obtained with equal or greater facility by the application of the *principle* to various other forms.

21.

DEVELOPMENT OF AN IDEA OF EISENSTEIN.

[*Quarterly Journal of Mathematics*, I. (1857), pp. 199—203.]

EISENSTEIN has remarked, in a note among his collected works, that the expansion of any negative power of a series of ascending powers of x may be made to depend upon the expansions of the positive powers of the same series. The following method which reposes upon the most elementary principles of algebra serves to establish this practically important proposition.

Let
$$u = 1 + A_1 x + A_2 x^2 + \&c.$$
Then
$$\frac{1}{u^i} = \frac{1}{\{1-(1-u)\}^i} = 1 + i(1-u) + \frac{i(i+1)}{2}(1-u)^2 + \frac{i(i+1)(i+2)}{2.3}(1-u)^3 + \&c.$$

If now we wish to express the nth power of x or, in fact, any power of x lower than the nth by means of this series, it is obvious that we may stop at the term containing the nth power of $1-u$. In general, then, denoting by $C_{i,\nu}$ the coefficient of x^ν in u^i, provided ν is greater than unity and not greater than n, we have

$$\begin{aligned} C_{-i,\nu} = &-i\left(1 + (i+1) + \frac{(i+1)(i+2)}{2} + \&c. + \frac{(i+1)(i+2)\ldots(i+n-1)}{1.2\ldots(n-1)}\right) C_{1,\nu} \\ &+ i.\frac{i+1}{2}\left(1 + (i+2) + \&c. + \frac{(i+2)(i+3)\ldots(i+n-1)}{1.2\ldots(n-2)}\right) C_{2,\nu} \\ &\qquad \&c., \&c., \&c. \\ &\pm \frac{i(i+1)\ldots(i+n-1)}{1.2\ldots n} C_{n,\nu} \\ = &-i.\frac{(i+2)(i+3)\ldots(i+n)}{1.2\ldots(n-1)} C_{1,\nu} + i\frac{i+1}{2}.\frac{(i+3)\ldots(i+n)}{1.2\ldots(n-2)} C_{2,\nu} \\ &\mp \&c. \pm \frac{i(i+1)\ldots(i+n-1)}{1.2\ldots n} C_{n,\nu}. \end{aligned}$$

In practice it will, of course, be always most expedient (on the score of brevity of expression) to assume $n=\nu$.

If $$u = 1 + A_\omega x^\omega + A_{\omega+1}x^{\omega+1} + \&c.$$
the condition for the truth of the above equation will be that ν shall not exceed ωn; and then in practice it will be expedient to take n equal to $\frac{\nu}{\omega}$, if that be an integer, or, if not, to take n, the integer next above $\frac{\nu}{\omega}$.

We may with propriety denote by $C_{0,n}$ the coefficient of x^n in $\log u$*; and since
$$\log u = -\{(1-u) + \tfrac{1}{2}(1-u)^2 + \tfrac{1}{3}(1-u)^3 + \&c.\}$$
we shall have, subject to the same conditions as before,
$$\begin{aligned} C_{0,\nu} &= (1+1+\&c. \text{ to } n \text{ terms})\, C_{1,\nu} \\ &\quad - \tfrac{1}{2}(1+2+\&c. + (n-1))\, C_{2,\nu} \\ &\quad + \tfrac{1}{3}\left(1+3+6+\&c. + \frac{(n-2)(n-1)}{2}\right) C_{3,\nu} \\ &\quad \&c., \&c. \\ &= nC_{1,\nu} - \frac{(n-1)n}{2^2} C_{2,\nu} + \frac{(n-2)(n-1)n}{2.3^2} C_{3,\nu} \;\&c. + (-)^{n-1}\frac{C_{n,\nu}}{n}. \end{aligned}$$

In the general theorem suppose $i=1$, $\nu=n$. Then
$$C_{-1,n} = -\frac{(n+1)n}{1.2} C_{1,n} + \frac{(n+1)n(n-1)}{1.2.3} C_{2,n} \;\&c. \pm C_{n,n}.$$
Thus, to take the example alluded to by Eisenstein, suppose
$$u = \frac{x}{1-e^{-x}} = 1 + \frac{1}{2}x + B_1\frac{x^2}{2!} - B_2\frac{x^4}{4!} + B_3\frac{x^6}{6!} \;\&c., \&c.,$$
so that by the formula
$$\frac{(-)^{n+1}B_n}{(2n)!} = -\frac{(2n+1)2n}{1.2} C_{1,2n} + \frac{(2n+1)(2n)(2n-1)}{1.2.3} C_{2,2n} \;\&c. + C_{2n,2n}.$$
Here
$$\begin{aligned} C_{u,2n} &= \text{coefficient of } x^{2n} \text{ in } \left(\frac{1-e^{-x}}{x}\right)^u \\ &= \text{coefficient of } x^{2n+u} \text{ in } 1 - ue^{-x} + u\,\frac{u-1}{2}\,e^{-2x} \;\&c. + (-)^u e^{-ux} \\ &= \frac{(-)^u}{(2n+u)!}\left\{-u + u\,\frac{u-1}{2}\,2^{2n+u} \mp \&c. + (-)^u u^{2n+u}\right\} \\ &= \frac{\Delta^u 0^{2n+u}}{(2n+u)!}. \end{aligned}$$

* The rule for any power, positive or negative, of $\log u$ deserves investigation; the case of $C_{0,n}$ (using that symbol in a more extended sense than in the text above) containing, as it were, a microcosmical reiteration of the whole theory under discussion.

Hence

$$(-)^{n+1}B_n = (2n\,!)\left\{-\frac{(2n+1)\,2n}{2\,!}\frac{\Delta 0^{2n+1}}{(2n+1)\,!}\right.$$
$$\left.+\frac{(2n+1)\,(2n)\,(2n-1)}{3\,!}\frac{\Delta^2 0^{2n+2}}{(2n+2)\,!}+\&c.\right\}$$
$$= 2n\left\{\frac{-\Delta 0^{2n+1}}{2\,!}+\frac{2n-1}{2n+2}\frac{\Delta^2 0^{2n+2}}{3\,!}-\frac{(2n-1)\,(2n-2)}{(2n+2)\,(2n+3)}\frac{\Delta^3 0^{2n+3}}{4\,!}+\&c.\right.$$
$$\left.+\frac{(2n-1)\,(2n-2)\dots 1}{(2n+2)\,(2n+3)\dots 4n}\frac{\Delta^{2n}0^{4n}}{(2n+1)\,!}\right\}.$$

Thus, if $n=1$,

$$B_1 = 2\left\{-\frac{\Delta 0^3}{2}+\frac{1}{4}\frac{\Delta^2 0^4}{6}\right\}$$
$$= 2\left\{-\frac{1}{2}+\frac{1}{24}(2^4-2\,.\,1^4)\right\}$$
$$= 2\left\{-\frac{1}{2}+\frac{14}{24}\right\} = \frac{1}{6}.$$

If $n=2$,

$$-B_2 = 4\left\{\frac{-\Delta 0^5}{2}+\frac{1}{2}\frac{\Delta^2 0^6}{6}-\frac{1}{7}\frac{\Delta^3 0^7}{24}+\frac{1}{56}\frac{\Delta^4 0}{120}\right\}$$
$$= 4\left\{-\frac{1}{2}+\frac{1}{12}(2^6-2)-\frac{1}{168}(3^7-3\,.\,2^7+3)\right.$$
$$\left.+\frac{1}{6720}(4^8-4\,.\,3^8+6\,.\,2^8-4)\right\}$$
$$= 4\left\{-\frac{1}{2}+\frac{31}{6}-\frac{43}{4}+\frac{243}{40}\right\}$$
$$= 4\left\{-\frac{1}{2}+\frac{1}{6}+\frac{1}{4}+\frac{3}{40}\right\}$$
$$= \frac{1}{30}\{-60+20+30+9\}$$
$$= -\frac{1}{30},\ \text{or}\ B_2 = \frac{1}{30},\ \text{and so on.}$$

The annexed independent demonstration of the formula for $\dfrac{1}{fu}$ appertains to Mr Cayley.

Write
$$x = u + hfx,$$
then, by Lagrange's theorem,

$$Fx = Fu + \frac{h}{1}F'u\,fu + \frac{h^2}{1\,.\,2}\frac{d}{du}F'u\,(fu)^2 + \frac{h^3}{1\,.\,2\,.\,3}\left(\frac{d}{du}\right)^2 F'u\,(fu)^3 + \dots,$$

or, differentiating with respect to u,

$$\frac{F'x}{1-hf'x}=F'u+\frac{h}{1}\frac{d}{du}F'ufu+\frac{h^2}{1.2}\left(\frac{d}{du}\right)^2F'u(fu)^2+\dots;$$

whence, putting $h=\dfrac{x-u}{fx}$,

$$\frac{F'x}{1-(x-u)\dfrac{f'x}{fx}}=F'u+\frac{x-u}{fx}\frac{d}{du}F'ufu+\frac{1}{1.2}\left(\frac{x-u}{fx}\right)^2\frac{d^2}{du^2}F'u(fu)^2+\dots,$$

which is true identically.

Suppose now $F'u=\dfrac{u}{fu}$, we have

$$\frac{x}{fx-(x-u)f'x}=\frac{u}{fu}+\frac{x-u}{fx}\frac{d}{du}u+\frac{1}{1.2}\left(\frac{x-u}{fx}\right)^2\frac{d^2}{du^2}ufu+\text{\&c.},$$

or, if $\qquad fu=1+bu+cu^2+du^3+\text{\&c.},\quad fx=1+bx+cx^2+\text{\&c.},$

and $x=0$, that is, $fx=1$, the formula becomes

$$0=\frac{u}{fu}-u+\frac{u^2}{1.2}\left(\frac{d}{du}\right)^2ufu-\frac{u^3}{1.2.3}\left(\frac{d}{du}\right)^3u(fu)^2+\dots$$

Whence

$$\frac{1}{fu}=1-\frac{u}{1.2}\left(\frac{d}{du}\right)^2ufu+\frac{u^2}{1.2.3}\left(\frac{d}{du}\right)^3u(fu)^2$$
$$-\frac{u^3}{1.2.3.4}\left(\frac{d}{du}\right)^4u(fu)^3+\text{\&c.},$$

which gives the expansion of $\dfrac{1}{fu}$ when the expansions of the positive powers $(fu)^2$, $(fu)^3$, &c. are known.

22.

NOTE ON THE ALGEBRAICAL THEORY OF DERIVATIVE POINTS OF CURVES OF THE THIRD DEGREE.

[*Philosophical Magazine*, XVI. (1858), pp. 116—119.]

TWO years and upwards have elapsed since I discovered the extraordinary theorem in the doctrine of cubic forms which I am about to state, but which has never yet been published by me, although communicated in confidence to a few friends, including Mr Cayley. It arose out of purely arithmetical speculations relating to such forms, to some of which I may make a brief allusion in the course of this note.

If we suppose the general homogeneous equation of the third degree in x, y, z reduced to the canonical form

$$x^3 + y^3 + z^3 + mxyz = 0,$$

any solution $x = a$, $y = b$, $z = c$ of this equation is of course one of a group of six obtained by the permutations of the three letters a, b, c, and having an obvious relation to one another through the medium of the points of inflexion. So, too, it is manifest if we take the equation to the curve in its most general form, from any given solution, a group of six, including the given one, may be formed, *the characteristics* of each of which will be linear functions of one another. For the purpose of the theorem about to be enunciated, such a group of solutions will be treated as a single solution; and then we can affirm the proposition following, in which a solvent system means a system of values of the variables x, y, z satisfying the equation $f(x, y, z) = 0$, and free from any common factor.

Let a, b, c *be any solvent system to a cubic homogeneous equation in* x, y, z; *then from* a, b, c *we may derive a new solvent system,* a′, b′, c′, *where* a′, b′, c′ *are each of them functions of the fourth degree of* a, b, c, *and another system* a″, b″, c″ *of the ninth degree in* a, b, c, *and another* a‴, b‴, c‴ *of the sixteenth*

degree, and so in general a new solvent system of the degree n^2 *in* a, b, c. *One such derivative system, and only one, of the degree* n^2 *can be formed, and none of any intermediate degree.*

Thus, for instance, the coordinates of the tangential (the name adopted from me by Mr Cayley to express the point of intersection of a tangent to a cubic curve at any point with the curve) being called a', b', c', these last letters are *biquadratic* functions of a, b, c*.

So again, as I also suggested to Mr Cayley, the point in which the conic of closest contact with a cubic curve cuts the curve will necessarily have a derivative system of coordinates of a square-numbered degree in respect of the original ones, which by actual trial Mr Cayley has found to be the 25th. Mr Salmon, I believe, has obtained in certain geometrical investigations derivatives of the 49th degree.

I am in possession of the equations by means of which the successive systems of the fourth, ninth, &c. degrees, which I incline to call the first or primary, the second, third, &c. derivative systems, may be formed explicitly by successive derivation from one another; so that, for instance, as soon as I am informed that the system investigated by Mr Cayley is of the twenty-fifth degree or fifth order, I can find them without any reference to the geometry of the question, the quantities belonging to the nth derivative being in fact a known algebraical function of n! I was led to the discovery of this surprising and unique law by a statement of a friend, *not since verified, and which, for aught that has yet been shown, may or may not be true*, that the number 5 *could be* divided into two rational cubes: assuming this to be the fact, it necessitated (by virtue of my investigations) the coincidence to a factor *près* of two functions obtained by apparently independent algebraical processes, which coincidence by actual comparison of the functions I found to obtain.

With reference to the connexion of this theory of derivation with the arithmetic of equations of the third degree between three variables with integer coefficients, it is after this kind. Fermat has taught us that a certain class of such equations, viz. the equation $x^3 + y^3 + z^3 = 0$, is absolutely insoluble in integers (abstraction made of the trivial solutions of the type $x = 0$, $y + z = 0$). I have greatly multiplied the classes of such known insoluble equations, as may be seen by a communication from me to Tortolini's *Annali* in 1856 [p. 63 above]. But over and above such equations I have ascertained the existence of a large class of equations, soluble, or possibly so, it is true, but enjoying the property that all their solutions in integers, when they exist, are *monobasic*; that is to say, all their solutions are known functions of one

* This derivative solution (though not as corresponding to the *tangential*) was known also to Euler for a particular case, as will be seen by reference to his *Algebra*.

of them, which I term the *base*, and which is characterized by this property—that of all the solutions possible it is the one for which the *greatest* of the three variables is the *smallest* number possible. If this solution be laid down as a point in the curve corresponding to the given cubic, all the other solutions possible in integers will be represented by points in this curve, which are derivatives (in the sense previously employed in this note) to the given point, having coordinates respectively of the 4th, 9th, 16th, &c. degrees, in respect of the coordinates of the *basic* point*.

If my memory serves me truly, I have found (as a particular case) that all cubic equations in numbers of the form

$$x^3 + y^3 + z^3 = imxyz,$$

where i is 1 or 3 or 6 (I cannot at the moment remember which), are either *insoluble* or *monobasic*. The case of $im = 3$ must of course be exceptional, being satisfied by $x + y + z = 0$. This doctrine of derivation evidently conducts to a new branch of the grand doctrine of invariance. I hope to have tranquillity of mind ere long to give to the world my memoir, or a fragment of it, "On an Arithmetical Theory of Homogeneous and the Cubic Forms," the germ of which, now, alas! many weary years ago, first dawned upon my mind on the summit of the Righi, during a vacation ramble.

* This theorem is analogous to that relating to the integer solutions of $x^2 - Ay^2 = 1$, in so far as there is a *basic* solution to this equation in integers of which all the other solutions are derivatives, and not more than one such derivative exists of any given degree, but with the difference that there does exist one of every degree, and not merely (as in my theorem for cubic forms) of every square degree.

23.

NOTE ON THE EQUATION IN NUMBERS OF THE FIRST DEGREE BETWEEN ANY NUMBER OF VARIABLES WITH POSITIVE COEFFICIENTS.

[*Philosophical Magazine*, XVI. (1858), 369—371.]

I PROPOSE to show that all the systems of values $(x, y, z \dots w)$ which satisfy a given equation in integers,

$$ax + by + cz + \dots + lw = n,$$

($(a, b, c \dots l)$ being all positive, and the number of systems therefore definite), may be made to depend on algebraical equations whose coefficients are known functions of $a, b, c \dots l$ and n. The fact is somewhat surprising, the proof easy, being an immediate consequence of the theorem I have given* in the *Quarterly Journal of Mathematics*, and also in Tortolini's *Annali* for January 1857, of the problem of the partition of numbers.

For my present purpose, this theorem may be with advantage presented under a somewhat modified form as follows:—Let $\Theta(Ft)$ be used to denote the coefficient of $\frac{1}{t}$ in the expansion of Ft in ascending powers of t. Let N stand for the number of solutions of the equation

$$ax + by + cz + \dots + lw = n;$$

let m be the least common multiple of $a, b, c, \dots l$,

ρ be any *primitive* root of $\rho^m = 1$,

and ρe^{-pt} be called Λp; then

$$N = \Sigma\Theta \left\{\frac{\Lambda(-n)}{(1 - \Lambda a)(1 - \Lambda b) \dots (1 - \Lambda l)}\right\}.$$

If now we call N' what N becomes when, in lieu of the equation

$$ax + by + cz + \dots + lw = n, \quad (1)$$

we write

$$ax' + ax'' + by + cz + \dots + lw = n, \quad (2)$$

[* p. 90 above.]

it is clear that

$$N' = \Sigma\Theta \left\{\frac{\Lambda(-n)}{(1-\Lambda a)^2(1-\Lambda b)\dots(1-\Lambda l)}\right\}.$$

But it is also clear that all the solutions of equation (2) may be derived from those of equation (1), by writing for each value of x

$$x' + x'' = x; \tag{3}$$

and as the number of solutions of equation (3) is evidently $x+1$, we have $N' = \Sigma x + N$, or

$$\Sigma x = \Sigma\Theta \left\{\frac{\Lambda a \,.\, \Lambda(-n)}{(1-\Lambda a)^2(1-\Lambda b)\dots(1-\Lambda l)}\right\}.$$

In like manner, if we write

$$ax' + ax'' + ax''' + by + cz + \dots + lw = n,$$

the solutions of this equation spring from those of equation (1) by making $x' + x'' + x''' = x$, the number of solutions of which equality is $\frac{1}{2}(x+1)(x+2)$: wherefore

$$\Sigma \frac{x^2+3x+2}{2} = \Sigma\Theta \left\{\frac{\Lambda(-n)}{(1-\Lambda a)^3(1-\Lambda b)\dots(1-\Lambda l)}\right\};$$

from which we may readily deduce, by aid of what has been already shown,

$$\Sigma x^2 = \Sigma\Theta \frac{(\Lambda a)(1+\Lambda a)\Lambda(-n)}{(1-\Lambda a)^3(1-\Lambda b)\dots(1-\Delta l)};$$

and so in general,

$$\Sigma x^i = \Sigma\Theta \frac{\Lambda(a)(1+\Lambda a)\dots\{(i-1)+\Lambda a\}}{(1-\Lambda a)^{i+1}(1-\Lambda b)\dots(1-\Lambda l)}.$$

Again, if we write

$$ax + by_1 + by_2 + \dots + by_\epsilon + cz + \dots + lw = n, \tag{4}$$

we shall find by parity of reasoning (seeing that in this last equation the solutions may be derived from those of equation (1) by keeping $x, z, \dots w$ all unaltered, whilst we give to $y_1, y_2 \dots y_\epsilon$ all the values compatible with $y_1 + y_2 + \dots + y_\epsilon = y$), the value of Σx^i in equation (4) will be the same as that of

$$\Sigma x^i . \frac{(y+1)(y+2)\dots(y+\epsilon)}{1.2\dots\epsilon}$$

in equation (1). Wherefore we shall evidently obtain

$$\Sigma x^i . y^\epsilon = \Sigma\Theta \frac{\Lambda a(1+\Lambda a)\dots\{(i-1)+\Lambda a\} \times \Lambda b(1+\Lambda b)\dots\{(\epsilon-1)+\Lambda b\}}{(1-\Lambda a)^{i+1}(1-\Lambda b)^{\epsilon+1}(1-\Lambda c)\dots(1-\Lambda l)};$$

the extension of the theorem to $\Sigma x^i . y^\epsilon . z^\omega \dots$ is too obvious to need further allusion.

Thus, then, to find $x_1, x_2 \dots x_N$, we may begin by forming an equation of the Nth degree, whose coefficients are known, because the sums of the powers of the roots are given. Supposing these roots to consist of N_1 values x_1, N_2 values $x_2, \dots N_\mu$ values x_μ, the solution of μ simple equations will enable us to find the sum of the N_1 values of y corresponding to x_1, the sum of the N_2 values of y corresponding to $x_2 \dots$, and the sum of the N_μ values of y corresponding to x_μ. To effect this, we have only to write down the values of $\Sigma xy, \Sigma x^2 y, \dots \Sigma x^\mu y$. In like manner we may find the sum of the N_1 values of y^2 corresponding to x_1, the N_2 values of y^2 corresponding to x_2, &c., and so in general for y^ω. Thus, then, we may obtain the requisite number of sets of equations for determining independently by means of equations of the degrees $N_1, N_2, \dots N_\mu$ respectively the values of y corresponding to each of the distinct values of x; and in like manner for all the other variables. The principal interest of this note consists, however, in the appreciation of the fact that we can represent algebraically, as has been shown above, the value of $\Sigma x^\alpha . y^\beta . z^\gamma \dots$, where the sign of summation extends over all the simultaneous solutions of

$$ax + by + cz + \&c. = n.$$

This is a considerable advance upon the conception (itself before my discovery entirely unrecognized*) of the explicit representability of the mere number of the solving systems $x, y, z \dots$ by general algebraical formulæ. By this new theorem we pass, as it were, from the shadow to the substance.

* As witness the comparatively unfructuous labours of Paoli, Herschel, Kirkman, and even of Cayley. But as honest labour is seldom entirely wasted, so in the present case it was my valued friend Mr Kirkman's Manchester memoir on partitions which first drew and fixed my attention on the subject.

24.

ON THE PROBLEM OF THE VIRGINS, AND THE GENERAL THEORY OF COMPOUND PARTITION.

[*Philosophical Magazine*, XVI. (1858), pp. 371—376.]

IN the *Opera Minora* of the great Euler, in the last page of his *second* memoir on the partition of numbers (Vol. I. p. 400), occur these words:—"Ex hoc principio definiri potest quot solutiones problemata quæ ab arithmeticis *ad regulam virginum* referri solent, admittunt; hujusmodi problemata huc redeunt ut inveniri debeant numeri p, q, r, s, &c., ita ut his duabus conditionibus satisfiat,

$$ap + bq + cr + ds + \&c. = n, \text{ et } \alpha p + \beta q + \gamma r + \delta s + \&c. = \nu;$$

et jam quæstio est quot solutiones in numeris integris positivis locum sint habituræ ubi quidem tenendum est numeros $a, b, c, d \ldots n$ et $\alpha, \beta, \gamma, \delta \ldots \nu$ esse integros"; and he then proceeds to observe that the number in question is the coefficient of $x^m y^n$ in the expansion of the expression

$$\frac{1}{(1 - x^a y^\alpha)(1 - x^b y^\beta)(1 - x^c y^\gamma)\ldots}$$

in terms of ascending positive powers of x and y.

Why the solution in integers of two simultaneous equations with an indefinite number of variables should be referred to "the rule of the Virgins" I am at a loss to conjecture, unless indeed it be supposed to have some mystical reference to the alligation or coupling of the coefficients of the two equations*. The problem in question may be otherwise stated as having

* Professor De Morgan has kindly furnished me with the following information as to the use of this singular phrase:—

"I have seen this process cited as the rule of—Ceres, Series, Verginum, Virginum, Ceres and Virginum, Series and Virginum, Ceres and Verginum, Series and Verginum. I do not think any one of the eight is missing. I cannot find that Ceres is attended by any maidens, and I cannot guess who the ladies were. It is applied by the arithmeticians to the rule of *alligation* when of an indeterminate number of solutions—just Euler's problem which you quote." Mr De Morgan subsequently writes, "I forget whether they wrote Series or Ceries; I think the latter"; and adds a pleasant caution against indulging a passion for one of these algebraical virgins; "for that though Jupiter did once animate a statue maiden at the prayer of an enamoured sculptor, yet even Jupiter himself could not impart a body to an algebraical abstraction."

for its object to discover the number of modes in which the couple m, n may be made up of the couples a, α; b, β; c, γ &c.

I need hardly remark that Euler's form of representation is no solution, but merely a transformation of the question. The problem in its most general form is to determine the number of modes in which a given set of conjoint partible numbers $l_1, l_2, \ldots l_r$ can be made up simultaneously of the compound elements,

$$a_1, a_2, \ldots a_r; \quad b_1, b_2, \ldots b_r; \quad c_1, c_2, \ldots c_r; \text{ \&c.}$$

The problem of simple partition has been already completely resolved by the author of this notice; but the resolution of the problem of double, and still more of multiple decomposition in general, seemed to be fenced round with insurmountable difficulties.

Let the reader imagine then with what surprise and joyful emotion, within a few days of despatching my previous paper on Partitions to this present Number of the Magazine, following out a train of thought suggested by the simple idea in that paper contained, I found myself led, as by a higher hand, to the marvellous discovery that the problem of compound partition in its utmost generality is capable of a complete solution—in a word, that this problem may in all cases be made to depend on that of simple partition. The theorem by which this is effected has been already confided to the great mathematical genius of England, and will be shortly committed to the 'Transactions' of one of our learned societies; for the present I shall confine myself to a disclosure of the general character of the theorem without going into any details. Thus, then, may the theorem be stated in general terms:—

Any given system of simultaneous simple equations to be solved in positive integers being proposed, the determination of the number of solutions of which they admit may in all cases be made to depend upon the like determination for one or more systems of equations of a certain fixed standard form. When a system of r *equations between* n *variables of the aforesaid standard form is given, the determination of the number of solutions in positive integers of which it admits may be made to depend on the like determination for*

$$\frac{\mathrm{n}\,(\mathrm{n}-1)\ldots(\mathrm{n}-\mathrm{r}+2)}{1\,.\,2\quad\ldots\quad(\mathrm{r}-1)}$$

single independent *equations derived from those of the given system by the ordinary process of elimination, with a slight modification; the final result being obtained by taking the sum of certain numerical multiples (some positive, others negative) of the numbers corresponding to those independent determinations. This process admits of being applied in a variety of modes, the resulting*

*sum of course remaining unaltered in value whichever mode is employed, only appearing for each such mode made up of a different set of component parts**.

In the Problem of the Virgins, where but two equations are concerned, the equations are reduced to the standard form when the two coefficients of every the same variable in the two equations are prime to one another, and when no two pairs of coefficients have the same ratio; and for this problem the process is always limited to only two modes of application. The method, however, in a very important class of cases admits of being applied in *one*, and only *one* mode when these conditions are not strictly fulfilled.

Thus the virgins who appeared to Euler, but with their forms muffled and their faces veiled, have not disdained to reveal themselves to me under their natural aspect. Wonderful indeed has been the history of this theory of partitions. Notwithstanding that the immortal Euler had written two elaborate memoirs on the subject, that Paoli, and I believe other Italian mathematicians, had taken it up from another but less advantageous point of view, so completely had it fallen into oblivion, as far as the mathematicians of this country are concerned, that Sir John Herschel has written a memoir upon it, inserted in the *Philosophical Transactions*, without any reference to, and evidently in complete unconsciousness of, the labours of his predecessors, and subsequently Professor De Morgan, so justly celebrated for his mathematical erudition, in a paper in the *Cambridge and Dublin Mathematical Journal*, refers to the doctrine of partitions as being of quite recent creation. The importance of the subject in these later times has been vastly augmented by the magnificent applications which our great mathematical luminary has made of it to the doctrine of invariants.

* Since the above was in print, I have discovered a much more specific theorem, which, indeed, is to be regarded as the fundamental theorem in the doctrine of compound partition, and the basis of that given in the text. It is as follows:—*If there be* r *simultaneous simple equations between* n *variables (in which the coefficients are all positive or negative integers) forming a definite system (that is, one in which no variable can become indefinitely great in the positive direction without one or more of the others becoming negative), and if the* r *coefficients belonging to each of the same variable are exempt from a factor common to them all, and if not more than* r − 1 *of the variables can be eliminated simultaneously between the* r *equations, then the determination of the number of positive integer solutions of the given system may be made to depend on like determinations for each of* n *derived independent systems, in each of which the number of variables and equations is one less than in the original system.*

This reduction in general can be effected in a great but limited variety of modes. When only two equations, however, are concerned, the number of modes is always two, neither more nor less. So that in fact we are still navigating in the narrows, and have not fairly entered upon the wide ocean of the theory of compound partitions until we have passed the case of double partition. When the given system supposed definite is one of three equations between four variables, the number of modes of reduction is twelve or sixteen, according to that type out of two (to one or the other of which it must of necessity belong) under which the system falls. The theory of types applicable to any system of simultaneous simple equations with rational coefficients, here faintly shadowed forth, constitutes, I apprehend, a new and important branch in the theory of inequalities.

Postscript. In the first instance I discovered the theorem above given by a method of induction, aided by an effort of imagination, and confirmed by numerous trials; but I have since obtained a very simple, although somewhat subtle general proof of it. Mr Cayley on his part, and independently, has also laid the foundation of a most ingenious and instructive method of demonstration entirely distinct from my own. I reason upon the equations, Mr Cayley upon the Eulerian generating function; but it was by operations performed upon this function that I was myself originally led to a perception of the transcendental analogies out of which I was enabled to evolve the law.

The very interesting case of the composition of a proposed integer out of elements given both in number and species, to which Euler has called particular attention, falls without preparation under the standard form; for this question is in fact merely that of determining the number of solutions of the binary system of equations,

$$ax + by + cz + \ldots + lw = m,$$

$$x + y + z + \ldots + w = \mu,$$

$a, b, c, \ldots l$ being supposed to be all different.

Thus, by way of very simple illustration, suppose it required to find in how many ways the number m can be made up of μ elements, limited to consist of the numbers 1, 2, 3. My method gives me at once the following solution. Call ν the number required. Then m must be not less than μ, and not greater than 3μ, or there will be no solutions. For all values of m between μ and 2μ, both inclusive,

$$\nu = \frac{m-\mu}{2} + \frac{3}{4} + (-1)^{\frac{m-\mu}{2}};$$

for all values of m between 2μ and 3μ, *still both inclusive*,

$$\nu = \frac{3\mu - m}{2} + \frac{3}{4} + (-1)^{\frac{m-\mu}{2}}.$$

It will be observed that when $m = 2\mu$, the two formulæ give the same value, so that either may be employed. Again, suppose we wish to express the number of modes of composition of m with the four elements 1, 2, 3, 4, the number of parts being μ, $\frac{m}{\mu}$ must be not less than 1 nor greater than 4, or there will be no solutions possible.

For all values of m from μ to 2μ inclusive,

$$\nu = \tfrac{1}{12}\{(m-\mu+3)^2 - \tfrac{7}{6}\} + \tfrac{1}{8}(-1)^{m-\mu} + \tfrac{1}{9}(\rho^{m-\mu} + \rho'^{m-\mu}),$$

ρ, ρ' being the prime cube roots of unity.

For all values of m from 2μ to 3μ inclusive,

$$\nu = \frac{(m-\mu+3)^2}{12} - \frac{(m-2\mu+3)^2}{4} + \frac{73}{36}$$
$$+ \tfrac{1}{8}\{(-1)^{m-\mu} + (-1)^m\}$$
$$+ \tfrac{1}{9}\{\rho^{m-\mu} + \rho'^{m-\mu}\}.$$

Finally, for all values of m from 3μ to 4μ inclusive,

$$\nu = \tfrac{1}{12}\{(4\mu - m - 3)^2 - \tfrac{7}{6}\} + \tfrac{1}{8}(-1)^m + \tfrac{1}{9}(\rho^{m-\mu} + \rho'^{m-\mu}).$$

At the joining points (so to say) between the successive cases, viz. where $m = 2\mu$ or $m = 3\mu$, the contiguous formulæ give like results whichever of them is applied, so that the discontinuity in the form of the solution resembles that arising from the juxtaposition of different curves*. This discontinuity (in itself a remarkable phænomenon to be brought to light), far from being a reproach to the method employed, is to be regarded as a quality inherent in the subject matter under representation, and inexpugnable, as such, in the very nature of things.

* The connexion between the contiguous formulæ is always closer than what is symbolized by the phrase used above. The curves must be regarded as not merely placed end to end, but to be, as it were, knit or spliced together through a certain finite portion of the extent of each of them. Thus the first and second formulæ in the text coincide [?] in value, not merely for $m=2\mu$, but also for $m=2\mu-1$ and $m=2\mu-2$; and the second and third formulæ coincide, not merely for $m=3\mu$, but also for $m=3\mu+1$ and $m=3\mu+2$. The adjacent curves have, so to say, in the instance above, the same tangents and circles of curvature at the points of union, so that we may be said to *modulate* from one formula into another. The *raison raisonnée* of this fact is easily explicable on *à priori* analytical principles.

25.

ON A GENERALIZATION OF PONCELET'S THEOREMS FOR THE LINEAR REPRESENTATION OF QUADRATIC RADICALS.

[*Oxford British Association Report*, Pt. II. (1860), p. 7.]

THE author explained the application of Poncelet's theorems to practical questions of mechanics in the case of forces acting in a single plane as in the theory of bridges.

He next referred to the mode of extension of this theorem, suggested by Poncelet, applicable to the case of forces in space, and pointed out its insufficiency, and, in a certain sense, its incorrectness.

The essential preliminary question to be resolved in the first instance (after which the matter became one of easy calculation), was shown to be that of cutting off by a plane the smallest possible segment of a sphere that should contain the whole of a given set of points lying on the sphere's surface. Some years ago Prof. Sylvester had proposed in the *Quarterly Mathematical Journal*, without any suspicion of its having any practical applications, the following question:—"Given a set of points in a plane, to draw the smallest possible circle that should contain them all." By a singular coincidence, Professor Peirce, of Cambridge University, U.S., had studied this question and obtained a complete solution of it, which he had communicated to the author during the present meeting of the British Association. A slight consideration served to show that precisely the same solution as Professor Peirce had found for the problem of points in a plane was applicable with a merely nominal change to the sphere also; and thus the solution of a question set almost in sport was found to supply an essential link for the complete development of a method of considerable importance in practical mechanics. The author stated that it would be easy to draw up tables of the values of the constants appearing in the linear function, representing the resultant of three forces at right angles to one another, for the principal cases likely to occur in practice, the values of these constants depending solely upon the condition of relative magnitude to which the component forces are supposed to be subjected.

26.

OUTLINES OF SEVEN LECTURES ON THE PARTITIONS OF NUMBERS.

[*Proceedings of the London Mathematical Society*, XXVIII. (1897), pp. 33—96.]

PREFACE.

THESE outlines appertain to lectures delivered by Prof. Sylvester at King's College, London, during the year 1859. The outline of each lecture was printed shortly before its delivery and handed to those in attendance, and a few copies also were privately circulated. They are now published for the first time. The Professor's attention was called away shortly afterwards to another department of mathematics, with the result that his researches on compound partitions were never published. As the lectures constitute the only serious attempt that has ever been made to deal with the subject, and as copies of the outlines are very scarce, Prof. Sylvester has yielded to the suggestion made to him in regard thereto by the Council of the London Mathematical Society, so far as to assent to their publication in the *Proceedings*, with all their imperfections on their heads. The present state of his health and the long lapse of time combine to render any revision upon the part of the Professor impossible. He desires it to be known that he cannot vouch for the correctness of all that appears in the notes, and that they were prepared in a hand-to-mouth manner during the process of investigation between the lectures, and that it is only on the opinion of the Council urgently expressed to him that the work should not entirely perish that he has consented at this late hour to the publication.

The Council desires to acknowledge the assistance it has derived from Prof. H. W. Lloyd Tanner, of University College, Cardiff, who kindly placed his annotated copy of the outlines at its disposal, and also to Mr R. F. Scott, of St John's College, Cambridge, who presented a copy to the Society.

FIRST LECTURE*.

Introductory Remarks.

Resolution of an integer into parts.

Resolution of an integer into parts limited in number.

Resolution of a number into parts limited in magnitude.

Euler's law of reciprocity, viz.,

As many ways as an integer n can be resolved into parts not exceeding m in number, so many ways can it be resolved into parts not exceeding m in amount.

Ferrers' Proof. Example. $n = 5$, $m = 3$,

$$\left.\begin{array}{cccc} 111 & 11 & 11 & 1 \\ 11 & 11 & 1 & 1 \\ & 1 & 1 & 1 \\ & & 1 & 1 \\ & & & 1 \end{array}\right\}\begin{array}{lllll} & & \text{may be read as} & & \\ & 3,\ 2; & 2,\ 2,\ 1; & 2,\ 1,\ 1,\ 1; & 1,\ 1,\ 1,\ 1,\ 1; \\ \text{or} & 2,\ 2,\ 1; & 3,\ 2; & 4,\ 1; & 5. \end{array}$$

Cayley's application of this law to the calculation of groups of symmetric functions.

Example. To find Σx^5, $\Sigma x^4 y$, $\Sigma x^3 y^2$, $\Sigma x^2 y^2 z$, where x, y, z are roots of $x^3 - p_1 x^2 + p_2 x - p_3 = 0$, $p_1 . p_1 . p_1 . p_1 . p_1$, $p_2 . p_1 . p_1 . p_1$, $p_2 . p_2 . p_1$, $p_3 . p_2$ will be linear functions of the quantities to be found.

Euler, Waring, Paoli, De Morgan, Warburton, Herschel, Kirkman, Ferrers, Cayley, in connexion with question of resolution.

The resolution of a number into parts is the problem of ascertaining the different modes of composing n with the elements

$$1, 2, 3, \ldots \text{ up to } n.$$

General problem of *simple* partition is to find in how many ways a given number n can be composed of given elements $a, b, c, \ldots k$.

General problem of *binary* partition is to find in how many ways the couple m, m' can be composed of the couples a, a', b, b', c, c', k, k'.

Statement of problem under form of equations.

Denumeration and denumerant defined.

Denumerant of $U = 0$ same as that of $kU = 0$.

Denumerant of $U = 0$, $V = 0$ same as that of

$$kU + lV = 0, \quad k'U + l'V = 0.$$

* Delivered at King's College, London, on the 6th June, 1859.

Coefficient groups and constant group defined.

How the resolution of an equation or system of equations with any *real* coefficients may be made to depend on the inverse problem of the centre of gravity of a system of points.

Example. A system of two equations.

Origin, coefficient points, primary defined.

Total of coefficient points is called a cluster.

Coefficient points may be denoted by the variables to which they belong.

Weighted cluster.

Weight of primary assumed to be positive unity.

If primary and cluster balance about the origin, the weights at the several points of cluster will satisfy the given system of equations.

Linear cluster; plane cluster; solid cluster.

The cluster origin and primary may be considered apart from the axes used in the construction.

Ray cluster; axis of cluster defined.

Derivative of an equation-system. An equation-system really consists of the universe of its derivatives.

How this universe is contained in the geometrical representation of the system.

A principal derivative of a binary system is the equation resulting from the elimination of any *one* of its variables.

A principal derivative of a ternary system is the equation resulting from the elimination of any *two* of its variables.

Universe or Plexus of Principal Derivatives.

How to construct geometrically the principal derivatives by aid of the cluster, primary, and origin.

(1) For binary system.

(2) For ternary system.

We can thus perform the process of elimination geometrically.

If more than the regular number of variables can be eliminated simultaneously out of the system, this will be evidenced in the plane cluster by three or more points lying in a line, and in the solid cluster by four or more points lying in a plane*.

* The general polyhedron *in solido* analogous to the polygon *in plano* is a polyhedron with triangular faces exclusively.

An equation is said to be homonymous when the coefficients of the variables are all positive or all negative.

It may be congruous or incongruous.

Example. $$2x + 3y + 4z = 10 \text{ congruous,}$$
$$2x + 3y + 4z = -10 \text{ incongruous.}$$

An *omni-positive* solution of an equation or system means a solution in which the variables are all positive.

An *omni-negative* solution is one in which the variables are all negative.

A *homonymous* solution is one which is either omni-positive or omni-negative.

An equation or equation-system may be *definite* or *indefinite.*

Indefinite when homonymous solutions can be found wherein the variables may be made indefinitely great.

Definite when the variables cannot be made indefinitely great in any homonymous solution.

The equations $ax - by = m$ and $ax + by - cz = m$, where $a, b, c, \ldots m$ are any real positive quantities whatever, are indefinite.

The character as to definite or indefinite depends only on the coefficients, and not on the constant term.

A single equation to be definite must be homonymous.

A system of equations to be definite must admit of a homonymous derivative.

If it admit of one, it must admit of an infinite number of such.

Definiteness and indefiniteness of systems depend only on the relative values of coefficients, and not on the constant terms.

Hence, the relative position of origin and cluster must suffice geometrically to determine this character.

Definition of boundary of a plane or solid cluster of points.

Lemma. The centre of gravity of any weighted cluster is contained inside the boundary, and may be made to lie at any point within it by a due adjustment of the relative magnitudes of the weights at the several points.

Theorem. If the origin lies within the cluster, the system is indefinite; if outside, definite.

In Fig. 1 the centre of gravity of the cluster may be brought to the position g or g' as near as we please to O on either side of it in a line with PO, and, the sum of the weights

$$x + y + z + t + u + v + w + \omega$$

being $\frac{PO}{gO}$ or $-\frac{PO}{g'O}$, may be made indefinitely great either on the positive or negative side of zero.

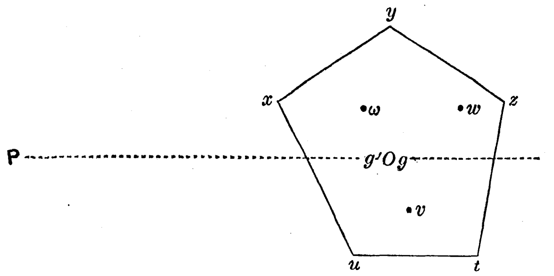


Fig. 1.

In Fig. 2, if origin is at O, Σx will lie between $\frac{PO}{gO}$ and $\frac{PO}{g'O}$; if origin is at O', Σx will lie between $-\frac{PO'}{gO'}$ and $\frac{PO'}{g'O'}$*; if at O'', the system cannot by any system of weights, all positive or all negative, be made to balance the weight at P about the origin.

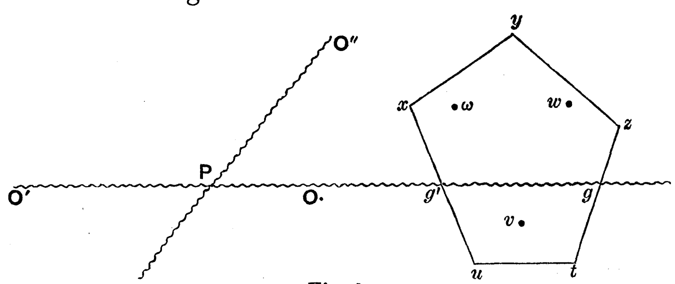


Fig. 2.

The same method is applicable to points *in solido*.

Indefinite systems in general admit of homonymous solutions of both kinds.

The only case of exception is when the origin is in the contour of cluster.

Definite systems admit only of solutions of one kind.

* Hence it may easily be shown that the greatest and least values of Σx in any definite system of equations

$$a_1x_1 + a_2x_2 + a_3x_3 + \ldots = m,$$
$$a_1'x_1 + a_2'x_2 + a_3'x_3 + \ldots = m',$$
$$a_1''x_1 + a_2''x_2 + a_3''x_3 + \ldots = m'',$$

will be the greatest and least values of ρ deduced successively from all the equations that can be formed after the type of the following:—

$$\begin{vmatrix} a_1, & a_2, & a_3, & 1 \\ a_1', & a_2', & a_3', & 1 \\ a_1'', & a_2'', & a_3'', & 1 \\ m, & m', & m'', & \rho \end{vmatrix} = 0,$$

and in like manner we may derive from the geometrical method a simple rule for determining algebraically the maxima and minima values of each separate variable.

Three Species of Definite Systems.

The system is *positive* or *negative* when the axis cuts the cluster according as primary or cluster lie on opposite or same side of origin.

It is *neuter* when the axis does not cut the cluster. (For example, origin O''.)

An analytical determination of the genus and species of a system may be deduced from the preceding construction.

Binary System.

In the indefinite case, if we draw lines from origin to every point in cluster, each such ray divides the cluster into two parts.

In the definite case there are two extreme rays leaving all the points in the cluster on the same side.

Hence, if a system is indefinite, the universe or plexus of principal derivatives will contain no homonymous equations.

If it be definite, it will contain two homonymous equations.

Again, as regards species—

If the system is positive definite, the two homonymous derivatives will be both congruous. If the system is negative, they will be both incongruous.

If the system be neuter, the homonymous will be one congruous, the other incongruous.

Ternary System.

If the system is indefinite, all the planes through the origin and any two points of the cluster divide the cluster into two parts.

If it be definite, the bounding planes of the pyramid formed by joining the origin with each point of the cluster will leave the other points of cluster all on one side.

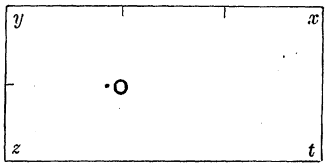
Hence, when the system is indefinite, the plexus of principal derivatives will contain no homonymous equations; when it is definite, there will be some homonymous derivatives, and the number cannot be less than three or greater than the number of variables. For, if we project the cluster from the origin on a plane cutting the rays all on the same side of origin, the number of sides in contour of this projection will be the number of planes in

the pyramid, and n points in a plane cannot form a figure bounded by less than three nor more than n sides*.

The different species of definite will be distinguishable by the homonymous principal derivatives being all congruous, all incongruous, or partly congruous, partly incongruous.

Examples of *indefinite* two-equation systems:

Let the cluster of coefficient points be at the four angles of a parallelogram x, y, z, t, the origin O being at the distance of one unit from yz, yx, zt, and two units from xt.



The system of equations will be

$$\begin{aligned} 2x - y - z + 2t &= n, \\ x + y - z - \ \ t &= m. \end{aligned}$$

The universe of principal derivatives will be

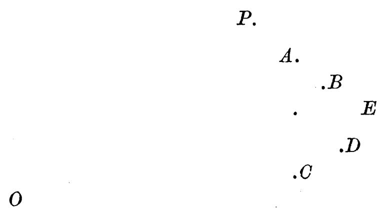
$$\begin{aligned} 3y - \ \ z - 4t &= 2m - \ \ n, \\ 3x - 2z + \ \ t &= \ \ m + \ \ n, \\ -x + 2y - 3t &= \ \ m - \ \ n, \\ 4x + \ \ y - 3z &= \ \ n + 2m, \end{aligned}$$

all of which are heteronymous or indefinite, showing that the system is indefinite.

* Thus we see in like manner that the number of homonymous principal derivatives to a definite quaternary system of n variables is some number intermediate to 4 and q (where q is the number of faces in a triangular polyhedron with n summits), that is, $2n - 4$.

This would be difficult to prove by a direct analytical process.

N.B. In any neuter binary system of which O is origin, P the primary and $ABCDE$ the



cluster, all the triangles OPA, OPB, &c., following the same order of rotation will represent determinants of the same sign.

This cannot be the case for definite positive or negative, or for indefinite systems.

Hence the neuter case may be recognised by the determinants, obtained by conjugating *in situ* each coefficient group in succession with the constant group, never changing sign.

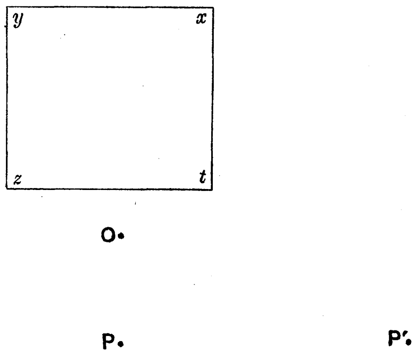
If y, z, x, t were a square, y, t as well as x, z would be brought into line with O; equations would become

$$x - y - z + t = m,$$
$$x + y - z - t = n,$$

and, on account of these two syzygies, there would be only two principal derivatives, viz.:

$$2y - 2t = n - m,$$
$$2x - 2z = n + m.$$

Examples of Definite Systems.*



Positive Case. Take x, y, z, t at the angles of a square, two units each way (breadth and depth).

Let the origin O be at an equal distance from z and t, and from x and y and *the primary P in a line with xt.* The system referred to OP and OQ at right angles to OP as axes of moment gives rise to the equations

$$x - y - z + t = 0,$$
$$(1 + c)\,x + (1 + c)\,y + cz + ct = m,$$

* In order that a system may be definite the points of the cluster, whether in line, plane, or solid, must be all *in front* to an eye at the origin. In the last two cases accordingly, a line or a plane may be drawn through the origin, leaving the cluster entirely on one side. Now, as a line in a plane will cut three out of any four quadrants in the plane made by two intersecting lines, and a plane *in solido* will cut seven out of any eight octants made by three intersecting planes, it follows that a binary system *may* be definite when of the four possible combinations of signs affecting the terms of the several coefficient groups, that is, $\begin{matrix}+ + - -\\ + - + -\end{matrix}$, three are found among the several groups, and so a ternary system may remain definite even when out of the 8 possible combinations of signs

$$\begin{matrix} + + + + & - - - - \\ + + - - & + + - - \\ + - + - & + - + - \end{matrix}$$

all but one are found among the several coefficient groups.

We may then safely infer that, in general, all but one of the possible combinations of sign may occur in the coefficient groups of any system without the system necessarily ceasing to be definite. But, if all possible combinations occur, the system will be necessarily indefinite.

c being the distance of O from zt, and m its distance from P. The extreme rays being Oz and Ot, the two homonymous principal derivatives will be the two resultants in respect to z and t, that is,

$$(1+2c)\,x+y+2ct=m,$$
$$x+(1+2c)\,y+2cz=m,$$

both of which are congruous.

Negative Case. Figure the same as the preceding, but *position of P reversed* (that is, *passed through origin to an equal distance from it on the other side*). The equations will be as above, with the exception of m becoming negative, so that the two homonyms will be incongruous.

Neuter Case. Same figure as above, but *the primary P moved horizontally through d to P′ lying to the right of zO produced.* This condition implies that $\frac{m}{d}<c$ or $m<cd$.

The two equations now become

$$x-y-z+t=d,$$
$$(1+c)\,x+(1+c)\,y+cz+ct=m,$$

and the two homonyms are

$$(1+2c)\,x+y+2ct=m+cd,$$
$$x+(1+2c)\,y+2cz=-(cd-m),$$

of which the first is congruous, the second incongruous, thereby indicating that the system is neuter.

The determinants

$$\begin{vmatrix} 1 & d \\ 1+c & m \end{vmatrix} \quad \begin{vmatrix} -1 & d \\ 1+c & m \end{vmatrix} \quad \begin{vmatrix} -1 & d \\ c & m \end{vmatrix} \quad \begin{vmatrix} 1 & d \\ c & m \end{vmatrix}$$

that is $\quad m-cd-d, \quad -m-d-cd, \quad -m-cd, \quad m-cd,$

being all negative, would also have served to prove the system to be neuter.

Scholium. If our equation or equation-system be now supposed to be integer equations, we see that the denumerant will be in all cases zero, if the system be negative or neuter. If it be indefinite, the denumerant in general will be infinite (according to the known theory of numbers), but it may be zero, namely, in the case where the coefficients of any equation or derived equation of the given system have a common factor which is not a factor of the constant term.

The plexus of principal derivatives affords an absolute criterion for determining whether the denumerant of a given indefinite system of equations is infinite or zero.

SECOND LECTURE*.

Definition of denumerant recalled,

$$\mathfrak{Q}\,U, \quad \mathfrak{Q}\,(U,\ V), \quad \mathfrak{Q}\,(U,\ V,\ W),$$

used as implicit symbols of denumeration.

$$\mathfrak{Q}\,U \text{ in its explicit form } \frac{n\,;}{a,\ b,\ c,\ \ldots\ l}\,;$$

$$\mathfrak{Q}\,(U,\ V) \qquad ,, \qquad ,, \qquad \frac{n,\ n'\,;}{a,\ a'\,;\ b,\ b'\,;\ c,\ c'\,;\ \ldots\,;\ l,\ l'\,;}$$

&c. &c.

Numeratives and denominatives defined.

Herschel's symbol r_n explained.

Its value as a linear function of nth powers of the rth roots of unity.

$$r_n \text{ in the new theory will be replaced by } \frac{n\,;}{r\,;}$$

Observation. $$\frac{n\,;}{r\,;} + \frac{n-1\,;}{r\,;} + \ldots \frac{n-(r-1)\,;}{r\,;} = 1.$$

More generally,

$$\frac{n\,;}{r'r\,;} + \frac{n-r\,;}{r'r\,;} + \frac{n-2r\,;}{r'r\,;} + \ldots \frac{n-(r'-1)\,r\,;}{r'r\,;} = \frac{n\,;}{r\,;}$$

for, if $\frac{n}{r}$ is fractional, so is $\frac{n}{r}-1,\ \frac{n}{r}-2,\ \ldots,$

and *a fortiori*,

$$\frac{1}{r'}\cdot\frac{n}{r},\ \ \frac{1}{r'}\left(\frac{n}{r}-1\right),\ \ \frac{1}{r'}\left(\frac{n}{r}-2\right),\ \ldots\ \frac{1}{r'}\left(\frac{n}{r}-r'-1\right),$$

and, if $\frac{n}{r}$ is integer, one and only one of the above quantities will be an integer.

What $E\left(\frac{n}{r}\right)$ is commonly used to denote; Herschel's notation $\underset{r}{\overset{n}{=}}$.

$$E\left(\frac{n}{r}\right) = \frac{(n-r)\,;}{1,\ r\,;}$$

Examples.

$$x+2y=7, \quad E\left(\frac{7}{2}\right) = 3,$$

$$x+3y=8, \quad E\left(\frac{8}{3}\right) = 2,$$

$$x+5y=15, \quad E\left(\frac{15}{5}\right) = 3.$$

* Delivered at King's College, London, on the 9th June, 1859.

N.B. In the partition theory, zero always counts as a *positive* integer*.

Of course the residue of n to modulus r is

$$n - r \times \frac{(n-r);}{1, r;};$$

but it may also be expressed as a binary denumerant.

Simplest class of indeterminate equations

$$x_1 + x_2 + \ldots + x_r - n = 0,$$

$$\frac{n}{1} = 1, \quad \frac{n}{1, 1} = n + 1, \quad \frac{n}{1, 1, 1} = \frac{(n+1)(n+2)}{2}, \text{ \&c.}$$

Generally $\mathfrak{A}\,U$ in above equation is coefficient of t^n in $\dfrac{1}{(1-t)^r}$.

The denumerant of the equation

$$ax_1 + ax_2 + \ldots + ax_r - n = 0,$$

$$\frac{n;}{a;} \times \left\{\frac{n+a}{a} \cdot \frac{n+2a}{2a} \cdots \frac{n+(r-1)a}{(r-1)a}\right\}.$$

Provisional Method of Simple Denumeration.

Any simple denumerant may be expressed in terms of denumerants of the class last treated of.

Example 1. $x + 2y = n;$

x must be of the form 2ξ or $2\xi + 1$, two suppositions mutually exclusive.

Hence the denumerant of the given equation is the sum of those of the two equations,

$$2\xi + 2y = n \quad \text{and} \quad 2\xi + 2y = n - 1.$$

Hence
$$\frac{n;}{1, 2;} = \frac{n;}{2, 2;} + \frac{n-1;}{2, 2;}$$

$$= \frac{n;}{2;} \times \frac{n+2}{2} + \frac{n-1;}{2;} \times \frac{n+1}{2}.$$

But
$$\frac{n;}{2;} + \frac{n-1;}{2;} = 1.$$

Hence
$$\frac{n;}{1, 2;} = \frac{2n+3}{4} + \left\{\frac{n;}{2;} - \frac{(n-1);}{2;}\right\}\frac{1}{4}.$$

Observe that
$$\left\{\frac{n;}{2;} - \frac{(n-1);}{2;}\right\}\frac{1}{4} = (-)^n \frac{1}{4}.$$

* Consequently, the equation $ax + by + cz \ldots = 0$ has the denumerant *unity*, and is not neuter; there being in fact no neuter cases for *simple* partition.

Example 2. $x+3y=n$;

x must be of the form 3ξ, or $3\xi+1$, or $3\xi+2$.

Hence
$$\frac{n;}{1,3;}=\frac{n;}{3,3;}+\frac{(n-1);}{3,3;}+\frac{(n-2);}{3,3;}$$
$$=\frac{n;}{3;}\cdot\frac{n+3}{3}+\frac{(n-1);}{3;}\cdot\frac{n+2}{3}+\frac{(n-2);}{3;}\cdot\frac{n+1}{3}$$
$$=\frac{1}{3}\left[(n+2)+\left\{\frac{n;}{3;}-\frac{(n-2);}{3;}\right\}\right].$$

Example 3. To find the denumerant of
$$x+2y+4z=n.$$

4 is the least common multiple of 1, 2, 4.

x is either 4ξ, $4\xi+1$, $4\xi+2$, or $4\xi+3$,

$2y$ is either 4η, or $4\eta+2$,

$4z$ is $4z$.

Thus there are eight cases, each giving rise to an equation of the form
$$4\xi+4\eta+4z+c=n.$$

I combine together those in which the constant on the left side is either the same, or leaves the same residue when divided by 4.

Thus, we obtain
$$\frac{n;}{1,2,4;}=\frac{n;}{4,4,4;}+\frac{n-4;}{4,4,4;}$$
$$+\frac{n-1;}{4,4,4;}+\frac{n-5;}{4,4,4;}$$
$$+2\frac{n-2;}{4,4,4;}+2\frac{n-3;}{4,4,4;}$$
and observing that
$$\frac{n-4;}{4;}=\frac{n;}{4;},\quad \frac{n-5;}{4;}=\frac{n-1;}{4;}$$
we obtain
$$\frac{n;}{1,2,4;}=\frac{n;}{4;}\left\{\frac{(n+4)(n+8)}{4.8}+\frac{n(n+4)}{4.8}\right\}$$
$$+\frac{n-1;}{4;}\cdot\left\{\frac{(n+3)(n+7)}{4.8}+\frac{(n-1)(n+3)}{4.8}\right\}$$
$$+\frac{n-2;}{4;}\cdot\frac{(n+2)(n+6)}{4.4}$$
$$+\frac{n-3;}{4;}\cdot\frac{(n+1)(n+5)}{4.4}$$
$$=\frac{1}{16}\left\{\frac{n;}{4;}(n^2+8n+16)+\frac{n-1;}{4;}(n^2+6n+9)+\frac{n-2;}{4;}(n^2+8n+12)\right.$$
$$\left.+\frac{n-3;}{4;}(n^2+6n+5)\right\}.$$

By aid of the identities,

$$\frac{n;}{4;} + \frac{n-1;}{4;} + \frac{n-2;}{4;} + \frac{n-3;}{4;} = 1,$$

$$\frac{n;}{4;} + \frac{n-2;}{4;} = \frac{n;}{2;}, \quad \frac{n-1;}{4;} + \frac{n-3;}{4;} = \frac{n-1;}{2;}$$

the above equation becomes

$$\frac{n;}{1,\,2,\,4;} = F + \left(\frac{n;}{2;} - \frac{n-1;}{2;}\right) G + \left(\frac{n;}{4;} + \frac{n-1;}{4;} - \frac{n-2;}{4;} - \frac{n-3;}{4;}\right) H,$$

where $$F = \frac{1}{16}\left(n^2 + 7n + \frac{21}{2}\right); \quad G = \frac{2n+7}{32}, \quad H = \frac{1}{8}.$$

F will be the mean value of the transcendental function $\dfrac{n;}{1,\,2,\,4;}$.

The mean value of any simple denumerant, by virtue of the theorem discovered by the lecturer, is always expressible directly as an algebraical function of n, and of the quantities $a_1, a_2, \ldots a_r$ left perfectly indefinite.

Observe that in the multipliers of G and H the sums of the coefficients are all zero.

General direct method of expressing every simple denumerant under a simple form is furnished by theorem above referred to.

The method above given substantially consists in making the denumeration of $a_1x_1 + a_2x_2 \ldots + a_rx_r = n$ depend on finding all the solutions of the congruence

$$a_1u_1 + a_2u_2 + a_3u_3 \ldots + a_ru_r - u_{r+1} = 0 \text{ to modulus } K,$$

K being the least common multiple of $a_1, a_2, \ldots a_r$, and $u_1, u_2, \ldots u_r$ being all limited to be positive integers less than K, but u_{r+1} being left indefinite.

Thus the numbering of the solutions in positive integers of an equation can be brought to depend upon finding the solutions themselves of a congruence in positive integers.

Euler's Method of Generating Fractions.

The denumerant of

$$ax + by + cz + \ldots + lt = n$$

is the coefficient of t^n in the expansion of

$$\frac{1}{(1-t^a)(1-t^b)\ldots(1-t^l)}$$

expanded in ascending powers of t.

Proof that the product of the series generated by $\frac{1}{1-t^a} \cdot \frac{1}{1-t^b}$, &c., gives $\frac{n;}{a, b, c, \ldots l;}$ as the coefficient of t^n.

Note that when $n=0$ the coefficient of t^n is 1.

Thus we see that the denumerant of $x_1+x_2+\ldots+x_r=n$ is the coefficient of t^n in $\frac{1}{(1-t)^r}$ as already found.

Necessity of attending to the *order* of terms in the denominators of generating fractions; $\frac{1}{p-q}$ and $\frac{1}{-q+p}$ distinguished; $\frac{1}{p \sim q}$ may be used to signify one or the other of the two previous forms, the choice being left subject to ulterior determination.

Euler's generating fraction continues to hold good even when any of the coefficients become negative, the expansion becoming *indefinite*.

Example. The denumerant of $x-y=n$ is generated by the product of $\frac{1}{1-t}$ by that of $\frac{1}{1-t^{-1}}$, that is to say, of the series

$$1+t \quad +t^2 \quad +t^3 \quad +\ldots \textit{ ad inf.},$$

by the series

$$1+t^{-1}+t^{-2}+t^{-3}+\ldots \textit{ ad inf.}$$

This product will consist of an ascending and descending branch, and the coefficients of every term in each branch will be infinite, showing that the denumerant of $x-y=n$ is infinite for all integer values of n whether positive or negative.

The cognate forms to a generating fraction defined.

Their number, if there are r factors in the denominator, is 2^r.

In above example $\frac{1}{(1-t)(1-t^{-1})}$ generates a double indefinite development, but the cognate form $\frac{1}{(1-t)(-t^{-1}+1)}$ will generate a series in which the indices of t ascend from 1 to ∞, and the coefficients for any finite value of an index remain finite.

So in general for $\frac{1}{(1-t^a)(1-t^{-b})}$; the coefficient of t^n in a cognate form to this is the coefficient of t^{n-b} in $\frac{1}{(1-t^a)(1-t^b)}$ with the sign changed. So again the coefficient of t^n in a cognate form to

$$\frac{1}{(1-x^a)\ldots(1-x^b)(1-x^{-c})(1-x^{-d})}$$

will be the coefficient of t^{n-c-d} in

$$\frac{1}{(1-x^a)\ldots(1-x^b)(1-x^c)(1-x^d)},$$

and so on.

Every generating fraction to a single equation contains two cognate forms (of which itself may be one), which admit of development in series with *finite* coefficients. One of these will be purely an ascending, the other purely a descending, series.

The coefficient of t^n in the ascending development I call the connumerant of the equation.

The connumerant is always finite; it may be positive or negative; when the coefficients are all positive the connumerant and denumerant are identical.

The meaning of the symbol $\dfrac{n;}{a_1,\, a_2,\, \ldots\, a_r;}$ extended and modified.

Rule for transforming a connumerant with some or all of its denominatives negative into one with all its denominatives positive.

Why connumerants are necessary.

When the numerative is a negative quantity the connumerant by virtue of the definition is always zero.

The denumerant of a binary system of equations

$$ax + by + cz + \ldots = m,$$
$$a'x + b'y + c'z + \ldots = m',$$

is the coefficient of $t^m t^{m'}$ in

$$\frac{1}{(1-t^a\,.\,t'^{a'})(1-t^b\,.\,t'^{b'})(1-t^c\,.\,t'^{c'})\ldots}.$$

Unnecessariness of the limitation imposed by Euler upon the signs of the coefficients.

How to exhibit geometrically, the limiting ratios to the values of the indices of t and t' which can appear in the development of the Eulerian fraction containing t and t'.

Hence we see that the series generated by such an Eulerian may consist of a single branch, or of two branches, or of three branches.

So the Eulerian of a definite ternary system developed may have any number of branches from one to seven inclusive.

Definition. A determinate series is one in which none of the coefficients of terms at a finite distance from the origin become infinite in value. A determinate generating function is one which generates a determinate series.

Then $\dfrac{1}{(1-t)^2}$ is determinate, but $\dfrac{1}{(1-t)(t-1)}$ indeterminate.

So, again, $\dfrac{1}{(1-t)(u-1)(ut-1)}$ is determinate, but

$$\frac{1}{(1-t)(u-1)(t-u)} \text{ indeterminate.}$$

Denumerative function distinguished from denumerant.

Reversal of a point in a cluster defined.

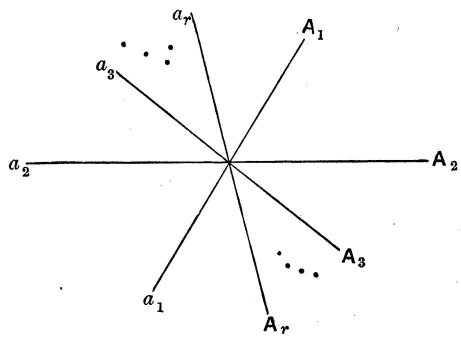
If we change any factor of an Eulerian of the second order

$$\frac{1}{1-t^a\,.\,t'^{a'}} \text{ into } \frac{1}{-t^a\,.\,t'^{a'}+1},$$

the equation-system by the denumeration of which the coefficient of $t^m t'^{m'}$ may be calculated undergoes a change in its constant terms as well as in its coefficients; but it is only the change in the latter which can influence the character of the system as to being definite or indefinite, and consequently the character of the coefficients of the developed Eulerian as to being finite or infinite.

Hence, it is easy to show geometrically that $2r$ out of the 2^r cognate forms to an Eulerian fraction of the 2nd order will give rise to series with finite coefficients.

For if there be r variables the total number of cognate forms will correspond to the 2^r clusters consisting of A_1 or a_1 (its reverse), combined with A_2 or a_2 its reverse, with A_3 or a_3 its reverse, and so on.



Now of all these clusters the only ones which do not enclose the origin are the pairs

$$\left.\begin{matrix} A_1A_2A_3 \ldots\ldots A_r \\ a_1A_r \ldots\ldots A_3A_2 \end{matrix}\right\},$$

$$\left.\begin{matrix} A_2A_3 \ldots\ldots A_ra_1 \\ a_2a_1A_r \ldots\ldots A_3 \end{matrix}\right\},$$

and so on, there being as many pairs of clusters outside the origin as there are points $A_1, A_2, \ldots A_r$.

In like manner an Eulerian fraction of the 3rd order and with r factors in its denominator will admit of as many cognate pairs of forms generating series with finite coefficients as there are combinations of r elements, 2 and 2 together, that is, $\frac{r \cdot r-1}{2}$ pairs, and so on, for any order whatever.

Hence it would not be possible without further specification to extend the definition of connumerants (if it were wished to do so) from simple equations to equation-systems.

Happily the necessity for the consideration of such does not arise, as it will be shown that denumerants of all orders may be expressed in terms of *simple* connumerants.

By the connumerant to

$$-ax - by - cz + dt + eu + \&c. = K,$$

I shall understand the expression

$$\frac{K;}{-a, -b, -c, d, e, \ldots;}.$$

This connumerant will be the same save as to sign (which is or is not to be changed, according as the number of negative coefficients $-a, -b, -c$ is odd or even) as the denumerant of

$$a(x+1) + b(y+1) + c(z+1) + dt + eu + \&c. = K.$$

THIRD LECTURE*.

Reduction.

Reduction explained.

Reduction in partitions analogous to elimination in equations.

A prime group defined. Examples.

Syzygy of variables; predicable also (elliptically) of groups.

In a plane cluster, syzygy is evinced by two or more points being in a line with the origin.

In a solid cluster, by three or more points being in the same plane with the origin.

* Delivered at King's College, London, on June 16th, 1859.

Analytical condition of two groups in a binary system being in syzygy is that the determinant formed by their coefficients vanishes.

Analytical condition of three variables in a ternary system being in syzygy is same as above; and so in general.

If $ab' - a'b = 0$,

$$\frac{a}{b} = \frac{a'}{b'};$$

and, if a, b is a prime group, and also a', b', either

$$\begin{matrix} a = a' \\ b = b' \end{matrix} \text{ or } \begin{matrix} a = -a' \\ b = -b'. \end{matrix}$$

On the latter supposition, the system would be indefinite (for the origin would either lie *on* the contour of the cluster or *within* it).

Hence two non-identical prime groups cannot be in syzygy.

The same will be true of three non-identical prime groups in a ternary system.

If, in a definite binary system, each of a certain set of groups is a prime group, and no two of the groups the same, the system will be asyzygetic so far as this set of groups or their variables is concerned.

Importance of the case of equal, that is, identical, coefficients or coefficient groups.

The symmetric functions of the roots of indeterminate equations may be expressed as denumerants to equations or equation-systems with equal coefficients or coefficient groups.

Scheme: its definition as collective name for cluster and primary.

Scheme: linear, plane, or solid.

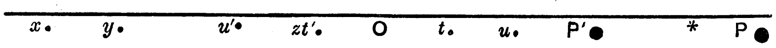


Fig. 1.

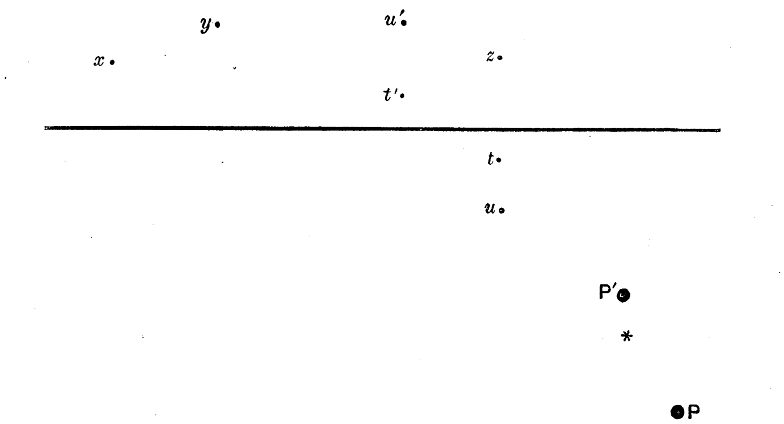


Fig. 2.

Centre: Axis: Balancing plane of scheme: denumerant of a linear scheme in respect to a given centre; of a plane scheme in respect to a given centre or axis; of a solid scheme in respect to a centre, axis, or plane.

Connumerant of a linear scheme in respect to a given centre; of a plane scheme in respect to a given axis*; of a solid scheme in respect to a given plane.

Notice the algebraical sign of the *connumerant*, which is positive or negative, according as an even number (including *zero* as one) or an odd number of transpositions of cluster-points is *transposed.*

Definitions of rays and planes of cluster recalled and applied to schemes.

The term beam substituted for ray of the primary.

The theory of the reduction of binary systems of equations may be geometrically stated.

Network. In line, plane, or solid.

Nodes and nodal lines.

Prime *point* or prime *ray* in network corresponds to prime groups of coefficients.

Prime *couples* or prime *planes* in network correspond to prime double-groups of coefficients, meaning a pair of groups whose minor determinants form a prime group.

Anticipatory statement, namely—

The denumerant of a plane scheme in respect to a given centre is the sum of its connumerants with respect to each in succession of the rays which lie on either side (chosen at will) of the beam, provided that all the rays on the side so chosen are prime rays, and the points to which they are drawn are no two of them coincident. If these conditions are satisfied on both sides of the beam, each of the segments of the ray-cluster into which it is divided by the beam will give a distinct solution, and the two sums of connumerants appertaining respectively to the rays in either cluster will be equal to one another.

Observe that, if the system be neuter, all the rays will be on one side of the beam, and there will be but one solution.

The denumerant of a solid scheme in respect to a given centre (corresponding to a ternary system of equations), which satisfies analogous conditions to the preceding, will be shown later on in the course to be expressible in very similar terms (cluster-planes being substituted for cluster-rays), with this remarkable difference, however, that, in lieu of a single dichotomous

* If the motion of P should carry it to P' on the opposite side of the axis, the transformed centre and primary will be brought to lie on one side of the axis, and consequently the cluster must have contrary signs to the primary, in order to balance about the axis; and, as there will thus be no omni-positive solution, the connumerant in that case will become zero. If P is sufficiently remote, this cannot take place.

division of the planes, there will be a considerable number of such, each of which will or may furnish a distinct pair of solutions.

The formation of these dichotomies involves the consideration of the doctrine of *normal orders*, or orders of *perspective sequence*—a branch of the doctrine of free geometry to which allusion was made in the opening address.

The problem of partitions stated as a problem in plane or solid network.

A system of equations in $x, y, z, \dots u$ may be denoted by $S(x, y, z, \dots u)$, or, when more convenient, by S alone, with implied reference to $x, y, z, \dots u$.

Resultants of systems. $R_x S$, where S is binary, defined. $R_{x,y} S$, where S is ternary, defined.

$R_x S$ is the equation which expresses that the coefficient cluster and primary of S balance about the axis Ox. This will remain good for a ternary system, so that $R_x S$ will then denote a specific binary system, that which corresponds to *projection* of cluster and primary on a plane through the origin perpendicular to O_x. $R_{x,y} S$ will denote that the centre of gravity of the cluster and primary of S is in the plane xy.

Interpretation of $\mathfrak{C} R_x S$ when S is binary. Interpretation of the same when S is ternary.

$\mathfrak{C} R_x S$ in the latter case is perfectly definite just as much as in the former, although the modes of expressing $R_x S$ are infinitely varied.

If S' is what S becomes when we write in S, $fx + g$, or more generally ϕx, in place of x we may denote S' symbolically by $\dfrac{\phi x}{\overline{x}} S$.

Note that $$R_x S = R_x \left(\frac{\phi x}{\overline{x}} S\right)$$

$$\left(\frac{\phi x}{\overline{x}} \cdot \frac{\phi y}{\overline{y}} \cdot \frac{\phi z}{\overline{z}} \dots S\right) \text{ explained.}$$

Order of operative symbols $\dfrac{\phi x}{\overline{x}}$, $\dfrac{\phi y}{\overline{y}}$, &c. is indifferent. The denumerant of any principal derivative $R_x S$, if homogeneous, will furnish a superior limit to the denumerant of S; for all the solutions of F must be solutions of $R_x S$.

Hence, to the denumerant of a definite binary system we can always, by *simple* denumeration, obtain two superior limits; to the denumerant of a ternary system, some number of superior limits, between 3 and n inclusive, such number depending upon the morphological character of the system (as will hereafter be explained).

Examples of superior limits to binary denumerants.

Examples of superior limits to ternary denumerants:—

1. By means of principal *simple* derivatives.
2. By means of principal derivative *binary systems*.

Hereafter we shall find that when the coefficient groups are all prime groups, and none of them alike, these two limits are the respective first terms of two distinct finite series of connumerants, each of which expresses the value of the denumerant of the given binary systems.

Lemma. If the x group in any system is a prime group, any omni-positive integer solution of $R_x S$ is in general an omni-positive integer solution either of S or of $\overset{-x}{\underset{x}{=}} S$.

Proof in case of binary system.

Proof in case of ternary or ultra-ternary system.

How an exception arises when the solution of $R_x S$, substituted in S, makes $x = 0$.

Were it not for this exception, the equation following would always subsist for any variable x corresponding to a prime group, namely,

$$\alpha S + \alpha \left(\frac{-x}{x} . S \right) = \alpha R S.$$

The number of omni-positive solutions of system $S(x, y, z, \ldots v)$, subject to the condition $x > k$, is the denumerant of

$$S\{(x+k), y, z, \ldots v\}.$$

Thus, if $x > 0$, for x we must substitute $1 + x$. Hence the true equation which connects the denumerants referred to *without* exception is

$$\alpha S + \alpha \overset{-1-x}{\underset{x}{=}} (S) = \alpha R_x S;$$

or, if we please,

$$= \alpha R_x \overset{-1-x}{\underset{x}{=}} S.$$

In future I shall denote

$$-x-1 \text{ by } \bar{x},$$
$$-y-1 \text{ by } \bar{y},$$

and so on.

We may therefore write

$$\alpha S = \alpha R_x \overset{\bar{x}}{\underset{x}{=}} S - \alpha \left(\overset{\bar{x}}{\underset{x}{=}} . S \right).$$

Now let the y group be also a prime group; we shall have

$$D \overset{\bar{x}}{\underset{x}{=}} S = S \alpha R_y \left(\overset{\bar{x}}{\underset{x}{=}} \overset{\bar{y}}{\underset{y}{=}} S \right) - \alpha \left(\overset{\bar{x}}{\underset{x}{=}} \overset{\bar{y}}{\underset{y}{=}} S \right);$$

therefore
$$\alpha S = \alpha R_x \overset{\bar{x}}{\underset{x}{=}} S - \alpha R_y \left(\overset{\bar{x}}{\underset{x}{=}} \overset{\bar{y}}{\underset{y}{=}} S \right) + \alpha \left(\overset{\bar{x}}{\underset{x}{=}} \overset{\bar{y}}{\underset{y}{=}} S \right);$$

and so, if the z group be a prime group,

$$\mathcal{Q}S=\mathcal{Q}R_x\underset{x}{\overset{\bar{x}}{=}}S-\mathcal{Q}R_y\left(\underset{x}{\overset{\bar{x}}{=}}\underset{y}{\overset{\bar{y}}{=}}S\right)+\mathcal{Q}\left(R_z\frac{\bar{x}}{x}\frac{\bar{y}}{y}\frac{\bar{z}}{z}S\right)-\mathcal{Q}\left(\frac{\bar{x}}{x}\frac{\bar{y}}{y}\frac{\bar{z}}{z}S\right),$$

and so on to any extent.

Extensions of this Equation to Systems of Equations of a Higher Order than the first indicated.

This I call the process of eduction. The question above indicated is always *true*, amounting in fact to the assertion of identity as regards the *solutions* themselves (not merely their number) of the systems on one side of the equation and those on the other. But, although true, it will be *nugatory* if any of the systems become indefinite, for then in general their denumerants will be infinite in magnitude.

The above equation applies to systems of any order. Its application will be first studied in respect to binary systems.

By continuing the process of eduction through a sufficient number of steps, we shall find that the equation $R_t\left(\underset{x}{\overset{\bar{x}}{=}}\underset{y}{\overset{\bar{y}}{=}}\ldots\underset{z}{\overset{\bar{z}}{=}}\underset{t}{\overset{\bar{t}}{=}}\right)S$ will become at length *incongruous*. Its denumerant will then vanish.

When this is the case, *a fortiori*, the denumerant of the system $\underset{x}{\overset{\bar{x}}{=}}\underset{y}{\overset{\bar{y}}{=}}\ldots\underset{z}{\overset{\bar{z}}{=}}$ will vanish. And thus the series is brought to a close, and the denumerant of S expressed entirely in terms of *simple* denumerants.

FOURTH LECTURE*.

THEORY OF EDUCTION (*continued*).

Process of eduction exemplified. Suppose the system $S(x, y, z, t)$; then

$$\mathcal{Q}S=\mathcal{Q}R_xS-\mathcal{Q}\underset{x}{\overset{\bar{x}}{=}}R_yS+\mathcal{Q}\underset{y}{\overset{\bar{y}}{=}}.\underset{x}{\overset{\bar{x}}{=}}R_zS-\mathcal{Q}\underset{z}{\overset{\bar{z}}{=}}.\underset{y}{\overset{\bar{y}}{=}}.\underset{x}{\overset{\bar{x}}{=}}R_tS+\mathcal{Q}\left(\underset{t}{\overset{\bar{t}}{=}}.\underset{z}{\overset{\bar{z}}{=}}.\underset{y}{\overset{\bar{y}}{=}}.\underset{x}{\overset{\bar{x}}{=}}\right)S;$$

but, if S is definite positive, $\underset{t}{\overset{\bar{t}}{=}}.\underset{z}{\overset{\bar{z}}{=}}.\underset{y}{\overset{\bar{y}}{=}}.\underset{x}{\overset{\bar{x}}{=}}S$ is definite negative. Hence its denumerant is zero, and

$$\mathcal{Q}S=\mathcal{Q}R_xS-\mathcal{Q}\underset{x}{\overset{\bar{x}}{=}}R_yS+\mathcal{Q}\underset{y}{\overset{\bar{y}}{=}}\underset{x}{\overset{\bar{x}}{=}}R_zS-\mathcal{Q}\underset{z}{\overset{\bar{z}}{=}}\underset{y}{\overset{\bar{y}}{=}}\underset{x}{\overset{\bar{x}}{=}}R_tS.$$

The same equation will subsist if S be definite neuter, but not if S be definite negative or indefinite.

* Delivered at King's College, London, on June 20th, 1859.

It is not necessary in general that *all* the coefficient groups should be prime groups, or *all* of them distinct from one another. Great importance of this observation.

Depression of order of denumerants by one degree.

Depression by several degrees :—(1) By successive eductions. (2) By one compound eduction.

Observe that successive eduction can only finally conduct to equations which are simple resultants of the original system, being resultants of its resultants.

Allusion to fundamental theorem for depression by two degrees, namely—

$$\mathfrak{A} R_{x.y}\,.\,S = \mathfrak{A} S + \mathfrak{A}\,.\underset{x}{\overset{\overline{x}}{=}}S + \mathfrak{A}\,.\underset{y}{\overset{\overline{y}}{=}}S + \mathfrak{A}\,.\underset{x}{\overset{\overline{x}}{=}}\,.\underset{y}{\overset{\overline{y}}{=}}S.$$

This equation is subject to the condition that the minor determinants of the matrix formed by the x and y coefficient groups conjoined shall form a prime group.

Observe the singular symbolical equations—

$$R_x = -\underset{x}{\overset{1}{=}}\,, \quad R_{x,y} = R_x\,.\,R_y = \underset{x}{\overset{1}{=}} \times \underset{y}{\overset{1}{=}}\,.$$

Notice that the lemma at p. [139] is true for systems of any order.

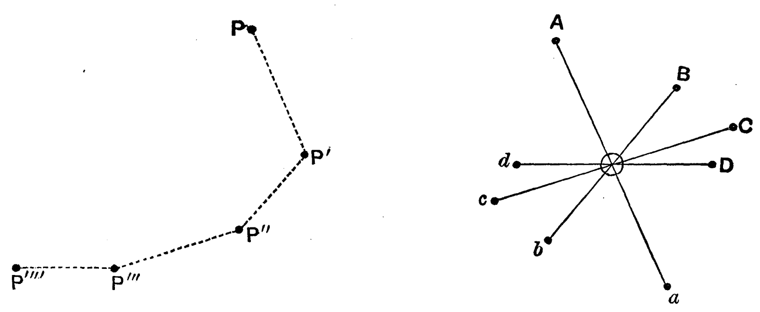
Problem of normal sequences stated. Its geometrical solution by means of the perspective to the cluster, (if a plane cluster) upon a line, (if a solid cluster) upon a plane, or (if a hypersolid cluster) upon a space.

Binary Systems.

Geometrical representation of the successive systems

$$S\,.\underset{x}{\overset{\overline{x}}{=}}S, \quad \underset{y}{\overset{\overline{y}}{=}}\underset{x}{\overset{\overline{x}}{=}}S, \text{ \&c.}$$

in which S is supposed to be definite and positive. (See figure.)



Reversal of successive points in cluster, accompanied with parallel motion of primary.

The systems $\frac{\overline{x}}{x}.S$, $\frac{\overline{y}}{y}\frac{\overline{x}}{x}.S$, &c. may be regarded as successive deformations of S, and then we may say the system, as it undergoes deformation, tends more and more to lose its positive character, until at length it becomes *neutral*, and immediately after changes into and continues *negative*.

The deformation may be commenced from either side. There are thus two courses of deformation.

If the primary were stationary, the deformed system in either course would become neuter after as many deformations as there are rays in S outside the beam, on that side of it from which the deformation proceeds.

The effect of the motion of the primary is to *accelerate* the tendency of the deformed system to become neuter.

If the primary be moved along the beam to a sufficient distance from the origin, the effect of such tendency will become at length insensible for such and for all greater distances.

The number of deformations in either course which may take place before the primary ray becomes denuded gives the number of terms in the development (by the eduction process) corresponding to that course.

When the primary is sufficiently remote, the sum of these numbers corresponding to the two courses of deformation will be the number of rays, that is, the number of variables in the system.

On account of the motion of the primary, the united sum may be *less* than this number; it cannot be greater.

Corollary. If all the groups are prime groups, the denumerant of a binary system may be expressed by a number of simple denumerants, *not greater* than half the number of variables in the system.

The question of partitions, given in number and species, is expressed by a binary system satisfying these conditions.

Example. To find the number of ways in which n can be made up of r values each limited to be either 1, 2, 3, or 4. This requires the denumeration of the system

$$x + 2y + 3z + 4t = n,$$
$$x + y + z + t = r.$$

The *skew-matrix* of elimination becomes

$$\begin{array}{cccc|l}
x & y & z & t & \\
\hline
0 & 1 & 2 & 3 & n - r \\
-1 & 0 & 1 & 2 & n - 2r \\
-2 & -1 & 0 & 1 & n - 3r \\
-3 & -2 & -1 & 0 & n - 4r.
\end{array}$$

If $n < r$, the system is neuter, and the number required is 0.

If $n > r$ but $< 2r$, the number required is the connumerant $\dfrac{n-r\,;}{1,\,2,\,3\,;}$ which is the same in value as the sum of the three complementary connumerants.

If $n > 2r < 3r$, the number required is the sum of the connumerants

$$\frac{n-r\,;}{1,\,2,\,3\,;}+\frac{n-2r\,;}{-1,\,1,\,2\,;},\ \text{or, if we please,}\ \frac{4r-n\,;}{1,\,2,\,3\,;}+\frac{3r-n\,;}{-1,\,1,\,2\,;}.$$

If $n > 3r$ but $< 4r$, the number required is $\dfrac{4r-n\,;}{1,\,2,\,3\,;}$, which is the same in value as the sum of the three complementary denumerants.

If $n > 4r$, the system is again neuter, and its denumerant is zero.

By aid of the above expressions we may give the general analytical representation of the number of partitions of the kind proposed.

These expressions, it will be observed, are discontinuous, the particular one to be employed depending on the comparative values of n and r. It may be shown, however, that they, as it were, melt or modulate into one another, so not only at the mere limiting values of n, which separate the several formulæ, but also for a short distance beyond these limits, either of the conterminous expressions may be used indifferently.

The proof of this depends upon the proposition that the coefficient of t^n in the *ascending* expansion of

$$\frac{1}{(1-t^a)(1-t^b)(1-t^c)\ldots(1-t^k)}$$

treated *as a function of* n remains fixed at zero, when n is made to become any negative number whose absolute value is inferior to

$$a+b+c+\ldots+k.$$

The truth of this proposition follows immediately from a theorem of Mr Cayley relating to the development of any rational fraction $\dfrac{1}{\phi t}$ in its two forms. If $\theta(n)$ is the fractional value of the coefficient of t^n in the positive, and $\pi(n)$ the fractional value of the coefficient of t^n in the negative, expansion of $\dfrac{1}{\phi t}$, $\theta(n) \pm \pi(n) \doteq 0$, the sign $+$ or $-$ being employed according as the degree of the rational function πt is an odd or an even number.

If the coefficient groups be arranged in natural order, the transformed systems will throughout remain definite.

The natural order for binary systems is indicated by the order in either direction in which the rays of the cluster succeed each other.

If the coefficients are all positive, this order will correspond with the order in which the variables must succeed each other, so that the ratios in

each group of the coefficient out of one given equation with the corresponding coefficient out of the other may continually increase or decrease.

But, if the coefficients are positive and negative intermingled, the rule is that the determinants formed by the combination of any group with each in succession of those which follow must all bear the same sign; or, as we may express it, the algebraical sign arising from the contact of the groups, with due regard to antecedence, must be always the same.

If the system were indefinite, such a uniformity of signs could not be established by any arrangement of the groups whatever.

Example (1):—

$\left.\begin{array}{l} x+y+z=50 \\ x+2y+3z=120 \end{array}\right\}$. In this system all the groups are prime groups.

First solution.

$$\begin{array}{ccc|c} x & y & z & \\ \hline 0 & 1 & 2 & 70 \\ -1 & 0 & 1 & 20 \\ -2 & -1 & 0 & 30. \end{array}$$

Observe that the coefficients form a skew-matrix.

$$R_x \cdot \frac{\overline{x}}{x} \cdot S \text{ is } y+2z=70.$$

$$R_y \cdot \frac{\overline{y}}{y} \cdot \frac{\overline{x}}{x} \cdot S \text{ is } (x+1)+z=20, \text{ or } x+z=19.$$

$$R_z \cdot \frac{\overline{z}}{z} \cdot \frac{\overline{y}}{y} \cdot \frac{\overline{x}}{x} \cdot S \text{ is } 2(x+1)+(y+1)=-30, \text{ or } 2x+y=-33.$$

The desired denumerant will be $\dfrac{70;}{1,\,2;}-\dfrac{19;}{1,\,1;}$ which is

$$36-20=16.$$

Second solution.

$$\begin{array}{ccc|c} z & y & x & \\ \hline 0 & 1 & 2 & 30 \\ -1 & 0 & 1 & -20 \\ -2 & -1 & 0 & -70. \end{array}$$

$$R_z \cdot \frac{\overline{z}}{z} \cdot S \text{ is } y+2x=30.$$

$$R_y \cdot \frac{\overline{y}}{y} \cdot \frac{\overline{z}}{z} \cdot S \text{ is } z+x=-21.$$

$$R_x \cdot \frac{\overline{x}}{x} \cdot \frac{\overline{y}}{y} \cdot \frac{\overline{z}}{z} \cdot S \text{ is } 2z+y=-73.$$

The desired denumerant will therefore be $\frac{30\,;}{1,\ 2\,;} = 16$, as before.

Observe, that $\mathfrak{C}S$ is the *sum* of the connumerants of R_xS, R_yS, R_zS, when these equations are written in the form in which they appear when the process of successive elimination is conducted by a uniform course of operations. This may be done in two ways, so as to give rise to two sets of equations differing from each other only in the signs. Observe, that the connumerant $L = \pm c$ is not the same as of $-L = \mp c$; of that one of these equations, in which the constant term is negative, the connumerant being zero, but of the other the connumerant being generally different from zero.

Corollary. The coefficient of $t^m\tau^\mu$ in

$$\frac{1}{(1 - t^a\tau^\alpha)(1 - t^b\tau^\beta)(1 - t^c\tau^\gamma)}$$

is the sum of the coefficients of

$$\rho^{a\mu-\alpha m} \text{ in } \frac{1}{(1 \sim \rho^{a\beta-b\alpha})(1 \sim \rho^{a\gamma-c\alpha})},$$

of

$$\rho^{b\mu-\beta m} \text{ in } \frac{1}{(1 \sim \rho^{b\alpha-a\beta})(1 \sim \rho^{b\gamma-c\beta})},$$

and of

$$\rho^{c\mu-\gamma m} \text{ in } \frac{1}{(1 \sim \rho^{c\alpha-a\gamma})(1 \sim \rho^{c\beta-b\gamma})},$$

subject to the interpretation that the preceding fractions are to be expanded all in terms of ascending, or all in terms of descending, powers of ρ, *provided* that the system

$$\left.\begin{aligned} ax + by + cz + dt &= m \\ \alpha x + \beta y + \gamma z + \delta t &= \mu \end{aligned}\right\}$$

is a definite non-negative system.

Allusion to Mr Cayley's proof of the proposition in this form. The proposition is, of course, general, that the denumerant of a definite binary system with r variables, in which the groups are all prime groups, admits of a double mode of representation as the sum of the connumerants of its principal derivatives. In the one mode of representation, only a certain number, at *utmost*, of these connumerants, say p, can differ from zero; in the other mode, only a certain number, at utmost, say q, can differ from zero; and, as we shall have $p + q = r$, these two modes of representation may be termed complementary to each other.

The denumerant of the system

$$\left.\begin{aligned} ax + by + cz + \ldots + dt &= m \\ x + y + z + \ldots + t &= \mu \end{aligned}\right\}$$

may therefore be represented in two modes as the sum of simple denumerants.

This is the system the denumeration of which constitutes the problem of the resolution of integers into parts given in number and species. See Euler's *Second Memoir on Partitions*.

The theorem of eduction may be put under the following form :—

$$\mathfrak{A}S = \mathfrak{A}R_x\frac{\overline{x}}{x}S - \mathfrak{A}R_y\frac{\overline{y}}{y}\frac{\overline{x}}{x}S + \mathfrak{A}R_z\frac{\overline{z}}{z}\frac{\overline{y}}{y}\frac{\overline{x}}{x}S \mp \text{\&c.},$$

provided that S is a definite system in which the groups of coefficients appertaining to the several variables, or at least to so many of them as are included between x and the last of them which appears as a suffix in a non-zero term of the above expression, are all distinct prime groups.

2nd Example:—

$$\left.\begin{aligned} x + y + z &= 10 \\ x + 2y + 3z &= 40 \end{aligned}\right\}.$$

$$\begin{array}{ccc|ccc|ccc} x & y & z & & & & x & y & z \\ \hline 0 & 1 & 2 & 30 & & -10 & 2 & 1 & 0 \\ -1 & 0 & 1 & 20 & & -20 & 1 & 0 & -1 \\ -2 & -1 & 0 & 10 & & -30 & 0 & -1 & -2. \end{array}$$

The solution corresponding to the right-hand matrix is evidently 0, the system in fact being neuter. The left-hand matrix gives the solution—

$$\frac{30;}{1, 2;} + \frac{20;}{-1, 1;} + \frac{10;}{-2, -1;}$$

$$= \frac{30;}{1, 2;} - \frac{19;}{1, 1;} + \frac{7;}{2, 1;}$$

$$= 16 - 20 + 4 = 0, \text{ as before.}$$

Illustrate effect of taking the variables in abnormal order.

3rd Example:— $2x + 5y + 2z - t = m$,

$$x + y - 2z - 2t = m,$$

the system will also be neuter, and we shall have

$$\frac{m;}{6, 9, 3;} + \frac{2m;}{-3, 6, 3;} + \frac{4m;}{-9, -12, 3;} + \frac{m;}{-3, -6, -3;} = 0,$$

or

$$\frac{m;}{3, 6, 9;} - \frac{2m - 3;}{3, 3, 6;} + \frac{4m - 21}{3, 9, 12;} - \frac{m - 12}{3, 3, 6;} = 0.$$

Allusion to importance and fertility of theory of neuter systems.

Example of a denumeration of a binary system containing unprime groups:—

$$x + y + 4z + 3t = m,$$
$$3x + 2y + 6z + 3t = n,$$

$\frac{m}{n}$ being supposed to be intermediate between $\frac{1}{2}$ and $\frac{2}{3}$. The denumerant will be

$$\frac{3m - n;}{1, 6, 6;} - \frac{2m - n;}{-1; 2; 3;}.$$

Cæsura (definition of, and how determined), accidental and universal, distinguished.

FIFTH LECTURE*.

Eduction and Reduction.

The cæsura for equation-systems generally falls after that coefficient group subsequent to the introduction of which, in the eduction process, the depressed systems whose denumerants are to be taken *must* cease to be positive, so that they may be neglected. It is determined for binary systems by the relation of the ratios of the terms in the coefficient groups to that of the terms in the constant group; the determinant formed by the apposition of the constant group with any group on one side of the cæsura being positive, on the other side negative.

The point after which the terms in an eduction process can be neglected *may* (if the constant terms are sufficiently small) be attained before the cæsura is reached. Such a point may be termed a turning-point, or pause. There may thus (in the case of binary systems) be two turning-points or pauses on each side of the cæsura corresponding to the two courses of eduction, but either or both of them may, and in general will, coincide with the cæsura.

For greater simplicity, we may suppose the constant terms given in ratio only, and not in magnitude, so as to obviate the necessity of paying any attention to the accidental pauses as distinguishable from the cæsura. The cases where they are so distinguishable are always exceedingly limited in number. Their existence arises solely from the fact of the introduction of $-x-1$, $-y-1$, &c., and not $-x$, $-y$, &c., in lieu of x, y, in applying the method of eduction.

A *per-reducible* binary system is one in which *all* the coefficient groups are *prime* groups *distinct* from each other.

Being prime and distinct, none of them can be in syzygy. Such a system admits of a double process of eduction, giving rise in general to two distinct forms of solution. But it may happen, in some very special cases, that these two solutions are identical in form as well as in value.

Example. The system

$$\left.\begin{array}{l} x+3y+7z+9t=5i \\ x+2y+4z+5t=3i \end{array}\right\}$$

gives rise to the bordered matrix

$$\begin{array}{cccc|c} 0 & 1 & 3 & 4 & 2i \\ -1 & 0 & 2 & 3 & i \\ -3 & -2 & 0 & 1 & -i \\ -4 & -3 & -1 & 0 & -2i \\ \hline -2i & -i & i & 2i. & \end{array}$$

* Delivered at King's College, London, June, 1859.

Here each solution is the same, namely, $\dfrac{2i;}{1,\,3,\,4;}+\dfrac{i;}{-1,\,2,\,3;}$, meaning

$$\frac{2i;}{1,\,3,\,4;}-\frac{i-1;}{1,\,2,\,3}.$$

But, if the constant terms in the above system were $11i$ and $7i$ respectively, the bordered matrix would be

$$\begin{array}{|cccc|c}
\hline
0 & 1 & 3 & 4 & 4i \\
-1 & 0 & 2 & 3 & i \\
-3 & -2 & 0 & 1 & -5i \\
-4 & -3 & -1 & 0 & -8i \\
\hline
-4i & -i & 5i & 8i &
\end{array}$$

giving rise to the two equal sums of connumerants,

$$\frac{4i;}{1,\,3,\,4;}+\frac{i;}{-1,\,2,\,3;}\ \text{ and }\ \frac{8i;}{1,\,3,\,4;}+\frac{5i;}{-1,\,2,\,3;}.$$

In this example the matrix happens to be *persymmetrical*, which is the reason of the denumeratives being the same in each solution.

This is avoided in the example below of the system

$$\left.\begin{aligned} x+2y+\ z+\ t&=7i\\ x+3y+2z+4t&=12i\end{aligned}\right\},$$

for which the bordered matrix is

$$\begin{array}{cccc|c}
0 & 1 & 1 & 3 & 5i \\
-1 & 0 & 1 & 5 & 3i \\
-1 & -1 & 0 & 2 & -2i \\
-3 & -5 & -2 & 0 & -16i \\
\hline
-5i & -3i & 2i & 16i &
\end{array}$$

giving rise to the two equivalent solutions

$$\frac{5i;}{1,\,1,\,3;}+\frac{3i;}{-1,\,1,\,5;}\ \text{ and }\ \frac{16i;}{2,\,5,\,3;}+\frac{2i;}{-2,\,1,\,1;},$$

meaning
$$\frac{5i;}{1,\,1,\,3;}-\frac{3i-1;}{1,\,1,\,5;}\ \text{ and }\ \frac{16i;}{2,\,3,\,5;}-\frac{2i-2;}{1,\,1,\,2;}.$$

A *simply-reducible* system is one for which the coefficient groups are prime and distinct on *one* side of the cæsura only.

Example:—
$$\left.\begin{aligned} x+2y&=4m\\ x+4y&=5m\end{aligned}\right\}.$$

The eduction from the x side gives rise to the equation of $2y=m$, of which the denumerant is $\dfrac{m;}{2;}$. This is the true solution, whereas the eduction

from the y side gives rise to the denumerant of $2x = 6m$, that is, 1, which is a false solution, owing to the group (2, 4) being a non-prime group.

If in a binary system the groups, which are either non-prime or repeated, or non-prime and repeated, represent ratios (between quantities given in algebraical sign) which are all less or all greater than the corresponding ratio of the constant terms, the system is still depressible by eduction commenced from that side of the system on which the groups of the kind mentioned do not fall.

Corollary. A single non-prime group, or a single sequence of any number of identical groups, can in no case hinder a binary system from being soluble by eduction.

The above remark is true also *à fortiori* for ultra-binary systems.

It should be noticed that $\left.\begin{matrix} a \\ 0 \end{matrix}\right\}$ is a non-prime group unless $a = \pm 1$. (For non-prime we may in future use the term composite.)

Example:—
$$\left.\begin{aligned} 3x + 2y + z + t &= i \\ 2z + 3t &= i \end{aligned}\right\}.$$

Here the coefficient groups of x and y are both of them composite; but, the cæsura falling between y and z, the denumerant required will be the sum of the connumerants of the two resultants in respect to t and z, that is,

$$\frac{2i\,;}{1,\,6,\,9\,;} + \frac{i\,;}{-1,\,4,\,6\,;},$$

meaning

$$\frac{2i\,;}{1,\,6,\,9\,;} - \frac{i-1\,;}{1,\,4,\,6\,;}.$$

If a system is affected with composite or repeated groups on *each* side of the cæsura, its denumeration may be made to depend on systems where such groups exist on only one side of their respective cæsuras*.

Example:—
$$\left.\begin{aligned} 10x + 2y + 3z &= 5i \\ 15x + 4y + 9z &= 11i \end{aligned}\right\}, \text{ which call } S.$$

If we form a ternary system as follows:—

$$\left.\begin{aligned} 10x + 2y + 3z \quad &= 5i \\ 15x + 4y + 9z \quad &= 11i \\ px + qy + rz - t &= -m \end{aligned}\right\}, \text{ which call } S',$$

* In certain special cases the composite groups may be reduced in number by substituting a connective of the equations in lieu of one of them, as in the example

$$\left.\begin{aligned} 10t - 7z - 8y &= 5i \\ 9z + 2y &= 7i \end{aligned}\right\},$$

which is apparently irreducible, but which, put under the equivalent form

$$\left.\begin{aligned} 10t - 7z - 8y &= 5i \\ 15t - 6z - 11y &= 11i \end{aligned}\right\},$$

becomes simply-reducible.

where p, q, r, m are any positive integers whatever, it is apparent that the omni-positive solutions of S' may be found from the omni-positive solutions of S, and to each of the latter will correspond one, and only one, of the former. Hence the denumerant of S is the same as the denumerant of S', and, if p, q, r are so chosen that 10, 15, p; 2, 4, q; 3, 9, r are all prime groups, $\mathfrak{C}S'$, and therefore $\mathfrak{C}S$, may be made to depend on the denumerants of a certain set of new binary systems obtained by the eduction of S'. Thus let $p=1$, $q=1$, $r=1$, $m=0$, so that the auxiliary equation becomes

$$x+y+z-t=0.$$

R_xS' may be represented by

$$\left.\begin{aligned}10t-\ \ 8y-7z&=\ \ 5i\\ 15t-11y-6z&=11i\end{aligned}\right\},$$

R_yS' by
$$\left.\begin{aligned}2t+\ \ 8x+\ \ z&=\ \ 5i\\ 4t+11x+5z&=11i\end{aligned}\right\},$$

R_zS' by
$$\left.\begin{aligned}3t+\ \ 7x-\ \ y&=\ \ 5i\\ 9t+\ \ 6x-5y&=11i\end{aligned}\right\},$$

R_tS' by
$$\left.\begin{aligned}10x+\ \ 2y+3z&=\ \ 5i\\ 15x+\ \ 4y+9z&=11i\end{aligned}\right\},\ \text{being the original system } S.$$

It will be seen therefore that

$$R_xS';\quad \overset{\overline{x}}{\underset{x}{=}}R_yS';\quad \overset{\overline{y}}{\underset{y}{=}}\,.\,\overset{\overline{x}}{\underset{x}{=}}R_zS';\quad \text{and}\quad \overset{\overline{z}}{\underset{z}{=}}\,.\,\overset{\overline{y}}{\underset{y}{=}}\,.\,\overset{\overline{x}}{\underset{x}{=}}R_tS'$$

respectively represent the systems following:—

$$\left.\begin{aligned}10t-\ \ 8y-7z&=\ \ 5i\\ 15t-11y-6z&=11i\end{aligned}\right\},\tag{1}$$

$$\left.\begin{aligned}2t+\ \ z-\ \ 8x&=\ \ 5i+\ \ 8\\ 4t+5z-11x&=11i+11\end{aligned}\right\},\tag{2}$$

$$\left.\begin{aligned}3t+\ \ y-7x&=\ \ 5i+6\\ 9t+5y-6x&=11i+1\end{aligned}\right\},\tag{3}$$

$$\left.\begin{aligned}-10x-2y-3z&=\ \ 5i+15\\ -15x-4y-9z&=11i+28\end{aligned}\right\}.\tag{4}$$

All these four systems are definite: in the first of them the natural order of the groups is

$$\begin{pmatrix}10 & -7 & -\ 8\\ 15 & -6 & -11\end{pmatrix};$$

in the others the natural order is that in which they are written. The first two only will be definite-positive, the last will be definite-negative, and the

last but one neuter if $i > 0$, negative if $i = 0$, and the denumerant required will be the difference between the denumerants of the two systems

$$\left.\begin{aligned} 10p - 7q - \ \ 8r &= \ \ 5i \\ 15p - 6q - 11r &= 11i \end{aligned}\right\},$$

$$\left.\begin{aligned} 2p + \ \ q - \ \ 8r &= \ \ 5i + \ \ 8 \\ 4p + 5q - 11r &= 11i + 11 \end{aligned}\right\}.$$

In each of these systems there is one, but only one, of the original non-prime groups, and no new ones have been introduced.

Consequently they admit of being depressed, and the final result will be an aggregate of simple denumerants.

If we had applied to S' a different course of eduction as follows :—

$$R_z S', \quad \overset{\bar{z}}{\underset{z}{=}} R_y S', \quad \overset{\bar{z}\,\bar{y}}{\underset{z\,y}{=\,=}} R_x S', \quad \overset{\bar{z}\,\bar{y}\,\bar{x}}{\underset{z\,y\,x}{=\,=\,=}} R_t S',$$

it may easily be seen that all these systems likewise would be definite, and only the first of them definite-positive. Hence a second solution of the question will be $\mathfrak{C} R_z S'$, that is,

$$\frac{5i,\ 11i\,;}{3,\ 9\,;\ 7,\ 6\,;\ -1,\ -5\,;},$$

which is of a depressible form, there being only one *affected* group (3, 9), and may be educed into a linear function of simple denumerants.

Dispersion process defined.

Cases which resist its application.

Theorem. The denumeration of any equation-system whatever may be made to depend upon the denumeration of systems that shall contain no composite groups, and at most only one set of repeated groups, and which will consequently be depressible.

Proof of this theorem in case of binary systems.

Definition of meaning of kG, and of $kG \pm lG'$, where G, G' represent any two coefficient groups of a system, and k, l are any two integers*.

Lemma 1. A system S, containing the coefficient group G, may be made to depend for its denumeration upon systems in each of which the coefficient groups are the same as in S, except that kG takes the place of G.

Lemma 2. A system S, containing the groups G and G', may be made to depend upon two systems, in one of which the coefficient groups are the same as in G, with the exception that H replaces G, and in the other the

* In a definite ternary system, where all the coefficient groups are prime groups, it may be shown that the only possible cases of syzygy are where $F = G$ or $F + G = H$ (F, G, H denoting coefficient groups of the system).

same as in G, with the exception that H replaces G', where H is $G - G'$ or $G' - G$.

Note that, if, instead of the coefficient group G in any definite system, first any other group H and then $-H$ be substituted, one *at least* of these substitutions must leave the deformed system definite.

Lemma 3 (*Corollary to Lemma* 1). Any equation-system may be made to depend for its denumeration on equation-systems in each of which one of the equations has all its coefficients positive units.

It follows from this lemma that, if a binary equation-system is free from syzygy (that is, from equalities of ratios between the coefficients of different variables), its denumeration may be made to depend upon that of systems which [their coefficient groups being all different and of the form $(a, 1)$] are per-reducible. But, if there be e sets of syzygies in the given system, there will be e sets of repetitions in the groups $(a, 1)$ in each of the deduced systems.

Lemma 4. In the case immediately above supposed, the e sets of syzygetic groups in the deduced systems may be replaced by e other syzygetic sets of groups of which all but one are of the form $(a, 1)$, $(a, 1)$, ...; $(b, 1)$, $(b, 1)$, $(b, 1)$, ..., &c., and that one of the form $(\sigma, 0)$; $(\sigma, 0)$; $(\sigma, 0)$, ...

Lemma 5. Any system of the form last supposed may (by virtue of Lemma 2) be replaced by two, in one of which $\pm(a - k\sigma, 1)$ takes the place of $(a, 1)$, and in the other $\pm(a - k\sigma, 1)$ takes the place of $(\sigma, 0)$, k being so chosen that $(a - k\sigma, 1)$ is distinct from every other coefficient group associated with it in the same system.

Lemma 6. Hence, by *repeated* application of this last process of replacement, the number of syzygetic groups in the deduced systems may be continually reduced until we arrive at systems in one class of which all the groups $(\sigma, 0)$ have disappeared, and in the other class of which all the syzygetic groups except those of the form $(\sigma, 0)$ have disappeared.

Lemma 7. Hence, so long as e is greater than 1, the deduced systems will eventually none of them contain more than $(e-1)$ sets of groups in syzygy, and thus we must eventually arrive at systems in none of which will be found more than a single set of groups in syzygy, which may be taken indifferently of the form $(a, 1)$ or $(a, 0)$.

Consequently the denumeration of every binary system, if free of syzygies, may be made to depend on the denumeration of per-reducible systems; and, if not free of syzygies, on the denumeration of simply reducible systems.

A similar demonstration may be extended to systems of a higher order than the second. Consequently every denumerant of an order higher than the first may be made to depend on denumerants of a lower order, and eventually upon simple denumerants.

Examples of Reduction of Persyzygetic Systems.

Let the given system S be

$$\left.\begin{array}{r} x+y \qquad\quad = m \\ z+t=n \end{array}\right\},$$

and suppose m not less than n. Making

$$x+z-u=0,$$

we obtain the systems

$$\left.\begin{array}{r} u+y-z \qquad = m \\ z+t=n \end{array}\right\},$$

which is $R_x S'$, say T; and

$$\left.\begin{array}{l} u+t+x \qquad = (n-1) \\ \qquad\quad x+y=m \end{array}\right\},$$

which is $\overset{\bar{x}}{\underset{x}{=}} R_z S'$, say U.

Since $\overset{\bar{x}}{\underset{x}{=}}\overset{\bar{z}}{\underset{z}{=}} S'$ contains the equation

$$-x-z-n=2,$$

the eduction is complete, and the required denumerant $= \mathfrak{a} T - \mathfrak{a} U$.

T arranged in natural order becomes

$$\begin{array}{l} z+t \qquad\qquad = n, \\ -z \qquad +u+y=m, \end{array}$$

the cæsura falling between t and u. Accordingly we obtain

$$\mathfrak{a} T = \frac{n+m\,;}{1,\,1,\,1\,;} + \frac{m\,;}{-1,\,1,\,1\,;} = \frac{n+m\,;}{1,\,1,\,1\,;} - \frac{m-1\,;}{1,\,1,\,1\,;}$$

$$= \frac{(n+m+1)(n+m+2)}{2} - \frac{m\,(m+1)}{2}.$$

In like manner

$$\mathfrak{a} U = \frac{n-1\,;}{1,\,1,\,1\,;} - \frac{n-1-m\,;}{-1,\,1,\,1\,;} = \frac{n-1\,;}{1,\,1,\,1\,;} = \frac{n\,(n+1)}{2},$$

for $n-1-m$ is negative. And we have

$$\mathfrak{a} S = \frac{(n+m+1)\,(n+m+2)}{2} - \frac{m\,(m+1)}{2} - \frac{n\,(n+1)}{2}$$

$$= \frac{2nm+2m+2n+2}{2} = (m+1)\,(n+1),$$

which is evidently the correct answer, being the product of the denumerants of the two given equations taken independently.

Second Example:— $$\left.\begin{array}{l} x+y+\theta \qquad\quad = m \\ \qquad\quad \theta+z+t=n \end{array}\right\}.$$

By the same method as above, we obtain

$$ȸS = ȸT - ȸU,$$

where T is $$\left.\begin{array}{l} z+t+\theta \qquad\qquad\; = n \\ -z \quad\; +\theta+u+y=m \end{array}\right\},$$

and U is $$\left.\begin{array}{l} u+t+\theta+x \qquad = n-1 \\ \qquad\quad \theta+x+y=m \end{array}\right\},$$

the cæsura in T falling between θ and u, and in U between x and y. Hence

$$ȸT = \frac{m+n\,;}{1,\,1,\,1,\,1\,;} - \frac{m-1\,;}{1,\,1,\,1,\,1\,;} + \frac{m-n-3\,;}{2,\,1,\,1,\,1\,;},$$

and $$ȸU = \frac{n-1\,;}{1,\,1,\,1,\,1\,;}.$$

Thus $$ȸS = \frac{m+n\,;}{1,\,1,\,1,\,1\,;} + \frac{m-n-3\,;}{1,\,1,\,1,\,1\,;} - \frac{m-1\,;}{1,\,1,\,1,\,1\,;} - \frac{n-1\,;}{1,\,1,\,1,\,1\,;}\,*.$$

Example of a composite group and a syzygy falling on opposite sides of the cæsura. *Problem*:—To express the residue of q in respect to p as a linear function of simple denumerants.

If we call x the required residue, we have

$$x+py=q, \quad x<p,$$

or

$$x+py=q,$$
$$x+z \;=p-1.$$

Hence the required residue is the denumerant of the system

$$py+t+u \qquad =q-1,$$
$$t+u+z=p-2,$$

in which the coefficient groups are in natural sequence.

In its present form the system is irreducible, because the cæsura falls between y and t (observe that p, 0 is a non-prime group); but, by the method above given, the denumerant of this system, by virtue of the subsidiary equation

$$y+t=v,$$

becomes the difference between the denumerants of

$$\left.\begin{array}{l} (1-p)\,t \qquad +u+pv=q-1 \\ \qquad\quad t+z+u \qquad =p-2 \end{array}\right\}$$

* The value of this expression will evidently be the sum

$$(m+1)\,(n+1)+mn+(m-1)\,(n-1)+\&c.+(m-n+1),$$

which is $$\left(m+1-\frac{n}{3}\right)\frac{(n+1)\,(n+2)}{2}.$$

and of
$$\left.\begin{aligned}(1-p)\,y+v+u+t&=(q-1)+(p-1)=q+p-2\\ y+z+u+t&=(p-2)-1 \qquad\;\;\, =p-3\end{aligned}\right\}.$$

The second system is neuter, for all the coefficient groups put in apposition with the constant group give determinants with negative values.

Hence the required expression is simply the denumerant of the first system, in which the cæsura falls between u and v. $(p, 0)$ being a composite group, the eduction must be commenced from the t side, and accordingly we obtain the series

$$\frac{(q-1)+(p-1)(p-2);}{(p-1),\,p,\,p;}+\frac{(q-1);}{-(p-1),\,1,\,p;}+\frac{q-p+1;}{-p,\,-1,\,p;}$$
$$=\frac{p^2-3p+q+1;}{p-1,\,p,\,p;}-\frac{q-p;}{1,\,p-1,\,p;}+\frac{q-2p;}{1,\,p,\,p;}$$

as the expression required for the residue of q in respect to p.

SIXTH LECTURE*.

Simple Partition.

Resolution of an integer into a defined number of parts.

With or without repetition.

$\dfrac{n-r;}{1,\,2,\,3,\,\ldots\,r;}$ expresses the r-ary partibility of n when repetitions are allowed;

$\dfrac{n-r\dfrac{r+1}{2};}{1,\,2,\,3,\,\ldots\,r;}$ the same when repetitions are excluded.

Example: $n=7$, $r=3$.

Proof of the above formulæ by Ferrers' method.

When n is great compared with r, these two functions approach to a ratio of equality.

The generating function for partitions without repetition.

Indefinite resolution of numbers with and without repetition.

Generating functions for both these kinds of indefinite resolutions.

Euler's *Series Mirabilis*, and its application.

Remark on indefinite *partition* with the elements 1, 2, 4, 8, &c.

Partition or composition. Partible number. Elements.

* Delivered at King's College, London, July 4th, 1859.

Construction of equation-system whose denumerant is the number of compositions of n with unrepeated elements.

The negation of the *possibility* (for integers) of the equation $x^i + y^i = z^i$ capable of being transformed into the affirmation of an analytical *identity* by the method of partitions.

Resolution of integers into a given number of parts, how treated by Sir John Herschel and others.

Formula of reduction $\dfrac{n;}{1, 2, \dots k;} = \dfrac{n-k;}{1, 2, \dots k;} + \dfrac{n-k;}{1, 2, \dots (k-1);}$.

Objections to this method:—(1) As inductive instead of direct (besides being limited to a mere special case of partition). (2) As excessively prolix and unmanageable. (3) As leading to an amorphous result.

Mr Cayley's improved method. The true form of representation.

The lecturer's discovery of the general analytical solution.

The provisional method superseded.

Fundamental Theorem in Simple Partition.

Axiom:— $$\frac{n;}{a, b, c \dots l;} = \frac{1}{abc \dots l} \Sigma H_n(\alpha, \beta, \gamma, \dots \lambda),$$

where $H_n(\alpha, \beta, \gamma, \dots \lambda)$ indicates the sum of the homogeneous powers and products of $\alpha, \beta, \gamma, \dots \lambda$ of the nth degree, and $\alpha, \beta, \gamma, \dots \lambda$ are respectively roots of

$$x^a = 1,\ y^b = 1,\ z^c = 1,\ \dots\ u^l = 1.$$

Example:— $$\frac{7;}{2, 3;} = \tfrac{1}{6} \Sigma (\rho^7 + \rho^6\sigma + \dots + \sigma^7),$$

where Σ includes six sums corresponding to the following six systems of values ρ, σ; namely,

$$1, 1;\ -1, 1;\ 1, \rho;\ -1, \rho;\ 1, \rho^2;\ -1, \rho^2,$$

ρ meaning a root of $$\rho^2 + \rho + 1 = 0.$$

In general,

$$H_n(p, q) = \frac{p^{n+1}}{p-q} + \frac{q^{n+1}}{q-p},$$

$$H_n(p, q, r) = \frac{p^{n+2}}{(p-q)(p-r)} + \frac{q^{n+2}}{(q-p)(q-r)} + \frac{r^{n+2}}{(r-p)(r-q)},$$

$$H_n(p, q, r, s) = \frac{p^{n+3}}{(p-q)(p-r)(p-s)} + \text{\&c.} + \text{\&c.} + \text{\&c.}$$

In applying this formula to the preceding axiom, several or all of the quantities p, q, r, &c., will become equal *inter se*, because the equations $a^\alpha=1$, $b^\beta=1$, $c^\gamma=1$, ... have the root unity in common, and will have other roots in common unless α, β, γ, ... are all prime to each other.

The value of $$\Sigma \frac{\phi p_1}{(p_1-p_2)(p_1-p_3)\ldots(p_1-p_{e+1})},$$

when $$p_1=p_2=\ldots p_{e+1},$$

is $$\frac{1}{1.2.3\ldots e}\left(\frac{d}{dp}\right)^e \phi p.$$

Every distinct root of $x^m=1$, where m is the least common multiple of $a, b, c, \ldots l$, furnishes a distinct expression to the sum and gives rise to a separate term, in the complete analytical expression for $\dfrac{n;}{a, b, c, \ldots l;}$. Such a term is called a wave. Reason for this name.

There are as many waves as distinct factors in $a, b, c, \ldots l$; every such factor as q giving rise to a term W_q.

If $a, b, c, \ldots l$ become the series of natural numbers $1, 2, 3, \ldots r$, the number of waves is r.

The value* of W_q for $\dfrac{n;}{a, b, c, \ldots l;}$ is the coefficient of $\dfrac{1}{t}$ in

$$\frac{1}{a\,b\,c\ldots l}\,\Sigma\,\frac{(\rho e^t)^n}{[1-(\rho e^t)^{-a}][1-(\rho e^t)^{-b}]\ldots[1-(\rho e^t)^{-l}]},$$

where ρ is a *primitive* root of $\rho^q=1$, that is, a root not belonging to $\rho^{q/i}=1$.

The only cases where the quantity under the sign of summation reduces to a single term is when $q=1$, for which case $\rho=1$, and when $q=2$, for which case $\rho=-1$.

W_1 considered. It is non-periodic. It is the coefficient of t^n in the development of $\dfrac{\phi(t)}{(1-t)^\varrho}$, when the Eulerian function

$$\frac{1}{(1-t^a)(1-t^b)\ldots(1-t^l)}$$

is supposed capable of being represented under the form

$$\frac{\phi t}{(1-t)^\varrho}+\frac{\psi t}{\dfrac{1-t^a}{1-t}\,\dfrac{1-t^b}{1-t}\cdots\dfrac{1-t^l}{1-t}}.$$

It is also the *mean* of the m algebraical forms, m being the least common

[* Cf. p. 91 above.]

multiple of $a, b, c, \ldots l$, which represent $\dfrac{n;}{a, b, c, \ldots l;}$ when n is made to go through the m forms

$$\frac{km;}{a, b, c, \ldots l;}, \quad \frac{km+1;}{a, b, c, \ldots l;}, \ldots \frac{km+(m-1);}{a, b, c, \ldots l;}.$$

It is therefore the mean value of $\dfrac{n;}{a, b, c, \ldots l;}$.

W_2 is $(-)^n B$, where B is the coefficient of $\dfrac{1}{t}$ in

$$\frac{1}{a\,b\,c \ldots kl}\,\frac{1}{(1-e^{-at})(1-e^{-bt})\ldots(1+e^{-kt})(1+e^{-lt})},$$

$a, b, \ldots$ being the odd, and $\ldots k, l$ the even, integers among $a, b, c, \ldots k, l$.

If the first wave is A, $A+B$ will be the mean of $\dfrac{n;}{a, b, c, \ldots l;}$ for even values of n, and $A-B$ the mean of the same for odd values of n.

Observe that the degree in n of W_1 is one unit less than the number of the elements $a, b, c, \ldots l$; in the algebraical part of W_2 is one unit less than the number of even elements among $a, b, c, \ldots l$, and in general W_q is one unit less than the number of elements which contain q as a factor.

Provisional notation co_{-1}, co_r explained.

The equations

$$co_l\,\phi(t) = co_{l+\omega}\,[t^{\omega}\,\phi(t)], \quad co_{2t}\,\phi(t^2) = co_t\,\phi(T)$$

identically true.

Mode of developing

$$\frac{(\rho e^t)^n}{[1-(\rho e^t)^{-a}]\,[1-(\rho e^t)^{-b}]\ldots \text{to } i \text{ terms}},$$

under the form $\rho^n . e^{nt-R}$; where

$$R = \Sigma \log\,[1-(\rho e^t)^{-a}].$$

This an essential part of the theorem.

The expression for $\dfrac{1}{1-ke^u}$ in terms of u being known, $\log(1-ke^u)$ is known by integration from the identity

$$\frac{d}{du}\log(1-ke^u) = \frac{-ke^u}{1-ke^u} = 1 - \frac{1}{1-ke^u},$$

so that in the first and second waves the only numerical constants to be determined are the numbers of Bernouilli.

Thus when $\rho = 1$, corresponding to W_1, R becomes

$$\Sigma \left\{ \log (at) - \frac{at}{2} + \frac{B_1}{2^2} a^2 t^2 - \frac{B_2}{2 \,.\, 3 \,.\, 4^2} a^4 t^4 + \frac{B_3}{2 \,.\, 3 \,.\, 4 \,.\, 5 \,.\, 6^2} a^6 t^6 - \text{\&c.} \right\}$$

$$= \log (a_1 a_2 \ldots a_i) + i \log t - \tfrac{1}{2} \Sigma a \,.\, t + \frac{B_1}{2^2} (\Sigma a^2) \, t^2 - \frac{B_2}{2 \,.\, 3 \,.\, 4^2} (\Sigma a^4) \, t^4 \pm \text{\&c.},$$

so that $nt - R$ becomes

$$- \log (a_1 a_2 \ldots a_i) - i \log t + \left\{ (n + \tfrac{1}{2} \Sigma a) \, t - \frac{B_1}{2^2} (\Sigma a^2) \, t^2 \pm \text{\&c.} \right\},$$

and
$$e^{nt - R} = \frac{t^{-i}}{a_1 a_2 \ldots a_i} e^{\nu t - (B_1/2^2) s_2 t^2 + \text{\&c.}};$$

and, finally, $W_1 = \dfrac{1}{a_1 a_2 \ldots a_i} co_{i-1} \{ e^{\nu t - \frac{1}{24} s_2 t^2 + \frac{1}{2880} s_4 t^4 - \frac{1}{181440} s_6 t^6 \pm \text{\&c.}} \}$;

where
$$\nu = n + \tfrac{1}{2} \Sigma a.$$

In like manner, when $\rho = -1$, corresponding to W_2, if $a_1, a_2, \ldots a_e$ are the even, and $b_1, b_2, \ldots b_\omega$ the odd, elements,

$$R = \Sigma \log (1 - e^{-at}) + \Sigma \log (1 + e^{-bt}),$$

and we shall obtain

$$W_2 = \frac{(-)^n}{2^\omega a_1 a_2 \ldots a_e} co_{e-1} \{ e^{\nu t - \frac{1}{24} (s_2 + 3\sigma_2) + \frac{1}{2880} (s_4 + 15 \sigma_4) \pm \text{\&c.}} \},$$

where
$$\nu = n + \tfrac{1}{2} (a_1 + a_2 + \ldots + a_e + b_1 + b_2 + \ldots + b_\omega),$$
$$s_l = \Sigma a^l, \quad \sigma_l = \Sigma b^l.$$

Calculation of Mean Values for any given Number of Elements.

Example 1. To find the mean value of $\dfrac{n\,;}{a, b, c, d\,;}$.

This will be the coefficient of t^3 in

$$\frac{1}{abcd} \{ e^{\nu t} \times e^{-\frac{1}{24} s_2 t^2} \}$$

$$= \frac{1}{abcd} co_{+3} \left\{ \begin{array}{l} 1 + \nu t + \dfrac{\nu^2 t^2}{1 \,.\, 2} + \dfrac{\nu^3 t^3}{1 \,.\, 2 \,.\, 3} + \ldots \\ \times \\ 1 - \dfrac{1}{24} s_2 t^2 \end{array} \right\}$$

$$= \frac{\nu}{abcd} co_{+1} \left\{ \left(1 + \frac{\nu^2 t}{1 \,.\, 2 \,.\, 3} \right) \left(1 - \frac{1}{24} s_2 t_2 \right) \right\}$$

$$= \frac{\nu}{abcd} \left\{ \frac{\nu^2}{6} - \frac{s_2}{24} \right\}.$$

Example 2. To find the mean value of $\dfrac{n\,;}{a,\,b,\,c,\,d,\,e\,;}$.

This will be the coefficient of t^4 in

$$\frac{1}{abcde}\{e^{\nu t}\times e^{\frac{1}{24}s_2t^2}\times e^{\frac{1}{2880}s_4t^4}\}$$

$$=\frac{1}{abcde}\,co_{+2}\begin{cases}1+\dfrac{\nu^2}{1\,.\,2}\quad t+\dfrac{\nu^4}{1\,.\,2\,.\,3\,.\,4}t^2\\ \times\\ 1-\dfrac{1}{24}\quad s_2t+\dfrac{1}{1152}s_2{}^2t^2\\ \times\\ 1+\dfrac{1}{2880}s_4t^2\end{cases}$$

$$=\frac{1}{abcde}\left\{\frac{\nu^4}{24}-\frac{s_2}{24}\nu^2+\left(\frac{s_2{}^2}{1152}+\frac{s_4}{2880}\right)\right\}.$$

Examples of Arithmetical Calculation of Simple Denumerants.

Example 1. To find the complete expression for $\dfrac{n\,;}{1,\,2,\,3\,;}$.

$$W_1=\frac{1}{1\,.\,2\,.\,3}\,co_2\{e^{(n+3)\,t}\times e^{-\frac{14}{24}t^2}\}$$

$$=\frac{1}{1\,.\,2\,.\,3}\,co_1\left\{1+\frac{(n+3)^2}{2}\,t\right\}\left\{1-\frac{14}{24}\,t\right\}\text{ (since }1+4+9=14)$$

$$=\frac{1}{12}\left\{(n+3)^2-\frac{7}{12}\right\};$$

$$W_2=co_{-1}\frac{(-)^n\,e^{nt}}{(1+e^{-t})\,(1+e^{-3t})\,(1-e^{-2t})}$$

$$=\frac{(-)^n\,1}{8}+\frac{1}{8}\left\{\frac{n\,;}{2\,;}-\frac{n-1\,;}{2\,;}\right\},$$

$$W_3=co_{-1}\,\Sigma\,\frac{\rho^n\,.\,e^{nt}}{(1-\rho^{-1}e^{-t})\,(1-\rho^{-2}e^{-2t})\,(1-e^{-3t})}\text{ (where }\rho^2+\rho+1=0)$$

$$=\frac{1}{3}\left\{\frac{\rho^n}{(1-\rho)\,(1-\rho^2)}+\frac{\rho'^n}{(1-\rho')\,(1-\rho'^2)}\right\}$$

$$=\frac{1}{9}\,(\rho^n+\rho'^n)$$

$$=\frac{1}{9}\left\{2\,\frac{n\,;}{3\,;}-\frac{n-1\,;}{3\,;}-\frac{n+1\,;}{3\,;}\right\}.$$

Thus the complete analytical value of $\dfrac{n;}{1, 2, 3;}$ is

$$\frac{(n+3)^2}{12} - \frac{7}{144} + \frac{1}{8}\left\{\frac{n;}{2;} - \frac{n-1;}{2;}\right\} + \frac{1}{9}\left\{2\frac{n;}{3;} - \frac{n-1;}{3;} - \frac{n+1;}{3;}\right\}.$$

Since
$$\frac{7}{144} + \frac{1}{8} + \frac{2}{9} = \frac{57}{144} < \frac{1}{2},$$

the arithmetical value of $\dfrac{n;}{1, 2, 3;}$ is the nearest integer to $\dfrac{(n+3)^2}{12}$, as had been early observed under a different form of statement by Mr De Morgan.

Example 2. The same process applied to $\dfrac{n;}{1, 4, 7;}$ will give

$$W_1 = \frac{1}{2\times 4\times 7}\left\{\left(n + \frac{1+4+7}{2}\right)^2 - \frac{1+16+49}{24}\right\}$$

$$= \frac{1}{56}\left\{(n+6)^2 - \frac{11}{4}\right\},$$

$$W_2 = \frac{(-)^n}{2^2 . 4} = \frac{1}{16}\left\{\frac{n;}{2;} - \frac{n-1;}{2;}\right\},$$

$$W_4 = co_{-1}\,\Sigma\,\frac{i^n e^{nt}}{(1-i^{-1}e^{-t})(1-i^{-7}e^{-7t})(1-e^{-4t})} \text{ (where } i^2+1=0)$$

$$= \frac{1}{4}\left\{\frac{i^n}{(1-i^3)(1-i)} + \frac{i'^n}{(1-i'^3)(1-i')}\right\}$$

$$= \frac{1}{8}\{i^n + i'^n\}$$

$$= \frac{1}{8}\left\{2\frac{n;}{4;} - 2\frac{n-2;}{4;}\right\}$$

$$= \frac{1}{4}\left\{\frac{n;}{4;} - \frac{n-2;}{4;}\right\}.$$

$$W_7 = co_{-1}\,\Sigma\,\frac{\theta^n e^{nt}}{(1-\theta^6 e^{-t})(1-\theta^3 e^{-nt})(1-e^{-7t})}$$
$$\text{(where } \theta^6+\theta^5+\theta^4+\theta^3+\theta^2+\theta+1=0)$$

$$= \frac{1}{7}\,\Sigma\,\frac{\theta^n}{(1-\theta^3)(1-\theta^6)}$$

$$= \frac{1}{49}\,\Sigma\,\theta^n(1-\theta)(1-\theta^2)(1-\theta^4)(1-\theta^5)$$

$$= \frac{1}{49}\{\theta^n + \theta^{n+5} - 2\theta^{n+1} - 2\theta^{n+6}\}$$

$$= \frac{1}{7}\left\{\frac{n;}{7;} - 2\frac{n+1;}{7;} - 2\frac{n+4;}{7;} + \frac{n+5;}{7;} + 2\frac{n+6;}{7;}\right\}.$$

Thus the complete value of $\dfrac{n\,;}{1,\,4,\,7\,;}$, expressed in terms exclusively of

$$\nu = n + 6,$$

is the following:—

$$\frac{1}{56}\left\{\nu^2 - \frac{11}{4}\right\} + \frac{1}{16}\left\{\frac{\nu\,;}{2\,;} - \frac{\nu-1\,;}{2\,;}\right\} + \frac{1}{4}\left\{\frac{\nu-2\,;}{4\,;} - \frac{\nu\,;}{4\,;}\right\}$$
$$+ \frac{1}{7}\left\{2\frac{\nu\,;}{7\,;} + \left(\frac{\nu+1\,;}{7\,;} + \frac{\nu-1\,;}{7\,;}\right) - \left(2\frac{\nu+2\,;}{7\,;} + 2\frac{\nu-2\,;}{7\,;}\right)\right\}.$$

The limiting values of the sum of the second, third, and fourth waves for any value of n will be

$$\frac{1}{16} + \frac{1}{4} + \frac{2}{7} = \frac{67}{112}$$

on the positive side, and

$$+\frac{1}{16} - \frac{1}{4} - \frac{2}{7} = \frac{-53}{112}$$

on the negative side.

Hence the difference between the exact value and $\dfrac{\nu^2}{56}$ must lie between $\dfrac{123}{224}$ and $\dfrac{-117}{224}$.

So that in the greatest number of cases the *nearest* integer to $\dfrac{(n+6)^2}{56}$ gives the value of $\dfrac{n\,;}{1,\,4,\,7\,;}$; and the result can never be in error by more than a single unit.

An analogous approximate form of representation can be made for the number of modes of composing an integer with any number of elements mutually prime to each other.

Observe in the foregoing expression that the form $\dfrac{\nu+i\,;}{q\,;}$ is always paired with $\dfrac{\nu-i\,;}{q\,;}$: and $\dfrac{\nu\,;}{q\,;}$ $\left(\text{which in the actual case under consideration is } \dfrac{\nu\,;}{7\,;}\right)$ affords no exception, for this may be expressed as

$$\frac{1}{2}\left\{\frac{\nu+0\,;}{q\,;} + \frac{\nu-0\,;}{q\,;}\right\}.$$

Neither does $\dfrac{\nu-r\,;}{2r\,;}$ $\left(\text{in the actual case } \dfrac{\nu-1\,;}{2\,;}\right)$, for this may be expressed as

$$\frac{1}{2}\left\{\frac{\nu-r\,;}{2r\,;} + \frac{\nu+r\,;}{2r\,;}\right\}.$$

The sign of the pairing may be positive or negative according to a rule which the exhibition of the result worked out in the following examples will render clear.

Example 3. The denumerant $\dfrac{n\,;}{1,\,3,\,5\,;}$, expanded in a similar manner, gives rise to the following expression:—

$$\frac{1}{30}\left\{(\nu)^2-\frac{35}{24}\right\}+\frac{2}{9}\left\{\frac{\nu+\frac{3}{2}\,;}{3\,;}+\frac{\nu-\frac{3}{2}\,;}{3\,;}-\frac{\nu+\frac{1}{2}\,;}{3\,;}-\frac{\nu-\frac{1}{2}\,;}{3\,;}\right\}$$
$$+\frac{1}{5}\left\{\frac{\nu+\frac{1}{2}\,;}{5\,;}+\frac{\nu-\frac{1}{2}\,;}{5\,;}-\frac{\nu+\frac{3}{2}\,;}{5\,;}-\frac{\nu-\frac{3}{2}\,;}{5\,;}\right\}.$$

And, since
$$\frac{2}{9}+\frac{1}{5}+\frac{7}{144}=\frac{330}{720}<\frac{1}{2},$$

the arithmetical value of $\dfrac{n\,;}{1,\,3,\,5\,;}$ is always the *nearest integer* to $\dfrac{(2n+9)^2}{120}$.

This arithmetical mode of statement, how capable of extension to any set of elements following the natural order of the prime numbers, and to other cases.

Example 4. The denumerant $\dfrac{n\,;}{1,\,2,\,3,\,4,\,5,\,6,\,7\,;}$ in its expanded form is expressed by the following function of ν, which here represents $n+14$, namely,

$$\frac{1}{725760}\left\{\frac{\nu^6}{5}-35\nu^4+\frac{13419}{10}\nu^2-\frac{190325}{126}\right\}$$
$$+\frac{1}{768}\left\{\frac{\nu^2}{2}-\frac{77}{6}\right\}\left(\frac{\nu\,;}{2\,;}-\frac{\nu-1\,;}{2\,;}\right)$$
$$+\frac{1}{162}\left\{\frac{\nu+1\,;}{3\,;}-\frac{\nu-1\,;}{3\,;}\right\}\nu-\frac{5}{972}\left\{\left(\frac{\nu\,;}{3\,;}+\frac{\nu\,;}{3\,;}\right)-\left(\frac{\nu-1\,;}{3\,;}+\frac{\nu+1\,;}{3\,;}\right)\right\}$$
$$+\frac{1}{64}\left\{\left(\frac{\nu+2\,;}{4\,;}+\frac{\nu-2\,;}{4\,;}\right)-\left(\frac{\nu\,;}{4\,;}+\frac{\nu\,;}{4\,;}\right)\right\}$$
$$+\frac{1}{25}\left\{\left(\frac{\nu+1\,;}{5\,;}+\frac{\nu-1\,;}{5\,;}\right)-\left(\frac{\nu+2\,;}{5\,;}+\frac{\nu-2\,;}{5\,;}\right)\right\}$$
$$+\frac{1}{36}\left\{\left(\frac{\nu+3\,;}{6\,;}+\frac{\nu-3\,;}{6\,;}\right)-\left(\frac{\nu\,;}{6\,;}+\frac{\nu\,;}{6\,;}\right)+\left(\frac{\nu-2\,;}{6\,;}+\frac{\nu+2\,;}{6\,;}\right)-\left(\frac{\nu-1\,;}{6\,;}+\frac{\nu+1\,;}{6\,;}\right)\right\}$$
$$+\frac{1}{49}\left\{3\left(\frac{\nu\,;}{7\,;}+\frac{\nu\,;}{7\,;}\right)-\left(\frac{\nu+1\,;}{7\,;}+\frac{\nu-1\,;}{7\,;}\right)-\left(\frac{\nu+2\,;}{7\,;}+\frac{\nu-2\,;}{7\,;}\right)-\left(\frac{\nu+3\,;}{7\,;}+\frac{\nu-3\,;}{7\,;}\right)\right\}*.$$

Observation. Provided that $\dfrac{-p\,;}{i\,;}$ shall be understood to signify the same thing as $\dfrac{p\,;}{i\,;}$, every wave in the above expansion remains entirely unaltered when ν becomes $-\nu$.

* The arithmetical value of $\dfrac{\nu-14\,;}{1,\,2,\,3,\,4,\,5,\,6,\,7\,;}$ is obviously the nearest integer to

$$\frac{1}{5760}\left\{\frac{\nu^6}{5}-35\nu^4+\frac{13419}{10}\nu^2\right\}+\frac{1}{1536}\left(\frac{\nu\,;}{2\,;}-\frac{\nu-1\,;}{2\,;}\right)\nu^2+\frac{1}{162}\left\{\frac{\nu+1\,;}{3\,;}-\frac{\nu-1\,;}{3\,;}\right\}\nu.$$

A priori view of the form of such expansions of $\dfrac{n;}{a, b, c, \dots l;}$.

First, the algebraical part is, as it ought to be, a *homogeneous* function of $n;\ a, b, c, \dots l$.

Secondly, the change of ν into $-\nu$ either leaves the expansion absolutely unaltered, or unaltered save as to algebraical sign.

This depends on the theory of the *denumerantive functions* as distinguishable from denumerants. The latter discontinuous quantities, the former continuous.

Binary denumerants have in general several functions attached to them, namely, one less than the number of their denominatives*.

All generated forms have arithmetical and functional values.

Example. The form u_n generated by

$$\frac{1}{(1-x)^2} = u_0 + u_1 x + \dots u_n x^n + \&c.$$

Property of the denumerantive function $\phi(n, a, b, c, \dots l)$ to $\dfrac{n;}{a, b, c, \dots l;}$; namely, that

$$\phi(n, a, b, c, \dots l) = \pm \phi(n', a, b, c, \dots l),$$

if

$$n + n' = -a - b - c \dots - l,$$

the + sign being used when the number of elements is odd, and the negative sign when it is even.

This explains the pairing of the terms observed in $\dfrac{n;}{1, 2, 3, \dots 7;}$. Great importance of this fact of pairing.

The number of modes of dividing n into seven parts is represented by the above formula, namely, with repetitions and zero values of parts allowed by making

$$\nu = n + 14,$$

with repetitions and zero values disallowed by making

$$\nu = n - 14,$$

with repetitions allowed, but zero values disallowed, by making

$$\nu = n + 7.$$

And so in general with the values $n + \dfrac{(r-1)r}{4}$; $n - \dfrac{(r-1)r}{4}$; $n - r$ respectively substituted for ν.

Mr Kirkman's representation of partitions to the modulus 7.

* This is when the form zero is not counted as a function. Zero occurs as a form once only in simple, but twice over in binary denumerants.

Example 5. Expansion of $\dfrac{n:}{1, 2, 3, 4, 5, 6:}$ as a function of n.

$$\frac{1}{17280}\left\{\frac{\nu^5}{5}-\frac{91}{6}\nu^3+\frac{9191}{48}\right\}+\frac{1}{768}\left\{\frac{\nu+\frac{1}{2}:}{2:}-\frac{\nu-\frac{1}{2}:}{2:}\right\}\left(\nu^2-\frac{161}{12}\right)$$
$$+\frac{1}{162}\left\{\left[\left(\frac{\nu+\frac{3}{2}:}{3:}+\frac{\nu-\frac{3}{2}:}{3:}\right)-\left(\frac{\nu+\frac{1}{2}:}{3:}+\frac{\nu-\frac{1}{2}:}{3:}\right)\right]\nu+\left(\frac{\nu+\frac{1}{2}:}{3:}-\frac{\nu-\frac{1}{2}:}{3:}\right)\right\}$$
$$+\frac{1}{32}\left\{\left(\frac{\nu+\frac{3}{2}:}{4:}-\frac{\nu-\frac{3}{2}:}{4:}\right)+\left(\frac{\nu+\frac{1}{2}:}{4:}-\frac{\nu-\frac{1}{2}:}{4:}\right)\right\}$$
$$+\frac{1}{25}\left\{\left(\frac{\nu-\frac{1}{2}:}{4:}-\frac{\nu+\frac{1}{2}:}{4:}\right)+\left(\frac{\nu-\frac{3}{2}:}{4:}-\frac{\nu+\frac{3}{2}:}{4:}\right)\right\}$$
$$+\frac{1}{18}\left\{\left(\frac{\nu+\frac{3}{2}:}{6:}-\frac{\nu-\frac{3}{2}:}{6:}\right)+\frac{1}{36}\left(\frac{\nu+\frac{5}{2}:}{6:}-\frac{\nu-\frac{5}{2}:}{6:}\right)+\left(\frac{\nu+\frac{1}{2}:}{6:}-\frac{\nu-\frac{1}{2}:}{6:}\right)\right\},$$

where $$\nu=n+\frac{21}{2}.$$

Observe the substitution of the colon for the semicolon above and below the line in the fraction-form to distinguish a denumerantive function from a denumerantive proper. The arithmetical value of the foregoing is the nearest integer to the sum of its first, second, and third waves; and in the two latter it is only necessary to retain those terms which contain ν^2 and ν respectively.

On the expression for the number of waves when the denominatives of a denumerant or the elements of a partition are given.

On the blending of waves, and its advantages in some cases, as when the elements are all prime to each other without all being absolute primes.

Easy mode of deducing the fundamental theorem, by the application of a formula in the calculus of residues to the Eulerian. Its capital importance in the theory of partitions.

Close of the analytical portion of the course.

SEVENTH LECTURE*.

The representation of systems of linear equations by clusters of points recalled.

A single equation by a cluster of points in a line, of two simultaneous equations by a cluster of points in a plane, three simultaneous equations by a cluster of points in space—geometrical criterion between definite and indefinite systems.

The linear cluster which corresponds to a single equation is unique in form, not so the plane cluster which corresponds to two, or the solid cluster

* Delivered at King's College, London, July 11th, 1859.

which corresponds to three equations. One arbitrary parameter enters into the former, three into the latter.

All such clusters balance about their respective origins with the same weights at corresponding points.

Clusters so related might be termed homobaric.

Mechanical representation of the property of homobarism by a series of jointed parallelograms having two axes in common (see Fig. *A*). [Plate I. at the end of the volume.]

Criterion between definite and indefinite systems recalled.

"To determine the chance that three points thrown anywhere within a parallelogram may contain the centre."

Solution of this problem by theory of definite and indefinite binary equation-systems with three variables alluded to.

The chance is $\frac{5}{32}$ths against the points including the centre, whatever the form or dimensions of the parallelogram.

Hence the chance is $\frac{5}{32}$ths against three points capable of being taken anywhere in an indefinite plane including a *given* fourth point in the same plane.

But it would be incorrect to infer from this that the chance of some three out of four points (capable of being taken anywhere in an indefinite plane), including the fourth, is $\frac{5}{8}$; it will be much less than this.

Explanation of this seeming paradox. Geometrical and analytical modes of treating this second question alluded to.

Experimental method of verification. New game of odd and even.

The natural order of the variables in a single homonymous equation recalled.

Unless definite there is no natural order.

The importance of obtaining such natural orders to the theory of compound partition, namely, in applying the process of eduction. Example of natural and disturbed order.

General *analytical* condition of normal sequence.

Difficulty of seeing any natural order among the rays of a solid cluster; it will presently appear that such orders do exist, but that instead of one natural order there are several; the number depending (1) upon the number of points in the cluster, (2) upon the mode in which the rays are grouped, subject to the observation that the distinct modes of grouping in the view of this theory are always limited in number, and determinable *a priori*.

Passage by perspective from the grouping of rays in a plane to the grouping of points in a line; from the grouping of rays *in solido* to the grouping of points in a plane; and from the grouping of rays in plu-space to the grouping of points *in solido*.

Observe that in studying the character of an equation-system no attention need in the first instance be paid to the primary, because the process of eduction concerns the variables only, and not the constant terms in the system; by the act of taking the perspective, the origin no longer appears—thus, nothing is left but a perspective cluster or group of points, as many in number as the variables of the system.

Theorem. The number of classes of definite binary systems of linear equations for any given number of variables is *one*, because there is but one species of arrangement of a given number of points in a line. The number of classes of definite ternary systems of equations with r variables is the number of distinct modes of grouping together r points in a plane; the number of classes of definite quaternary systems of equations with r variables is the number of distinct modes of grouping together r points in space.

Observe the fact of space being made subservient through the method of perspective to systems of linear equations greater in number than the so-called dimensions of space.

In determining the natural order in a binary system, the perspective group may be substituted for the cluster, provided the line of projection cuts all the rays on the same side of the origin, so that a line through the origin parallel to the line of projection falls *outside* the cluster; but, if this condition is not observed, the order will be disturbed. (See Fig. *B.*) [Plate I. at the end of the volume.]

In like manner, for a ternary system, the plane of projection must be supposed to be drawn parallel to a plane through the origin external to the solid cluster.

Observation on Plane and Solid Groups, considered as representing definite Ternary and Quaternary Equation-systems respectively. If we suppose a ternary system of which one of the equations is of the form

$$x + y + z + \ldots + u - 1 = 0,$$

the others being

$$ax + by + cz + \ldots + lu - m = 0,$$

$$\alpha x + \beta y + \gamma z + \ldots + \lambda u - \mu = 0,$$

such a system may evidently be represented by a group of points in a plane whose coordinates are

$$(a, \alpha)\,(b, \beta)\,(c, \gamma)\ldots(l, \lambda);\ (m, \mu),$$

and

$$x, y, z, \ldots u;\ -1,$$

will be the weights to be placed at these points respectively in order to *balance* each other.

Moreover, if we start with any *definite* ternary system, we may substitute for one of the equations in it a homonymous equation reducible to the form of

$$x' + y' + z' + \dots + u' - 1 = 0,$$

on taking $$x', y', z', \dots u',$$

all homonymous multiples of

$$x, y, z, \dots u.$$

Consequently, the *form* of the plexus of principal derivatives which depends essentially only on the relations of algebraical signs in the coefficients of these derivatives will be the same whether the system be considered as involving explicitly $x, y, z, \dots u$, or $x', y', z', \dots u'$, and, consequently, every definite ternary system of equations whatever may be represented in its essential properties of form by a plane group of points, in lieu of a solid cluster of rays. And in like manner, without going from hypersolidity to the solid, we see that any definite quaternary system may be represented by a group of points *in solido*.

The property indicated of the self-balancing group *in plano* being substitutable for the solid cluster with its centre of gravity at the origin may be deduced easily from this more general theorem, that if two groups of weighted points are in perspective about a given point G, and the weights at the corresponding points in the two groups are in the inverse ratios of their distances from G, if one of them has its centre of gravity at G, the other also will have its centre of gravity there. Hence, if one of these groups be considered as a derivative from the first, and all the points of the derivative group be brought to lie within the same plane, it must become self-balancing, since otherwise a plane group of statical points would have its centre of gravity outside the plane.

Notice that the geometrical construction for determining whether a system of equations is definite or indefinite would fail for a quaternary system, but the analytical method operative through the principal plexus continues to hold good.

Ternary Systems, and Plane Groups.

Imagine a sphere to be drawn with the origin of a solid cluster as its centre; the general arrangement of the points on the sphere will correspond to the arrangement of the points on the perspective plane, and, when convenient to do so, the one may be substituted for the other.

Illustration by examples with four and five points.

Recall eduction and the condition of its giving rise to definite systems, namely, that the systems deduced by the successive deformations of the given system shall remain definite, that is, external to the centre. If a group of

points on a sphere be contained within the boundary of a hemisphere, the centre will be external to such group; but, if the bounding contour of the group formed by arcs of great circles cover more than half the sphere, the centre will be contained within the group. The effect upon the spherical perspective of a cluster representing a ternary system of equations due to the change of a variable x into $-x$ is to make the point corresponding to the coefficients of x pass to the opposite end of the diameter passing through it; such a change may be termed a *reversal* of the point, and the point so obtained the opposite of the original point. The problem of normal orders for ternary systems may therefore be stated geometrically as follows:—

A given number of points being contained within a hemisphere, to discover what orders of sequence of these points will possess this property that on the first, second, third, &c., to the last of them, one after the other undergoing *reversal*, the transformed group shall never occupy more than half the surface of the sphere.

From this it follows that, if

$$xyz \ldots tuv$$

be a normal sequence, $\quad vut \ldots xyz$

will be so likewise, for, if we denote the opposite points to

$$xyz \ldots uv \text{ by } x'y'z' \ldots u'v',$$

respectively, it is clear that, if the groups

$$\left.\begin{array}{l} x'yz \ \ldots\ uv \\ x'y'z \ \ldots\ uv \\ x'y'z' \ \ldots\ uv \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ x'y'z' \ \ldots\ u'v \end{array}\right\}$$

are respectively contained, each within their own hemispheres, the groups

$$\left.\begin{array}{l} v'u \ \ldots\ zyx \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ v'u' \ \ldots\ zyx \\ v'u' \ \ldots\ z'yx \\ v'u' \ \ldots\ z'y'x \end{array}\right\}$$

will each also be contained in hemispheres opposite to the former, taken in reverse order.

The *contour* of a spherical group defined.

What is meant by a peripheral and what by an internal point to a group on a sphere.

Again, to obtain the law of normal sequences, we have the following propositions:—

(1) Any sequence $x, y, z, t, \ldots u, v, w$ of points in a sphere will be a normal order of sequence, provided the following condition is satisfied, namely, that, on joining those points with each other in the order of their succession by arcs of great circles, the broken line or spherical zigzag so formed shall be capable at every one of its angular points of being divided into two parts by a great circle which does not cut the line at any other point; for evidently in such case, if we draw a great circle through a point u, which does not cut any of the sides, or any other angle except u, the points x, y, z, t being all reversed, will lie together with v, w in a hemisphere bounded by the great circle so drawn.

(2) A normal sequence of points in a group cannot be bounded at either extremity by an interior point of the group. For, on joining the opposite of such exterior point with the closed figure surrounding that point, we evidently obtain a figure clasping the hemisphere bounded by the great circle perpendicular to the diameter through these points, and stretching into the hemisphere beyond. On the other hand, a normal order may always be commenced from any peripheral point in the group at pleasure, for, if u, v, y, z, x, t be any group contained within a hemisphere H, and u' a point in the contour, it is apparent that u, v, y, z, x, t, u' will also be contained within the same hemisphere H, so that, in fact, one way of characterizing a normal sequence would be as a sequence in which each point in turn becomes a peripheral point, alike when all the points preceding as when all the points following it are *reversed*.

(3) No arc joining y, z, two consecutive points in a normal sequence $x, y, z, t, u, v, w, \omega$, can cut uv any other such arc, for it is clear that, if yz crosses uv, x will be contained within the triangle $y'uv$ (y' meaning the opposite point to y), and, consequently, z will not be external to $tuvw\omega x'y'$, as it must be if the given order is normal.

(4) It follows also, as an immediate corollary from 2, that no point t can be contained within the contour of xyz or within that of $uvw\omega$.

Hence (5) it follows from (3) and (4) that the two spherical areas bounded respectively by the contours of xyz, $uvw\omega$ have no part whatever in common, and, consequently, may be separated by a great circle drawn through t.

Hence, combining the conclusions of (1) and (5), we arrive at the theorem that the sole necessary and sufficient condition for determining $x, y, z, t, u, v, w, \omega$ to be in normal sequence upon a sphere is that through any point as t a great circle can be drawn upon the sphere not cutting this line in any other point; and, consequently, the sole necessary and sufficient condition

for a number of points in a plane group being in normal sequence is that the zigzag formed by drawing straight lines from any one point to the next in the sequence shall be capable of being cut in twain at any of its angles by a right line.

General definition of a diatomic line continuous or discontinuous in a *plane* or in *space* (see plate) [at the end of the volume].

The condition of normal sequence may be extended from plane to solid groups, that is, from ternary to quaternary equation-systems, the sole necessary and sufficient condition for determining a normal order of points, as well *in solido* as *in plano*, being that the zigzag following the succession of the points in the order shall be a *diatomic* line.

In order to depress a ternary system so as to make its denumerant depend upon binary denumerants, we must be able to form orders of normal sequence among its variables.

Every such order will or may furnish two distinct forms of solution, provided the requisite conditions of relative primeness and asyzygeticism are satisfied.

The easiest way of determining such normal orders is by means of diatomic lines drawn from point to point of the representative plane group.

Every ternary system may be identified by means of its principal plexus, as will presently be shown with some specific form of group, corresponding in number to the number of the variables in the system. It becomes necessary, therefore, to facilitate the solution of the problem of denumeration of ternary systems, to classify and register the distinct forms of arrangement of plane groups (and in like manner, in order to make denumerants of the fourth order depend upon those of the third, we must begin with classifying and registering the various dispositions of which a given number of points is susceptible in space).

Plane Groups.

For three points only one species of arrangement is possible, and all the orders are normal orders.

For four points two distinct arrangements only are possible, namely, of four points external to one another, or three points with a fourth point in the interior.

Morph defined—its geometrical and analytical meaning.

Exclusion of syzygetic cases.

The morph corresponding to the one case (see plate) [at the end of the volume] will be the following:—

$$xy : \quad yz : \quad zt : \quad tx :$$
$$x : z \quad y : t.$$

The morph corresponding to the other case (see plate) will be

$$\begin{matrix} zt: & xt: & yt: \\ y:z & z:x & x:y. \end{matrix}$$

Accordingly, the plexus of principal derivatives to every definite ternary system with four variables, omitting the constant terms, will be of one or the other of the two following forms:—

$$\left.\begin{matrix} ax+by=? & lx-mz=? \\ cy+dz=? & ny-pt=? \\ ez+ft=? & \\ gx+ht=? & \end{matrix}\right\} \text{1st form,}$$

$$\left.\begin{matrix} az+bt=? & \alpha y-\beta z=? \\ cx+dt=? & \gamma z-\delta x=? \\ ey+ft=? & \epsilon x-\phi y=? \end{matrix}\right\} \text{2nd form.}$$

The constants are of course not independent of each other, nor are the *signs* of the quantities left unexpressed, but symbolized by ?, independent. How to assign the laws of the dependence is not material to our immediate object.

By means of these two forms, we can by examination of the plexus of principal derivatives to a ternary system of linear equations with four variables, ascertain whether it corresponds to the form of group in Figs. 2, 3, or to that in Fig. 4.

The normal orders belonging to these groups respectively are shown in the plate [at the end of the volume].

If we proceed now to the groupings of five points, we shall find that the only essentially distinct arrangements are the following:—

Five points all external to each other.

Four points with one inside.

Three points with two inside.

No differences of any other kind than those above denoted will give rise to any difference in the character of the morph resulting. The three morphs corresponding to these three arrangements are the following:—

$$\left.\begin{matrix} ztu: & tux: & uxy: & xyz: & yzt: \\ zt:x & tu:y & ux:z & xy:t & yz:u \end{matrix}\right\} \text{1st form,}$$

$$\left.\begin{matrix} ztu: & xtu: & xyu: & yzu: \\ & zu:x & tu:y & \\ yz:t & xt:z & xy:t & xy:z \end{matrix}\right\} \text{2nd form,}$$

$$\left.\begin{matrix} ztu: & xtu: & ytu: \\ uz:y & uz:x & yu:x \\ yt:z & xt:z & xt:y \\ & xz:y & \end{matrix}\right\} \text{3rd form.}$$

And by simply observing whether the principal plexus has three, four, or five homonymous derivatives, the perspective representation of any definite ternary system of linear equations with five variables can be identified with one or the other of the Figures 5–7, 8–11, or 12–15. The diatomic zigzags expressing the normal orders to these figures are given in the plate.

So far we have found that the number of morphs has followed the progression 1 ; 2 ; 3 ; arising from the fact that there are no essential differences of position of a point or a couple of points within a triangle or of one point within a quadrilateral. Very different is the case for a figure of six points, corresponding to a definite ternary system, with six variables. The following cases arise :—

(1) The six points may be at the angles of a hexagon.

(2) Five of the six points may be at the angles of a pentagon, and the interior point may occupy any one of three essentially distinct regions, each such position giving rise to a distinct species of morph confined to the inner.

These three distinct kinds of position defined.

(3) Four of the points may be external, and the pair of points within occupy six distinct kinds of position.

These six different relative positions described.

Finally, three of the six points may be external, so that the figure may be viewed as a triangle within a triangle, and there will be six distinct relative positions of triangles so related.

These six different relative positions described.

There are thus in all sixteen different classes of arrangement of six points in a plane, and therefore sixteen different classes of ternary systems with six variables.

The normal orders for each of these sixteen cases have been completely worked out by Captain Noble, R.A., and the numbers for each figure attached to them in the plate (16–31).

On the arrangement of plane groups in natural families.

Classes belong to the same natural family which are capable of appearing simultaneously in the same course of the same process of eduction.

Any two classes of systems belong to the same family which may be obtained from one or other by altering the signs concurrently of one or more of such of the coefficient groups as may be so altered without the system ceasing to be definite. If in the morph of any class we make any letter pass from right to left of the colon, and *vice versa* throughout, and after such change the morph still contains characters of the form $xyz \ldots u$: corresponding to homonymous principal derivatives, the morph so obtained will belong

to the same family with that from which it is derived. Thus from one morph all others of the same family may be derived by simple inspection and transposition.

Conversion defined.

Scales of derivation in general are divaricative, but for principal family are lineal.

Example four-, five-, and six-point systems.

How to determine *a priori* what letters in a given morph are convertible.

Examples in six-point systems.

The number of families for six-point arrangements is four.

The two classes of four-point systems, and the three classes of five-point systems, belong respectively to a single family. Proof.

Numerical and natural modes of classifying groups contrasted.

The principal class and principal family of any r-point group defined.

The tactical rule which serves to define any normal order in the principal class. Example in five-point system. Example in six-point system.

In four- and five-point systems all the classes belong to the principal family, there being no other.

The numerical system of arrangement in families gives rise to a new question in the partition of numbers.

Thus a seven-point system, and an eight-point system arranged after the numerical system, consist respectively of families which may be typified as follows:—

$$7 \quad 6,1 \quad 5,2 \quad 4,3 \quad 3,4 \quad 3,3,1$$
$$8 \quad 6,2 \quad 5,3 \quad 4,4 \quad 4,3,1 \quad 3,5 \quad 3,4,1 \quad 3,3,2$$

Theorem. All classes of the same family may be derived from one another by perspective projection.

Conversion balls and their use.

Theorem. Normal orders are orders of perspective sequence (see plate).

Application of perspective regions to finding normal orders by exhaustive method.

The position of the eye must be external to the group.

The entire plane outside the group may be divided into as many distinct perspective regions as there are normal orders (see plate).

A ship tacking along a diatomic zigzag is continually making angular way in reference to a point taken anywhere in some determinate region.

Normal orders of points in space are also orders of double perspective sequence, a line of view and planes of light being substituted for the point of view and rays of light.

Four-, five-, and six-point systems in space, like three-, four-, and five-point systems in planes, are reducible respectively to one, two, and three classes.

The classes of quaternary point-systems like those of ternary, and by the same method, may be arranged in natural families.

Reasons for believing a higher or more complex colligation of classes possible for quaternary systems.

Although the geometry of dispositions does not explicitly recognise distinctions grounded on magnitude, still the relations which it contemplates must admit of quantitative discrimination.

The *cæsura* in the *eduction* process following any normal order of the variables; how determined geometrically for ternary or quaternary systems by the principle of *denudation*; conformity of this with the rule for binary systems.

How the neutral region in a normal order which does not exist for binary arises for ternary and higher systems (see plate).

Example. The distances from each other of four points in a plane being given (six quantities connected by one equation) it must be possible to form one or more rational functions of these quantities of which the values as positive or negative must serve to discriminate between the two kinds of disposition in which four points may be grouped.

General character of the new geometry of disposition.

"It is the theory of permutation of space."—*Cayley*.

27.

THÉORIE DES NOMBRES.

(*Extrait d'une Lettre adressée à* M. HERMITE *par* M. SYLVESTER.)

[*Comptes Rendus de l'Académie des Sciences,* L. (1860), p. 367.]

...EN désignant par $(n\,;\,a,\,b)$ le nombre des solutions entières et positives de l'équation

$$ax + by = r$$

pour la série des valeurs $r = 0, 1, 2, \ldots, n$, j'ai obtenu ces deux théorèmes :

1°. Soit $n + 1 = kab + n'$, on aura

$$(n\,;\,a,\,b) = k\frac{kab + a + b + 2n' - 1}{2} + (n'\,;\,a,\,b).$$

Cette relation permet déjà de remplacer n par son résidu minimum suivant le module ab dans $(n\,;\,a,\,b)$.

2°. Soit ν un nombre entier inférieur à ab ; on pourra déterminer les entiers positifs a' et b' de manière à avoir

$$ab' - ba' = 1,$$

a' étant moindre que a, et b' moindre que b. Cela posé, si l'on désigne par $E(x)$ l'entier compris dans une quantité quelconque x, et qu'on pose

$$E\left(\frac{b'\nu}{b}\right) = \nu',$$

on aura
$$(\nu\,;\,a,\,b) = (\nu'\,;\,a',\,b') - \mathfrak{R},$$

ou
$$\mathfrak{R} = \left[\nu' - E\left(\frac{a'\nu}{a}\right)\right] E\left(\frac{a\nu' - \nu a' + 1}{a'}\right).$$

Par ce second théorème on peut diminuer les deux coefficients a et b, en les remplaçant par a' et b' ; donc en le joignant au précédent et appliquant successivement les deux propositions, on voit qu'on pourra exprimer $(n\,;\,a,\,b)$ par une série contenant au plus autant de termes qu'il a de fonctions convergentes vers $\dfrac{a}{b}$.

28.

THÉORIE DES NOMBRES.

(*Extrait d'une Lettre adressée à* M. HERMITE *par* M. SYLVESTER.)

[*Comptes Rendus de l'Académie des Sciences,* L. (1860), p. 489.]

LA somme $\sum_{i=1}^{i=\frac{q-1}{2}} E\left(i\frac{p}{q}\right)$ où $E(x)$ désigne, suivant l'usage, l'entier contenu dans la quantité x, et qui joue un si grand rôle dans la théorie des résidus quadratiques, peut se calculer complétement par la méthode suivante, plus simple et plus facile que celle d'Eisentein pour déterminer seulement si la somme est paire ou impaire. Je développe $\frac{p}{q}$ sous la forme d'une fraction continue avec ces conditions, que le nombre des quotients soit impair et que chaque quotient de rang impair après le premier soit pair, ce qu'on réalisera en faisant le premier quotient congru à p suivant le module 2. Soit donc ainsi :

$$\frac{p}{q} = a_0 + \cfrac{\epsilon_1}{a_1 + \cfrac{\epsilon_2}{a_2 + \cfrac{\vdots}{+\cfrac{\epsilon_{2\omega}}{a_{2\omega}}}}}.$$

Les quotients $a_2, a_4, \ldots, a_{2\omega}$ étant pairs, a_1, a_3, etc., quelconques, et ϵ_1, ϵ_2, etc., égaux à ± 1, on aura, en faisant $\lambda_i = \epsilon_1, \epsilon_2, \ldots \epsilon_i$,

$$\sum_{i=1}^{i=\frac{q-1}{2}} E\left(i\frac{p}{q}\right) = \frac{1}{8}\left[(p-2)\,q - \Sigma\,\{2\lambda_{2i-1} + (a_{2i}-2)\,\lambda_{2i}\}\right].$$

A cette proposition je joindrai la suivante qui en est une généralisation.

Soit k un diviseur quelconque de $q-1$; et supposons que dans le développement en fraction continue

$$\frac{p}{q} = a_0 + \cfrac{\epsilon_1}{a_1 + \cfrac{\epsilon_2}{a_2 + \cfrac{\vdots}{+\cfrac{\epsilon_{2\omega}}{a_{2\omega}}}}},$$

tous les quotients à partir de a_1 soient multiples de k, le premier a_0 étant congru à p module k; alors on aura

$$E\left(\frac{p}{q}\right) + E\left(2\frac{p}{q}\right) + \ldots + E\left(\frac{q-1}{k}\cdot\frac{p}{q}\right)$$
$$= \frac{(q-2)\,p + k\,(p-q) - \Sigma\left[k\lambda_{2i-1} + \{(k-1)\,a_{2i} - k\}\,\lambda_{2i}\right]}{2k^2}.$$

Les conditions énoncées seront toujours d'ailleurs possibles, si l'on a $k < 5$.

29.

NOTE SUR CERTAINES SÉRIES QUI SE PRÉSENTENT DANS LA THÉORIE DES NOMBRES.

[*Comptes Rendus de l'Académie des Sciences*, L. (1860), p. 650.]

Soit $E\left(\frac{p}{q}\right)$ le plus grand nombre entier contenu dans la fraction $\frac{p}{q}$, et faisons

$$F(p, q, k, l) = E\left(\frac{p}{q}\right) + E\left(2\frac{p}{q}\right) + \ldots + E\left(l\frac{q-1}{k}\frac{p}{q}\right),$$

en supposant $q-1$ divisible par k. Il existe entre trois fonctions quelconques F, qui ont les mêmes valeurs de p, q, k, mais où la quantité l varie en restant moindre que k, l'équation algébrique suivante:

$$\frac{k-l'-l''}{(l-l')(l-l'')}F(p, q, k, l) + \frac{k-l''-l}{(l'-l'')(l'-l)}F(p, q, k, l') + \frac{k-l-l'}{(l''-l)(l''-l')}F(p, q, k, l'') = \frac{(p-1)(q-1)}{2k}.$$

Quand $l'+l''=k$, cette relation devient

$$F(p, q, k, l) - F(p, q, k, k-l) = (2l-k)\frac{(p-1)(q-1)}{2k},$$

ce qu'on peut vérifier par un procédé tout élémentaire. Il existe aussi entre les fonctions F, où k et l restent les mêmes, p et q étant changés entre eux, l'équation

$$F(p, q, k, l) + F(q, p, k, l) = \frac{l^2(p-1)(q-1)}{2k^2}.$$

Pour le cas de $l=1$, ce théorème a été déjà donné par Eisenstein, qui a exprimé alors la fonction F par une série trigonométrique finie. Mais quel que soit l, je suis parvenu à exprimer d'une manière analogue cette fonction, et dans le même ordre d'idées, c'est également par une série trigonométrique que j'ai été amené à représenter les valeurs de p' et q', moindres que p et q, satisfaisant à l'équation

$$p'q - q'p = 1,$$

valeurs qu'on obtient habituellement par le procédé du plus grand commun diviseur.

30.

SUR LA FONCTION $E(x)$.

[*Comptes Rendus de l'Académie des Sciences,* L. (1860), pp. 732—734.]

SOIENT p et q deux quantités positives quelconques, λ une quantité moindre que la plus petite valeur a qui rende en même temps ap et aq entiers, de sorte que, p et q étant incommensurables, λ est arbitraire; mais si l'on suppose que ces quantités aient un plus grand commun diviseur k, on aura

$$\lambda < \frac{1}{k}.$$

Cela étant, je dis qu'on aura l'égalité suivante:

$$\sum_{\omega=0}^{\omega=E(\lambda q)} E\left(\omega\frac{p}{q}\right) + \sum_{\omega=0}^{\omega=E(\lambda p)} E\left(\omega\frac{q}{p}\right) = E(\lambda p)\, E(\lambda q).$$

Supposons-la satisfaite, en effet, pour toutes les valeurs de λ inférieures à une certaine limite, et faisons croître λ par degrés insensibles à partir de cette limite. Aucun des membres de l'équation ne changera de valeur qu'autant que λp ou λq deviendront des nombres entiers, ce qui, par hypothèse, n'arrivera jamais en même temps. Supposons que λp le premier devienne entier: à ce moment la seconde somme du premier membre s'augmente de $E\left(\lambda p\,.\,\frac{q}{p}\right)$, c'est-à-dire de $E(\lambda q)$, la première ne changeant pas. Quant au second membre de l'équation, il est évident que $E(\lambda q)$ ne change pas, mais $E(\lambda p)$ est augmenté d'une unité, donc le second membre comme le premier s'accroît de $E(\lambda p)$. Donc le théorème subsiste pour la première valeur de λ qui fait varier les deux membres de l'équation, par conséquent, pour la seconde, la troisième, etc., et enfin pour toutes les valeurs inférieures à la plus petite quantité qui rend en même temps λp et λq entiers. Donc, étant vrai pour $\lambda = 0$, le théorème a lieu également pour toutes les valeurs de λ moindres que la limite supposée.

Si l'on supprime la restriction admise à l'égard de λ, j'observe que toutes les fois que, cette quantité croissant d'une manière continue, λp et λq deviennent entiers en même temps, l'expression

$$\sum_{\omega=0}^{\omega=E(\lambda q)} E\left(\omega\frac{p}{q}\right) + \sum_{\omega=0}^{\omega=E(\lambda p)} E\left(\omega\frac{q}{p}\right),$$

recevra un accroissement

$$E(\lambda p) + E(\lambda q) = \lambda p + \lambda q,$$

tandis que $E(\lambda p) + E(\lambda q)$ ne recevra que l'accroissement

$$\lambda p \lambda q - (\lambda p - 1)(\lambda q - 1) = \lambda p + \lambda q - 1.$$

Par conséquent, on aura pour toutes les valeurs de λ, l'égalité suivante :

$$\sum_{\omega=0}^{\omega=E(\lambda q)} E\left(\omega \frac{p}{q}\right) + \sum_{\omega=0}^{\omega=E(\lambda p)} E\left(\omega \frac{q}{p}\right) = E(\lambda p) E(\lambda q) + L,$$

où L désigne combien de fois px et qx deviennent entiers lorsque x croît de zéro à λ, ou, si l'on veut, le nombre des solutions positives moindres que λ de l'équation

$$(p+q)\,x = E(px) + E(qx).$$

Supposons maintenant p et q entiers, et $\lambda = \frac{k'}{k}$, k et k' étant aussi entiers avec la condition $k' < k$. En désignant par e et f les résidus minima positifs de p et q suivant le module k, les quantités $k'e$ et $k'f$ soient toutes deux moindres que k et le théorème se présente sous la forme suivante :

$$\sum_{\omega=0}^{\omega=\frac{k'(q-f)}{k}} E\left(\omega \frac{p}{q}\right) + \sum_{\omega=0}^{\omega=\frac{k'(p-e)}{k}} E\left(\omega \frac{q}{p}\right) = \left(\frac{k'}{k}\right)^2 (p-e)(q-f).$$

Lorsque $e=1$, $f=1$, les inégalités $k'e < k$, $k'f < k$ ont séparément lieu et on obtient l'équation

$$\sum_{\omega=0}^{\omega=\frac{k'(q-1)}{k}} E\left(\omega \frac{p}{q}\right) + \sum_{\omega=0}^{\omega=\frac{k'(p-1)}{k}} E\left(\omega \frac{q}{p}\right) = \frac{k'^2 (p-1)(q-1)}{k^2},$$

qui donne le théorème d'Eisenstein en posant $k'=1$. On voit aussi qu'on aura toujours si e et f sont les résidus minima de p et q par rapport au module k,

$$\sum_{\omega=0}^{\omega=\frac{q-f}{k}} E\left(\omega \frac{p}{q}\right) + \sum_{\omega=0}^{\omega=\frac{p-e}{k}} \Sigma\left(\omega \frac{q}{p}\right) = \frac{(p-e)(q-f)}{k^2}.$$

Il m'a paru qu'une démonstration tellement simple, on peut presque dire intuitive, de la proposition fondamentale de la théorie des résidus quadratiques, par l'emploi d'une variable continue, ne serait pas sans intérêt pour les géomètres.

31.

ON PONCELET'S APPROXIMATE LINEAR VALUATION OF SURD FORMS.

[*Philosophical Magazine*, XX. (1860), pp. 203—222.]

M. PONCELET'S method of approximately representing surd forms, and more particularly the square roots of homogeneous quadratic functions, by linear functions of the variables, is given in *Crelle's Journal*, Vol. XIII. 1834, pp. 277—291, under the title "Sur la Valeur approchée des radicaux." By this method, as applied to two variables, the resultant of two forces in a plane may be approximately expressed as a linear function of its two components, a case fully considered by M. Poncelet; and tables have been worked out applicable to this case, which appear to have been found of great utility in some important problems of mechanical and practical engineering. But the illustrious author of this beautiful method has left his theory imperfect in respect of its application to three variables.

To supply this slight but not unimportant omission, and to indicate how this more general case admits of being treated, more especially with reference to the approximate representation of the resultant of three forces in space as a linear function of its three components, is the object of this communication. At the close of the memoir referred to, M. Poncelet uses these words:—"Il serait inutile de pousser plus loin cet examen (referring to a discussion of the form $\sqrt{(a^2-b^2)}$), attendu que dans les applications de la mécanique aux machines les radicaux de la forme $\sqrt{(a^2-b^2)}$ sont rarement à considérer. Nous en dirons autant de ceux de la forme $\sqrt{(a^2+b^2+c^2)}$, qui représentent la résultante de trois forces rectangulaires entre elles et situées dans l'espace. D'ailleurs, si l'on connait les limites entre lesquelles demeurent compris les rapports des composantes a, b, c, ou de leurs résultantes partielles $\sqrt{(a^2+b^2)}$, &c., on pourra toujours ramener ce cas au premier de ceux que nous avons examinés," meaning to the case of $\sqrt{(a^2+b^2)}$. Now, in the first place, it is not clear how this reduction can be effected in general, or indeed in the vast majority of cases that might be proposed. For instance, if we have given

$a < \sqrt{(b^2+c^2)}$, $a > b$, $a > c$, I do not see how after, according to M. Poncelet's process, $\sqrt{(a^2+b^2+c^2)}$ is put under the form $\alpha a + \beta\sqrt{(b^2+c^2)}$ by aid of the limit $a < \sqrt{(b^2+c^2)}$, any use can be made of the other limits $a > b$, $a > c$ in further reducing this to the ultimate form $\alpha a + \alpha'\beta b + \beta'\beta c$. Or if we take the still simpler case, where a, b, c are left unlimited, in whatever way we attempt to proceed we shall obtain different approximations, according to the order in which we effect the successive reductions.

Furthermore, in those few exceptional cases where the process indicated by M. Poncelet leads to the use of all the limits given, the form arrived at is not and never can be the true *best* form, defined as such, according to M. Poncelet's own principles, as that which within the given limits has its *maximum* proportional error the least possible. Thus M. Poncelet indicates as the linear form for $\sqrt{(a^2+b^2+c^2)}$, when the given limits are $a^2 > b^2+c^2$, $b^2 > c^2$, $\cdot 96046a + \cdot 38201b + \cdot 15827c$, with a maximum error textually quoted from his memoir, ·0507. It will be seen hereafter that the true best linear form gives a maximum error about one-tenth less than this. But it would be quite easy to give examples in which the maximum error by Poncelet's process should exceed in an indefinite proportion the necessary maximum error. This, for instance, would be the case if we imposed the limitations

$$x^2+y^2 > \lambda z^2, \quad y^2+z^2 > \lambda x^2, \quad z^2+x^2 > \lambda y^2,$$

on taking λ inferior but indefinitely near to 2.

The geometrical method of demonstration given by M. Poncelet for the case of two variables, labours under the inconvenience of *beginning* with a figure of three dimensions, and consequently does not admit of being carried beyond that case, although the results for three variables geometrically stated, when the conditions of the question are set under an appropriate form, are precisely analogous to that obtained by M. Poncelet for two variables; for whilst his construction is begun in space, his result subsides to a representation *in plano*. But between these two cases there is a very marked distinction; which is, that whilst for a surd radical with two variables every change in the limits proposed gives rise to a change in the corresponding linear form, such is never the case with a surd form with three or more variables, unless the limits be expressed by a *single linear* inequality between the variables which enter into the surd form, and the surd form itself. Thus, for instance, if $\sqrt{(x^2+y^2+z^2)}$ is to be represented linearly within the limits $z > x$, $z > y$ (for greater conciseness I throughout suppose the variables to be positive), the linear representation will be precisely the same as for the single limit $z > \sqrt{(x^2+y^2)}$, or, which is the same thing, $z - \sqrt{\tfrac{1}{2}}\,\sqrt{(x^2+y^2+z^2)} > 0$; and accordingly for the problem with three variables there is usually a preliminary question to be solved, namely, to find the single inequality of the

kind proposed which involves the satisfaction of the given limits, and is capable of being substituted for them without increasing the maximum proportional error. This preliminary question may be reduced, as will be seen, to an elementary geometrical form, and is strictly tantamount to the problem following:—Imagine a pincushion with a number of pins stuck into it, to find the least ring which can be made to take them all in,—a problem proposed by myself some four or five years ago with reference to points in a plane, in the *Quarterly Mathematical Journal*, and of which Professor Peirce of Cambridge University, U.S., has favoured me with a complete solution, which is equally applicable to the sphere, the case with which we shall be principally concerned in what follows.

I shall begin, then, with supposing R to be an integer homogeneous quadratic function of x, y, z, where x, y, z, R are subject to the linear inequality $Ax + By + Cz - \sqrt{R} > 0$. The geometrical solution, as such, will be seen to be equally applicable to the case of two, and the analytical representation to which it leads to any number of variables.

The problem to be solved is to find a linear form $Lx + My + Nz$ such that the greatest value of $\dfrac{Lx + My + Nz}{\sqrt{R}} - 1$ shall have the least possible arithmetical magnitude, without regard to *sign* as positive or negative, for all values of x, y, z satisfying the proposed inequality.

It is clear that, as the entire question is one of ratios, we may subject x, y, z to the condition expressed by $R=1$ without affecting the result; in other words, we may consider x, y, z as the coordinates of a point limited to lie on the segment of the surface $R=1$ cut off by the plane $Ax+By+Cz=1$. Suppose, then, that $Lx + My + Nz$ is the linear form sought. The proportional error is $Lx + My + Nz - 1$; so that if we draw the plane

$$Lx + My + Nz - 1 = 0,$$

the error is expressible geometrically (paying no attention to sign) as the quotient of the perpendicular upon this plane from any point x, y, z in the segment, namely, $\dfrac{Lx + My + Nz - 1}{\sqrt{(L^2 + M^2 + N^2)}}$, divided by the perpendicular from the origin to the same plane, namely, $\dfrac{1}{\sqrt{(L^2+M^2+N^2)}}$. Hence, then, the geometrical question to be resolved is simply to draw a plane for which the greatest value of this quotient, restricted to points within the segment, shall be the least possible. From this it is immediately seen to follow, that the portion of the surface cut off by the plane $Lx + My + Nz - 1 = 0$ must be a portion of the segment cut off by the given plane $Ax + By + Cz - 1 = 0$. And its actual position may be determined by means of a principle generally known, but which, as it will occupy but a few words, it may be well to deduce from first principles.

Suppose there are $(r+1)$ quantities, each containing the same system of r parameters; for greater brevity, say three quantities, p, q, r, each functions of the same two parameters λ, μ: let us call the greatest of the quantities p, q, r, corresponding to assigned values of λ, μ, the *dominant*; so that, according as we change λ, μ, the name of the dominant is liable to change; and that we wish to find M the minimum value of the dominant upon the supposition that the variations of p, q, r in respect to λ or μ are never simultaneously zero, and may be made positive or negative at will; then M will be found from the equations $M=p=q=r$. For if we had $M=p$ and $p>q$, $p>r$, by varying at will λ or μ we could make δp negative; and consequently since by hypothesis p differs sensibly from q and r, the dominant of $p+\delta p$, $q+\delta q$, $r+\delta r$ would necessarily be less than that of p, q, r, and thus M would not be the minimum dominant.

In like manner, if $M=p=q$, $p>r$, we could by means of the equations

$$\frac{dp}{d\lambda}\delta\lambda+\frac{dp}{d\mu}\delta\mu=-\epsilon,$$

$$\frac{dq}{d\lambda}\delta\lambda+\frac{dq}{d\mu}\delta\mu=-\eta,$$

so determine $\delta\lambda$, $\delta\mu$ as to diminish simultaneously p and q; and thus the dominant of $p-\epsilon$, $q-\eta$, $r+\delta r$ would, as before, be less than that of p, q, r. The same reasoning applies to any number $(r+1)$ functions of r variables. And if the number of functions should exceed $r+1$, it would still serve to show that when the dominant is a minimum, $(r+1)$ out of the whole number of the functions must all alike represent that dominant. Thus leaving for a moment in our original problem the case of three variables, and going down to that of only two variables, in which case we have to deal with a curve of the second order in lieu of a surface, and are to suppose that a segment of such curve is cut off by a right line A, and are required to draw another right line B such that the maximum square of the quotient of a perpendicular upon B from any point in the segment by the perpendicular from the centre upon B is to be a minimum, we evidently have to solve the same problem as if we had to find the least value of the dominant of three quantities involving two parameters, two being the number of constants required to fix the line B; those three quantities being the squares of the fractions whose numerators are the three perpendiculars from the extremities of A, and from the vertex of the arc cut off by B upon B, and their denominators the perpendicular upon B from the origin; accordingly the line B must be so chosen as to make the three perpendiculars in the numerators, without reference to sign, all equal, so that B is parallel to A, and bisects the sagitta of the segment cut off by A, that is, the longest perpendicular from any point in the segment upon A.

In the case of R being, as originally supposed, a function of x, y, z, we may take an indefinite number of points in the section of the surface $R=1$ made by the plane $Ax+By+Cz-1=0$, and the summit of the segment made by the plane to be determined $Lx+My+Nz=1$, and may show by the same reasoning as above (there being now three parameters) that four of these perpendiculars must be equal *inter se*, which proves, to begin with, that at all events the two planes must be parallel; and then the reasoning applied to two functions of one parameter will further show that this plane must bisect the sagitta of the segment cut off by the *given plane* $Ax+By+Cz-1=0$*. And we have now a geometrical solution of the question, which it is important to observe is in general, but, as will be presently seen, not universally applicable to the case when the limiting relations of x, y, z are defined by means of the position of a variable point limited to lie within a triangular area upon the surface $R=1$, whose sides are determined by the traces upon that surface of three planes drawn through the origin; the plane drawn through the angular points of this triangle will then take the place of the plane $Ax+By+Cz-1=0$ in the preceding investigation.

The next thing to be done is to obtain the quantities L, M, N in terms of A, B, C, and the coefficients of R, which is an easy matter to accomplish. Let

$$R = ax^2+by^2+cz^2+2fyz+2gzx+2hxy=\phi(x, y, z),$$

and call ξ, η, ζ the coordinates at the summit of the segment; the equation to the tangent plane at that point, which is of the form $Ax+By+Cz=0$, will be identical with

$$(a\xi+h\eta+g\zeta)X+(h\xi+b\eta+f\zeta)Y+(g\xi+f\eta+c\zeta)Z=1.$$

Hence
$$a\xi+h\eta+g\zeta=\frac{A}{\sigma},$$

$$h\xi+b\eta+f\zeta=\frac{B}{\sigma},$$

$$g\xi+f\eta+c\zeta=\frac{C}{\sigma},$$

and
$$\frac{A}{\sigma}\xi+\frac{B}{\sigma}\eta+\frac{C}{\sigma}\zeta=1;$$

* The absolute liberty of the plane sought for $(Lx+My+Nz=1)$ to take up all positions in space, and the absence of singular points in the segment cut off by the plane $Ax+By+Cz=1$, suffice to show that the conditions of variation necessary for the legitimate application of the theorem employed above are satisfied. If the minimum dominant is not at one of the *points of equality* given by the theorem, it must lie either at some minimum, or at all events at some singular point of one of the functions of the system to which the dominant belongs, or else at some point corresponding to the contour, so to say, if there be one, of the space within which the parameters are contained. In the case before us, the parameters, however chosen, to fix the position of the plane are perfectly independent, so that there is no limiting contour; and it is obvious that the functions representing the distances concerned from this variable plane have no

and therefore
$$\frac{1}{\sigma^2}\frac{P\phi(A, B, C)}{\Delta\phi(A, B, C)}=1,$$

where $\Delta\phi$ is the *discriminant*, and $P\phi$ the *polar reciprocal* of $\phi(A, B, C)$. Hence
$$\sigma=\sqrt{\frac{P}{\Delta}}\,^*,$$

and the perpendicular upon the tangent plane is
$$\frac{1}{\sqrt{(A^2+B^2+C^2)}}\sqrt{\frac{P}{\Delta}}.$$

Consequently the mean between this and the perpendicular upon the given plane is
$$\frac{1}{\sqrt{(A^2+B^2+C^2)}}\frac{\sqrt{P}+\sqrt{\Delta}}{2\sqrt{\Delta}};$$

and therefore the equation to the plane required is
$$Ax+By+Cz=\frac{\sqrt{P}+\sqrt{\Delta}}{2\sqrt{\Delta}},$$

so that
$$L=\frac{2\sqrt{\Delta}}{\sqrt{P}+\sqrt{\Delta}}A,\quad M=\frac{2\sqrt{\Delta}}{\sqrt{P}+\sqrt{\Delta}}B,\quad N=\frac{2\sqrt{\Delta}}{\sqrt{P}+\sqrt{\Delta}}C,$$

$Lx+My+Nz$ being the approximate representation of $\sqrt{\{\phi(x, y, z)\}}$, and the maximum error being evidently
$$\frac{\sqrt{P}-\sqrt{\Delta}}{\sqrt{P}+\sqrt{\Delta}}.$$

These results are perfectly general, and apply to a quadratic radical of an integer homogeneous quadratic function of any number of variables; thus for $\sqrt{\{\phi(x, y, z, t)\}}$ the linear representative form is
$$\frac{2\sqrt{\Delta}\,.\,A}{\sqrt{P}+\sqrt{\Delta}}x+\frac{2\sqrt{\Delta}\,.\,B}{\sqrt{P}+\sqrt{\Delta}}y+\frac{2\sqrt{\Delta}\,.\,C}{\sqrt{P}+\sqrt{\Delta}}z+\frac{2\sqrt{\Delta}\,.\,D}{\sqrt{P}+\sqrt{\Delta}}t,$$

maxima or minima values. I do not (nor ought I to) pretend to have presented the theoretical principles involved in the limitation of the general *law of equality* with all the logical rigour and precision of which the subject might admit, as this would be beside my present object, which is not to call in question the grounds of admitted truth applicable to the question in hand, but to advance it one step further in the direction of practical application.

* We see from the above, that if $Ax+By=1$, or $Ax+By+Cz=1$ be the equation to the chordal line or plane of a segment of a line or surface of the second degree, the ratio of the perpendiculars to such line or plane from the centre of the line or surface and the vertex of the segment respectively, or, which is the same thing, of a ray to any point in the segment to the portion of this ray produced, intercepted between the line or surface and the tangent at the vertex, is expressed by $\sqrt{\Delta}:\sqrt{P}$. It may at first sight appear strange that P should be of the *form* of a *contravariant* (in lieu of a covariant); but it must be remembered that the axes to which the line or surface and its chord are referred are supposed to be orthogonal, and for orthogonal substitutions, contravariants and covariants are indistinguishable.

and the greatest proportional error is still

$$\frac{\surd P - \surd \Delta}{\surd P + \surd \Delta};$$

D signifying the discriminant, and P the polar reciprocal of $\phi(A, B, C, D)$.

For the sphere, the perpendicular upon any tangent plane being 1, the linear form ought to be that obtained from the equation $Ax + By + Cz = K$, where

$$\frac{K}{\surd(A^2+B^2+C^2)} = \frac{1}{2}\left(1 + \frac{1}{\surd(A^2+B^2+C^2)}\right),$$

or

$$K = \tfrac{1}{2}\{\surd(A^2 + B^2 + C^2) + 1\},$$

that is to say, the approximation is

$$\frac{2A}{1+\surd(A^2+B^2+C^2)}\,x + \&c.,$$

the maximum error being

$$\frac{\surd(A^2 + B^2 + C^2) - 1}{\surd(A^2 + B^2 + C^2) + 1},$$

which is easily seen to agree with the general formulæ above given.

When, as is usually the case in applying these results, the plane $Ax + By + Cz - 1 = 0$ is not directly given, but is to be found as the plane passing through three given points whose coordinates are a, b, c; a', b', c'; a'', b'', c'' respectively, we may use the equations

$$A = \frac{F}{Q}, \quad B = \frac{G}{Q}, \quad C = \frac{H}{Q},$$

where

$$F = (b'c'' - b''c') + (b''c - bc'') + (bc' - b'c),$$

$$G = (c'a'' - c''a') + (c''a - ca'') + (ca' - c'a),$$

$$H = (a'b'' - a''b') + (a''b - ab'') + (ab' - a'b),$$

$$Q = \begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix}.$$

But it may also sometimes be needful in practice, as will presently appear, to determine the plane with immediate reference to only two points upon the surface.

Application to the surd form which represents the resultant of three forces at right angles to each other.

Here $R=\sqrt{(x^2+y^2+z^2)}$, and $R=1$ represents a sphere. Two cases will be shown to arise. The first, the more frequent one, is that already alluded to, where a limiting plane has to be drawn through three given points. For this case, using F, G, H in the sense in which they have immediately above been employed, the linear representation of $\sqrt{(x^2+y^2+z^2)}$ becomes

$$\frac{2F}{Q+N}x+\frac{2G}{Q+N}y+\frac{2H}{Q+N}z,$$

with a maximum proportional error

$$\frac{N-Q}{N+Q},$$

N representing

$$\sqrt{(F^2+G^2+H^2)}.$$

The second case is where the limiting plane has to be drawn through two points upon the sphere so as to cut it in a circle, of which the line joining the two points is a diameter.

In this case, calling the coordinates of the two points respectively α, β, γ; α', β', γ', and writing $\alpha\alpha'+\beta\beta'+\gamma\gamma'=m$, it is easily seen that the perpendicular upon the limiting plane is $\sqrt{\dfrac{1+m}{2}}$, and consequently the perpendicular upon the plane

$$Lx+My+Nz=1 \text{ is } \frac{1}{2}\left\{1+\frac{\sqrt{(1+m)}}{2}\right\}.$$

Also this plane being parallel to the limiting plane, is perpendicular to the line joining the origin to the point

$$x:y:z::\frac{\alpha+\alpha'}{2}:\frac{\beta+\beta'}{2}:\frac{\gamma+\gamma'}{2},$$

and therefore

$$L=\frac{\alpha+\alpha'}{\rho},\quad M=\frac{\beta+\beta'}{\rho},\quad N=\frac{\gamma+\gamma'}{\rho},$$

and
$$\frac{\rho}{\sqrt{\{(\alpha+\alpha')^2+(\beta+\beta')^2+(\gamma+\gamma')^2\}}}=\frac{1}{2}\left\{1+\sqrt{\frac{1+m}{2}}\right\};$$

that is to say,

$$\rho=\sqrt{\{2(1+m)\}}\cdot\frac{1}{2}\left(1+\sqrt{\frac{1+m}{2}}\right)$$
$$=\tfrac{1}{2}\left[\sqrt{\{2(1+m)\}}+(1+m)\right];$$

so that the linear form required is

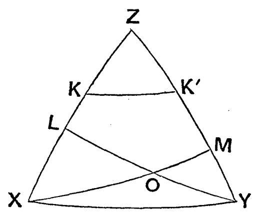
$$[\sqrt{\{2(1+m)\}}+1+m]\left\{\frac{\alpha+\alpha'}{2}x+\frac{\beta+\beta'}{2}y+\frac{\gamma+\gamma'}{2}z\right\},$$

with a maximum proportional error

$$\frac{\sqrt{2}-\sqrt{(1+m)}}{\sqrt{2}+\sqrt{(1+m)}}.$$

(m is of course identical with the cosine of the angle between the radii joining the two given points.)

The conditions of inequality which obtain between x, y, z may be, and usually will be, such as correspond to the limitation of the point (x, y, z) to an area contained within a triangle or polygon upon the surface of the sphere. Thus take X, Y, Z each a quadrant apart from the other, the points where the surface of the sphere $x^2+y^2+z^2=1$ is pierced by the axes. If no limitation is placed upon the values of x, y, z further than the one throughout supposed of their remaining always positive, the limiting area will be XYZ. If we suppose



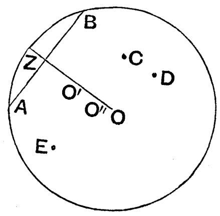
$$z > k\sqrt{(x^2+y^2)},$$

we may take $\tan XK = k$, and drawing the small circle KK', ZKK' will be the limiting area; if, again, $z < k\sqrt{(x^2+y^2)}$, $KK'YX$ will be the limiting area; if, again, $z < k\sqrt{(x^2+y^2)}$, $z > lx$, $z > my$ be the limiting conditions, taking $\tan LX = l$, $\tan MY = m$, and drawing LY, XM to intersect in O, $KK'MOL$ will be the corresponding area, and so in general. Even so simple a set of conditions as $z > x$, $z > y$ it is seen will give rise to a quadrilateral area, limited in the figure by $ZLOM$, when $ZL = ZM = 45°$. Thus, then, we approach the preliminary question to which allusion has been already made, which is to determine the *least circle* that will cut off from a given sphere a segment containing a given system of points lying upon it. The solution is precisely the same, substituting arcs of great circles for right lines, as the problem of drawing upon a plane the least circle containing a set of points given in the plane.

We may, in the first place, obviously reject all those points that are contained within the contour formed by arcs joining the remaining points, so that the case of points lying at the angles of a convex polygon alone remains to be studied. Now if we confine our attention even to the simplest case of a system of three points, we shall see at once that two cases arise. If a circle be drawn through them, and these three points do not lie in the same semicircle, no smaller circle than this can be drawn to contain the

three; but if they do lie in the same semicircle, it is obvious that a circle described upon the line joining the outer two as a diameter will be smaller than the circle passing through all three, and will contain them all. It was this simple but striking fact in the geometry of situation which led me to propose the question for any number of points in the *Quarterly Mathematical Journal*; and as Prof. Peirce's exhaustive method of solution has not appeared in print, I may take this occasion of presenting it.

Let A, Z, B, C, D, E be the given points. Let AZB be a circle whose centre is drawn through A, Z, B, chosen so as to include all the others; then if A, Z, B are not contained in the same semicircle, AZB is the circle required. But if AZB be less than a semicircle, as in the figure, we may first reject the consideration of all the points contained between the arc AB and its chord. We must then find O', O'', &c., the centres of the circles passing through A, B, C; A, B, D, &c.: these will all lie in the same straight line $O'O''O$. Selecting the one nearest to O, say O'', we describe the corresponding circle, in which AC will now take the place of AB in the former circle. If the points A, B, C are not contained in less than a semicircle, that is, if ABC is an acute-angled or right-angled triangle, ABC is the circle required; but if they do lie within the same semicircle so that ABC forms an obtuse angle, B will now have to be rejected, and we must find a new centre as before, and so on continually. By this process we must inevitably at last exhaust all the given points; and the final circle so obtained will be the circle sought, unless the three points through which it has been drawn are distributed over the same semicircle, in which case the circle required is that described upon the chord joining the two extreme points as its diameter. The solution will evidently be *unique*, and (as already hinted at) merely require the construction upon the sphere either of a circle passing through a certain set of three out of all the given points, or else passing through only two of them, so as to be perpendicular to the radius bisecting their joining line.



If we imagine an india-rubber band (similar, we may suppose, in form to a "parlour quoit" but more elastic) having the faculty of maintaining its figure always circular, or which is more simple in the case before us, capable of maintaining itself in the same plane, and imagine this sufficiently stretched over the surface of the sphere to contain all the given points (represented by very minute pins' heads given upon it), this band will by its contraction upon the surface of the sphere, however originally placed, imitate the steps of Prof. Peirce's method of solution; and after (it may be) passing through and quitting successive sets of three points, come to a position of *geometrical equilibrium*, either when its circumference contains a triad of the

given points lying at the angles of an acute-angled triangle, or a duad at the extremities of one of its diameters*.

The following observation, which constitutes a veritable theorem, and is presupposed in Prof. Peirce's solution, is very important:—"Any circle being found which, either passing through three of the given points such that no two of their joining lines form an obtuse angle, or which described upon the line joining two of the given points as a diameter, includes all the rest, is the minimum circle which contains all the points of the given cluster; so that one, and only one, circle exists satisfying the above *alternative* condition."

It may be instructive to proceed to the application of the method now fully explained to some of the more salient cases of inequality, it being understood that these cases are given to afford some general notion of the precision of the method, and by no means as specimens of such as it would be applied to in practice, for which the limits I shall suppose would be far too wide to furnish any useful result.

Example 1. x, y, z unlimited. Here the values of F, G, H, Q are the minor determinants of the matrix,

$$\begin{vmatrix} 1, & 0, & 0, & \bar{1} \\ 0, & 1, & 0, & \bar{1} \\ 0, & 0, & 1, & \bar{1} \end{vmatrix}$$

$F=G=H=1$, $Q=1$, and the linear approximation to $\sqrt{(x^2+y^2+z^2)}$ becomes $\frac{2}{\sqrt{3}+1}x+$ &c., or $(\sqrt{3}-1)x+(\sqrt{3}-1)y+(\sqrt{3}-1)z$, or say

$$\cdot 73025x+\cdot 73025y+\cdot 73025z,$$

* The annexed is a more complete and, I think, a correct account of what would happen to the band under the supposed conditions. It will begin to move parallel to its own plane, and continue so to do until it comes in contact with one of the physical points (call it A) upon the surface of the sphere. Supposing that the position of equilibrium is not then attained by the band passing at the same moment through *one* other point at the opposite extremity of a diameter to A, or through *two* other of the given points forming a non-obtuse-angled triangle with A, it will begin to revolve (always contracting the while) about a tangent at A to its intersection with the sphere as an axis, until it meets a second of the given points, say B. If the line AB is a diameter of the band, *cadit quæstio*, the problem is solved. If not, the band will go on further contracting, revolving meanwhile round AB as an axis until either AB *becomes* a diameter in virtue of the contraction of the band's dimensions (and so the problem is solved), or else before this can take place the band is arrested at a third point C, either forming a non-obtuse-angled triangle with AB and so solving the problem, or else an obtuse-angled triangle with AB and lying exterior to the arc AB on one side of it or the other; on the latter supposition the line joining C with the extremity of AB nearest to it, will (it appears to me) form a new axis of rotation for the band, which will quit the further extremity of the old axis, and thus the motion will continue with an intermitting change of axes, until at last the band either finds out for itself an axis which in the course of the contraction becomes a diameter, or else brings the band into contact with a third point forming a non-obtuse-angled triangle with such axis, in either of which cases the minimum periphery is attained, the contraction comes to an end, and the problem is solved.

with a maximum proportional error

$$\frac{\sqrt{3}-1}{\sqrt{3}+1}, \text{ or } 2-\sqrt{3} = \cdot 26895.$$

The corresponding error for $\sqrt{(x^2+y^2)}$ under the form $\cdot 8284x + \cdot 8284y$ is $\cdot 17160$, or about two-thirds of the one in question*.

Example 2. $z > \sqrt{(y^2+x^2)}$. Here the determining matrix is

$$\begin{vmatrix} 0, & 0, & 1, & \bar{1} \\ 0, & \sqrt{\tfrac{1}{2}}, & \sqrt{\tfrac{1}{2}}, & \bar{1} \\ \sqrt{\tfrac{1}{2}}, & 0, & \sqrt{\tfrac{1}{2}}, & \bar{1} \end{vmatrix}$$

$$F = G = \sqrt{\tfrac{1}{2}} - \tfrac{1}{2} = \cdot 207107$$

$$H = \tfrac{1}{2}$$

$$Q = \tfrac{1}{2}$$

$$N^2 = F^2 + G^2 + H^2 = 1 - \sqrt{\tfrac{1}{2}} = \cdot 292893$$

$$N = \cdot 541196$$

$$N + Q = 1\cdot 041196 \qquad N - Q = \cdot 041196.$$

Thus the linear approximation becomes

$$\cdot 397825x + \cdot 397825y + \cdot 960430z,$$

with a maximum error $\cdot 039493$.

Example 3. $z > \sqrt{(y^2+x^2)}, y > x$. This is M. Poncelet's example (*Crelle*, Vol. XIII. p. 291). His a, b, c correspond respectively with my z, y, x; there are some misprints in line 6 of this page (in M. Poncelet's Memoir) which may perplex the reader; it is intended to stand thus:

$$\delta\sqrt{(a^2+b^2+c^2)} + \beta\delta'\sqrt{(b^2+c^2)} = \sqrt{(a^2+b^2+c^2)} \cdot \left(\delta + \beta\delta'\sqrt{\frac{b^2+c^2}{a^2+b^2+c^2}}\right).$$

Here the determining matrix corresponds to the area ZKN (the coordinates of N being found from the equations $z^2 = x^2+y^2$, $y = x$, $z^2+x^2+y^2 = 1$), and the matrix will be as subjoined.

* It would have been more exact to have treated this as a case of a circle to be drawn through four points, namely, Z the middle points of ZX, ZY and the middle or lowest point (in reference to Z) of the small circle drawn through these two, and having Z for its pole. But it is easily seen that the small circle drawn through the three former will contain the one last named, for the tangent of its circular radius will be $\sqrt{2} \times \tan\frac{45^\circ}{2}$, and consequently its summit will be further from Z than from the point in question. A similar remark applies to the subsequent and some other examples.

$$\begin{vmatrix} 0, & 0, & 1, & \bar{1} \\ 0, & \sqrt{\tfrac{1}{2}}, & \sqrt{\tfrac{1}{2}}, & \bar{1} \\ \tfrac{1}{2}, & \tfrac{1}{2}, & \sqrt{\tfrac{1}{2}}, & \bar{1} \end{vmatrix}.$$

$$F = \sqrt{\tfrac{1}{2}} + \tfrac{1}{2}\sqrt{\tfrac{1}{2}} - \tfrac{1}{2} - \tfrac{1}{2} = 3\sqrt{\tfrac{1}{8}} - 1 = \cdot 060660$$

$$G = \tfrac{1}{2} - \tfrac{1}{2}\sqrt{\tfrac{1}{2}} = \tfrac{1}{2} - \sqrt{\tfrac{1}{8}} = \cdot 146447$$

$$H = \tfrac{1}{2}\sqrt{\tfrac{1}{2}} = \cdot 353553$$

$$Q = \tfrac{1}{2}\sqrt{\tfrac{1}{2}} = \cdot 353553$$

$$\begin{aligned} N^2 = F^2 + G^2 + H^2 &= \tfrac{17}{8} + \tfrac{3}{8} + \tfrac{1}{8} - 7\sqrt{\tfrac{1}{8}} \\ &= \tfrac{21}{8} - \tfrac{1}{2}\sqrt{24\cdot 5} \\ &= 2\cdot 625 - 2\cdot 474874 \\ &= \cdot 150126 \end{aligned}$$

$$N = \cdot 387461, \quad N + Q = \cdot 741014, \quad N - Q = \cdot 033908.$$

The maximum error therefore is $\dfrac{33908}{741014} = \cdot 0457$, or about one-tenth less than that given by M. Poncelet's form.

$$\frac{2F}{N+Q} = \frac{6066}{37051} = \cdot 1637,$$

$$\frac{2G}{N+Q} = \frac{14645}{37051} = \cdot 3953,$$

$$\frac{2H}{N+Q} = \frac{35355}{37051} = \cdot 9542.$$

The last of these quantities is less, the first two greater, than the corresponding coefficients in M. Poncelet's form.

Examples 4 *and* 5. The inequality system, $\sqrt{(x^2+y^2)} > z > y > x$, is represented by the triangle KNQ, and the corresponding determining matrix will be

$$\begin{vmatrix} 0, & \sqrt{\tfrac{1}{2}}, & \sqrt{\tfrac{1}{2}}, & \bar{1} \\ \tfrac{1}{2}, & \tfrac{1}{2}, & \sqrt{\tfrac{1}{2}}, & \bar{1} \\ \sqrt{\tfrac{1}{3}}, & \sqrt{\tfrac{1}{3}}, & \sqrt{\tfrac{1}{3}}, & \bar{1} \end{vmatrix}.$$

So, too, the inequality system, $\sqrt{(x^2+y^2)} < z < y > x$, has for its locus the triangle ZKN, its determining matrix

$$\begin{vmatrix} 0, & \sqrt{\tfrac{1}{2}}, & \sqrt{\tfrac{1}{2}}, & \bar{1} \\ \tfrac{1}{2}, & \tfrac{1}{2}, & \sqrt{\tfrac{1}{2}}, & \bar{1} \\ 0, & 0, & 1, & \bar{1} \end{vmatrix}.$$

It would be superfluous to go on multiplying numerical examples, that may be left to those who feel the want of the Tables which this method affords. If the limiting conditions were supposed to be $z > y$, $z > x$, this

would correspond to the quadrilateral $ZK'OK$ in the last figure: it may easily be ascertained that a circle passing through $K'ZK$ would contain O, and would have its centre between N and Z. Hence by the application of Peirce's law, we know that the minimum circle in this case is that which can be drawn through $K'ZK$, and consequently the linear form and maximum error will be precisely the same as for the simpler case already considered, $z > \sqrt{(x^2 + y^2)}$. On the other hand, if the conditions imposed were simply $z < x$, $z < y$ (conditions, be it remembered, far wider than ever would be admitted in practice), the limiting figure becomes XOY; and since $MO < MX$ or MY, the centre of the circle through XOY would fall under XY, so that the limiting circle in this case would be that having M for its pole; the linear substitutive form would not contain z, but would be the same as if z did not appear, namely $\cdot 96046x + \cdot 960467y$, with $\cdot 03954$ as the maximum proportional error. The same remark would apply to the system of conditions $z < \lambda x$, $z < \lambda y$ for any value of λ not inferior to $\sqrt{\tfrac{1}{2}}$.

The conditions $z > x$, $z > y$, $z < \sqrt{(x^2 + y^2)}$ would correspond to the limiting area $KK'O$, which would give rise to the determining matrix,

$$\begin{vmatrix} 0, & \sqrt{\tfrac{1}{2}}, & \sqrt{\tfrac{1}{2}}, & \bar{1} \\ \sqrt{\tfrac{1}{2}}, & 0, & \sqrt{\tfrac{1}{2}}, & \bar{1} \\ \sqrt{\tfrac{1}{3}}, & \sqrt{\tfrac{1}{3}}, & \sqrt{\tfrac{1}{3}}, & \bar{1} \end{vmatrix}.$$

The condition $z < \sqrt{(x^2 + y^2)}$ would correspond to a limiting area, $KK'XY$. If KY be bisected in G, and $K'X$ in G', and $G'YGX$ intersect in H, it is obvious that a small circle may be described with H as its pole passing through all four points X, Y, K, K', which will be the minimum circle of limitation. To assign the determining matrix, we may take any three of these four points, as, for example, Y, X, K, which will give

$$\begin{vmatrix} 0, & 1, & 0, & \bar{1} \\ 1, & 0, & 0, & \bar{1} \\ \sqrt{\tfrac{1}{2}}, & 0, & \sqrt{\tfrac{1}{2}}, & \bar{1} \end{vmatrix}.$$

This gives

$$Q = \sqrt{\tfrac{1}{2}} = \cdot 70711,$$
$$F = \sqrt{\tfrac{1}{2}},\ G = \sqrt{\tfrac{1}{2}},\ H = 1 - \sqrt{\tfrac{1}{2}} = \cdot 29289,$$
$$N^2 = \tfrac{5}{2} - \sqrt{2} = 1\cdot 085786,$$
$$N = 1\cdot 04200,$$
$$N + Q = 1\cdot 74911, \quad N - Q = \cdot 33489.$$

The linear approximation is accordingly

$$\cdot 8090x + \cdot 8090y + \cdot 3351z,$$

with a maximum proportional error $\cdot 1914$.

Finally, for $z > y$, $y > x$ the limiting triangle will be ZKO, the determining matrix

$$\left|\begin{array}{cccc} 0, & 0, & 1, & \bar{1} \\ 0, & \sqrt{\tfrac{1}{2}}, & \sqrt{\tfrac{1}{2}}, & \bar{1} \\ \sqrt{\tfrac{1}{3}}, & \sqrt{\tfrac{1}{3}}, & \sqrt{\tfrac{1}{3}}, & \bar{1} \end{array}\right|.$$

$$F = \sqrt{\tfrac{1}{2}} - \sqrt{\tfrac{1}{3}} = \cdot 1297, \quad G = \sqrt{\tfrac{1}{3}}\,\{1 - \sqrt{\tfrac{1}{2}}\} = \cdot 1692,$$

$$H = \sqrt{\tfrac{1}{6}} = \cdot 4082,$$

$$N^2 = \tfrac{3}{2} - \sqrt{\tfrac{2}{3}} - \sqrt{\tfrac{2}{9}} = \cdot 21207,$$

$$N = \cdot 4605, \qquad N + Q = \cdot 8687,$$

$$Q = \sqrt{\tfrac{1}{6}} = \cdot 4082, \qquad N - Q = \cdot 0523.$$

The linear approximation is $\cdot 2986x + \cdot 3895y + \cdot 9397z$, with a maximum error $\cdot 06$ (more precisely $\cdot 0602$). This is a trifle beyond half as much again as the maximum error of the best linear approximation to $\sqrt{(x^2 + y^2)}$, subject to the limitation $x > y$, which (see Poncelet's Memoir, p. 280) is a little under $\cdot 04$.

Poncelet has shown that for $\sqrt{(x^2 + y^2)}$, when x, y are the coordinates of a point limited within a sector whose bounding radii make angles ϕ and ψ with the axis of X, the approximate linear form is

$$\frac{\cos \frac{1}{2}(\phi + \psi)}{\cos^2 \dfrac{\phi - \psi}{4}}\,x + \frac{\sin \frac{1}{2}(\phi + \psi)}{\cos^2 \dfrac{\phi - \psi}{4}}\,y,$$

with a maximum error $\tan^2 \dfrac{\phi - \psi}{4}$.

In like manner it follows immediately from the method given in the text, that if the summit of the limiting segment make angles λ, μ, ν with the axes of X, Y, Z, and its spherical radius be ρ, the approximate expression for $\sqrt{(x^2 + y^2 + z^2)}$ is

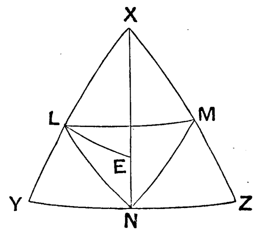
$$\frac{\cos \lambda}{\cos^2 \dfrac{\rho}{2}}\,x + \frac{\cos \mu}{\cos^2 \dfrac{\rho}{2}}\,y + \frac{\cos \nu}{\cos^2 \dfrac{\rho}{2}}\,z,$$

with a maximum error $\tan^2 \dfrac{\rho}{2}$, which expressions are the precise analogues of the former, as will immediately appear from the consideration that the summit of the spherical segment corresponds with the centre of the circular arc.

As an example of the use of these formulæ, suppose the given limits to be

$$x < \sqrt{(y^2 + z^2)}, \quad y < \sqrt{(z^2 + x^2)}, \quad z < \sqrt{(x^2 + y^2)}.$$

If we bisect the quadrants XY, YZ, ZX in L, N, M respectively, the variable point will be limited to lie in LMN, and the base of the corresponding segment will be the circle passing through LMN whose summit will be at E, the point where the perpendicular to XY at L and the arc bisecting the angle X meet.



Here then we have

$$\rho = LE, \quad \lambda = \mu = \nu = XE,$$

$$\tan\rho = \cos 45° = \sqrt{\tfrac{1}{2}}, \qquad \cot\lambda = \sqrt{\tfrac{1}{2}},$$

$$\cos\rho = \sqrt{\tfrac{2}{3}}, \quad \cos^2\frac{\rho}{2} = \tfrac{1}{2}\{1 + \sqrt{\tfrac{2}{3}}\}, \quad \cos\lambda = \sqrt{\tfrac{1}{3}},$$

$$\tan^2\frac{\rho}{2} = \frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}.$$

Hence the linear approximation is

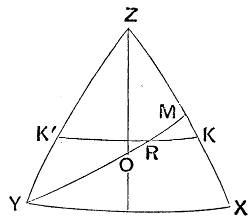
$$\frac{2}{\sqrt{3}+\sqrt{2}}(x+y+z) = 2(\sqrt{3}-\sqrt{2})\{x+y+z\}$$
$$= {\cdot}6356744\,(x+y+z),$$

with a maximum proportional error $5 - \sqrt{24} = {\cdot}10102$.

More generally, if we assume the system of conditions

$$\sqrt{(x^2+y^2)} > cz, \quad \sqrt{(y^2+z^2)} > cx, \quad \sqrt{(z^2+x^2)} > cy,$$

c being any number intermediate between 1 and $\sqrt{2}$, if in the figure annexed, we take $\tan ZK = \tan ZK' = c$, and join KK' by a small circle intersecting YM, which bisects ZX, in R, O remaining still the summit of XZY, it is easy to perceive that the limiting area will be included within the triangular space cut out between KK' and the two other analogous small circles; λ, μ, ν will remain the same as before, and OR will represent ρ. Accordingly we have from the quadrantal triangle ZYR,



$$\cos ZR = \sin RY \cos RYZ,$$

that is
$$\sin RY = \sqrt{\frac{2}{c^2+1}};$$

therefore
$$RY = \tan^{-1}\sqrt{\frac{2}{c^2-1}},$$

$$\tan\rho = \tan RO = \tan(RY - OY) = \frac{\sqrt{\left(\frac{2}{c^2-1}\right)} - \sqrt{2}}{1 - \frac{2}{\sqrt{(c^2-1)}}}$$

$$= \sqrt{2}\left\{\frac{1-\sqrt{(c^2-1)}}{\sqrt{(c^2-1)}-2}\right\}.$$

When $c = \sqrt{2}$, this vanishes; and when $c > \sqrt{2}$, the conditions become incompatible.

The equations $\tan\phi = \sqrt{\dfrac{2}{c^2-1}}$, or $\cos 2\phi = \dfrac{c^2-3}{c^2+1}$, and

$$\rho = \phi - \tan^{-1}\sqrt{2} = \phi - 54^\circ\, 44',$$

are well adapted for logarithmic computation. Suppose

$$c = \tfrac{4}{5}, \quad \cos 2\phi = -\tfrac{11}{25} = -\cdot 44, \quad 2\phi = 180^\circ - 63^\circ\, 54' = 116^\circ\, 6',$$

$$\phi = 58^\circ\, 3', \quad \rho = 3^\circ\, 19',$$

giving a maximum error $\tan(1^\circ\, 39'\, 30'')^2 = \cdot 0008375$. The linear form corresponding to this is

$$\frac{2\sqrt{\frac{1}{3}}}{1+\cos\rho}\{x+y+z\} = \cdot 5778x + \cdot 5778y + \cdot 5778z.$$

If $c < 1$, the formula changes; the limiting area, from a triangle, becoming a hexagon through all the angles of which a circle will admit of being drawn, which circle will give the limiting segment. ρ becomes the third side of a spherical triangle of which the other two sides are $\tan^{-1}\sqrt{2}$ and $\tan^{-1} c$ respectively, and the included angle 45°; so that

$$\cos\rho = \sqrt{\frac{1}{3(1+c^2)}} + \sqrt{\frac{c}{3(1+c^2)}} = (1+\sqrt{c})\sqrt{\frac{1}{3(1+c^2)}},$$

and the maximum error, that is $\tan^2\dfrac{\rho}{2}$, becomes

$$\frac{\sqrt{\{3(1+c^2)\}} - 1 - \sqrt{c}}{\sqrt{\{3(1+c^2)\}} + 1 + \sqrt{c}}.$$

The only real difficulty in extending M. Poncelet's method in the manner pursued in the above unpretending study, consisted in forming a clear preconception of the mode in which any given system of limits require for the purpose in view to be regarded, namely, as enveloped, so to say, in a single condition (no wider than absolutely necessary) expressed by a linear equation between the given surd function and the variables which enter into it.

I may in conclusion just observe that if the relative values of the variables be limited, not by a system of conditions giving rise to a polygonal area of limitation, but by a condition expressed by the positivity of a single homogeneous function of the variables of any degree, the variable point will then be limited by the intersection of the sphere with a cone, and we should have to solve a preliminary geometrical problem of circumscribing a spherical curve by the least possible circle,—a question which I have neither leisure nor inclination to discuss, but to which I believe Mr Cayley has paid some attention.

Before taking final leave of my readers and the subject, I devote a word to the *inverse case* of *Three Rectangular Forces.* This is the case where the resultant and two of the rectangular components are given, and it is the third component which is to be expressed linearly in terms of them. In this case an approximate expression is to be found for $\sqrt{(z^2-y^2-x^2)}$, and the geometrical locus which replaces the sphere becomes an equilateral hyperboloid of revolution of two sheets.

If the variable point be supposed to be limited to a segment of one sheet of the hyperboloid cut off by the plane $Ax+By+Cz=1$, the discriminant of $z^2-y^2-x^2$ being 1, and its polar reciprocal of the same form as itself, the approximate linear form of the surd becomes

$$\frac{2Cz}{\sqrt{(C^2-B^2-A^2)}+1}+\frac{2By}{\sqrt{(C^2-B^2-A^2)}+1}+\frac{2Ax}{\sqrt{(C^2-B^2-A^2)}+1},$$

with a maximum proportional error

$$\frac{1-\sqrt{(C^2-B^2-A^2)}}{1+\sqrt{(C^2-B^2-A^2)}}.$$

To *envelope*, however, any given arbitrary system of inequalities between the coordinates x, y, z on the hyperboloid within a single condition,

$$Ax+By+Cz-1>0,$$

becomes a geometrical problem of somewhat greater difficulty than the corresponding one for the sphere, and I do not propose to enter upon the discussion of it here.

I shall content myself, as M. Poncelet has done in the corresponding case *in plano*, with exhibiting a single numerical application of the method.

Suppose the given limits to be defined by the equations

$$z^2>\tfrac{3}{2}(y^2+x^2), \quad y>x.$$

Here it is obvious that the *enveloping condition* will be expressible by means of the equation to a plane drawn through three points on the hyperboloid, the coordinates of one of which are found by writing

$$y=0, \quad x=0;$$

of a second by writing

$$z^2-\tfrac{3}{2}y=0, \quad x=0;$$

and of the third by writing

$$z^2-\tfrac{3}{2}(y^2+x^2)=0, \quad y-x=0;$$

and for all three

$$z^2-y^2-x^2=1.$$

Hence we obtain the matrix

$$\begin{vmatrix} 1, & 0, & 0, & \bar{1} \\ \sqrt{3}, & \sqrt{2}, & 0, & \bar{1} \\ \sqrt{3}, & 1, & 1, & \bar{1} \end{vmatrix}.$$

And if we call the minors obtained by leaving out the first, second, third, fourth columns respectively H, G, F, Q, the linear form becomes

$$\frac{2Hz}{\sqrt{(H^2-G^2-F^2)}+Q}+\frac{2Gy}{\sqrt{(H^2-G^2-F^2)}+Q}+\frac{2Fx}{\sqrt{(H^2-G^2-F^2)}+Q},$$

with a maximum error

$$\frac{Q-\sqrt{(H^2-G^2-F^2)}}{Q+\sqrt{(H^2-G^2-F^2)}}.$$

And since

$$Q=\sqrt{2},\quad H=\sqrt{2},\quad -G=\sqrt{3}-1,\quad -F=(\sqrt{2}-1)(\sqrt{3}-1),$$

we have

$$\sqrt{(H^2-G^2-F^2)}=1\cdot1714 \text{ and } Q=1\cdot4142,$$

so that the representative form becomes $1\cdot093z-\cdot566y-\cdot089x$, with a maximum relative error of about $\cdot094$.

32.

MEDITATION ON THE IDEA OF PONCELET'S THEOREM.

[*Philosophical Magazine,* XX. (1860), pp. 307—316.]

HITHERTO Poncelet's theorem has been regarded as a method *sui generis* and complete in itself; but in truth it is but the first germ or rudiment of a vast and prolific algebraical theory; and not only so, but the principle which it contains admits of applications of the utmost value in various dynamical and analytical questions, which it is surprising should have been allowed to lay so long dormant. For the present, however, I mean to confine myself to a very brief indication of one direction in which the theorem admits of being generalized. And first I will make a remark upon so simple a matter as the extraction of the square root, which seems to have escaped observation, and at all events is so far from being generally known, that two of the highest authorities for mathematical erudition in this country whom I have consulted on the subject provisionally accept it as new.

Let r be an approximate value of $\sqrt{N}$; then by that mode of application of Newton's method of approximation to the equation $x^2 = N$ which is equivalent to the use of continued fractions, we may easily establish the following theorem, namely, that

$$\frac{r^2+N}{2r},\ \frac{r^3+3rN}{3r^2+N},\ \frac{r^4+6r^2N+N^2}{4r^3+4rN},\ \frac{r^5+10r^3N+5rN^2}{5r^4+10r^2N+N^2},\ \ldots^*$$

will be successive approximations to $\sqrt{N}$, whose limits of error can be

* In other words, if r be the first approximation to $\sqrt{N}$, the ith approximation will be

$$\frac{(r+\sqrt{N})^i+(r-\sqrt{N})^i}{(r+\sqrt{N})^i-(r-\sqrt{N})^i}\sqrt{N},$$

so that the relative error becomes

$$\frac{2(r-\sqrt{N})^i}{(r+\sqrt{N})^i-(r-\sqrt{N})^i},$$

in which form the theorem is self-subsistent, and needs no proof. But the fact remains interesting, that the application of Newton's method of approximation to the equation $x^2=N$ will be found to lead to the form above written at the ith step of the process conducted after the continued-fraction fashion.

assigned when a limit to the error of the first approximation r is given. The coefficients of the qth approximation, it will be observed, are for the numerator the alternate binomial coefficients

$$1,\ q\frac{q-1}{2},\ q\frac{q-1}{2}\frac{q-2}{3}\frac{q-3}{4},\ \text{\&c.};$$

and for the denominator the intermediate ones,

$$q,\ q\frac{q-1}{2}\frac{q-2}{3},\ \text{\&c.}$$

Mr Cayley has reminded me that the third approximation, $\frac{r^3+3rN}{3r^2+N}$, is a special case of a formula for *any* root of N given in the books; and to Mr De Morgan I am indebted for a hint which has led me to notice that all these forms may be deduced from the Newtonian method of approximation*.

If we call the ith approximation $\phi(i, r)$, we shall find that the functional equation $\phi\{j, \phi(i, r)\} = \phi(ij, r)$ will be satisfied; which is not so mere a truism as might at first sight be supposed, as any one may satisfy himself by studying the analogous theory for cubic or higher roots, a part of the subject to which I may hereafter return.

Now as to the limits of accuracy afforded by the successive approximations. Let e be a known limit to the relative error of the first approximation r, by which I mean that $\left(\frac{\sqrt{N}-r}{\sqrt{N}}\right)^2 < e^2$. For greater simplicity, I take separately the cases where r is too great and r is too small.

1. Let $\sqrt{N} < r < (1+\epsilon)\sqrt{N}$; then the errors will be throughout in excess; and we may assign as a limit of error to the ith approximation a quantity, say ϵ_i, which is a known function of ϵ, namely, $\frac{2}{(2\epsilon^{-1}+1)^i-1}$, which it may be noticed is less than $\frac{\epsilon^i}{2^{i-1}}$.

2. Let $\sqrt{N} > r > (1-\eta)\sqrt{N}$; then the errors will be alternately in defect and excess, and to the ith approximation we may assign a limit of error η_i, where $\eta_i = \frac{2}{(2\eta^{-1}-1)^i-(-1)^i}$ †.

* The expansion (after Newton) of $\sqrt{N}$ introduces the binomial coefficients—a curious fact! What are the analogous integers which the continued-fraction process applied to $\sqrt[3]{N}$ will produce?

† If we write

$$\epsilon_i = \theta(\epsilon, i) \text{ and } \eta_i = \vartheta(\eta, i),$$

then if i be any *odd* number,

$$\theta\{\theta(\epsilon, i), j\} = \theta(\epsilon, ij),$$
$$\vartheta\{\vartheta(\eta, i), j\} = \vartheta(\eta, ij);$$

and if i be any *even* number,

$$\vartheta\{\theta(\epsilon, i), j\} = \theta(\epsilon, ij),$$
$$\theta\{\vartheta(\eta, i), j\} = \vartheta(\eta, ij).$$

We may now apply these results to Poncelet's linear approximate representation of $\sqrt{(a+bx+cx^2)}$. Suppose $f+gx$ is the first approximation, as found by Poncelet's method, with a maximum relative error e, then

$$\frac{(f+gx)^2+(a+bx+cx^2)}{2(f+gx)}$$

will be a much closer approximation, with a relative error never exceeding $\frac{e^2}{2+e}$ in excess, nor $\frac{e^2}{2-e}$ in defect. So a still nearer approximation will be $\frac{(f+gx)^3+3(f+gx)(a+bx+cx^2)}{3(f+gx)^2+a+bx+cx^2}$, with a relative error never exceeding $\frac{e^3}{4+6e+3e^2}$ in excess, nor $\frac{e^3}{4-6e+3e^2}$ in defect, and so on. The marvellous facility which these formulæ afford for the calculation of elliptic and ultra-elliptic functions, and not merely for their computation as by a method of quadratures, but (which is of far greater importance) their quasi-representation under circular and logarithmic forms, with assignable limits of proportional error, will be illustrated in a future communication. As regards the idea of substituting rational for irrational functions, I have only to-day learned from Mr Cayley that I am anticipated in this by Mr Merrifield*,

Or more simply, if the error in excess be treated as positive, and in defect as negative, and δ be the first and δ_i the ith limit of error, we shall have

$$\delta_i=\frac{2\delta^i}{(2+\delta)^i-\delta^i};$$

and calling $\delta_i=\theta(i,\delta)$,

$$\theta\{j,\theta(i,\delta)\}=\theta(ij,\delta).$$

Thus, then, if we call $\frac{N+x^2}{2x}=\psi x$, $\psi^q x$ will correspond to the (2^q)th order of approximation, and the absolute value of the error will be less than

$$\frac{2\delta^{2^q}}{(2+\delta)^{2^q}-\delta^{2^q}}.$$

By way of example, suppose we take 6 as our first approximation to $\sqrt{31}$, then

$$\delta<\frac{\frac{1}{2}}{5\frac{1}{2}}<\frac{1}{11};$$

and if we make $\psi x=\frac{31+x^2}{2x}$, we shall have

$$\psi^4 6:\sqrt{31}::1+\omega:1,$$

where

$$\omega<\frac{2}{23^{16}-1},$$

which serves to exemplify the prodigious rapidity of the approximation in this method of extracting the square roots of numbers.

* I quite concur with Mr Merrifield, and in fact before being made acquainted with the existence of his paper, had emitted the same opinion (among others to Dr Borchardt of Berlin), that the substitutive method, consisting in the employment of rational functions in place of the radical, affords by far the most expeditious means for the calculation of elliptic functions of all orders, especially the third, and supersedes the necessity for the construction of special

in a paper very recently read before the Royal Society, but not yet printed in the *Transactions**.

auxiliary tables. I believe, however, that my substitutions, founded on Poncelet's views, are in general the best that can be employed for the purpose. In addition to other advantages they possess this, which deserves notice—that as we know *à priori* a superior limit to the proportional error, the arithmetical values of the integrals to which they are applied may be brought out correct to any required place of decimals, without its being necessary to calculate and compare a superior and inferior limit to the integral, either one of these being sufficient in my method to indicate its own reliable degree of precision.

* In general it is obvious, if ϕx between the limits a and b retain always the same sign, and ψx within these limits be sometimes greater and sometimes less than ϕx, but the difference between them be always less than $\epsilon\phi x$, then $\int_b^a dx\,\psi x$ will differ from $\int_b^a dx\,\phi x$ by *considerably* less than $\epsilon\int_b^a dx\,\phi x$. Paradoxical, however, as it may at first sight appear, there are extreme cases where this difference tends to a ratio of equality with $\epsilon\int_b^a dx\,\phi x$. The complete elliptic function of the first order may be made to furnish an example of this. Let

$$\phi x = \frac{1}{\sqrt{\{(1-x^2)(1-c^2x^2)\}}} = \frac{\sqrt{\{(1-x^2)+b^2x^2\}}}{(1-c^2x^2)\sqrt{(1-x^2)}}$$

(so that $b^2 = 1 - c^2$), and let

$$\psi x = \frac{f\sqrt{(1-x^2)} + gbx}{(1-c^2x^2)\sqrt{(1-x^2)}};$$

if we make

$$f = \frac{2}{1+\sqrt{2}}, \qquad g = \frac{2}{1+\sqrt{2}}, \qquad \epsilon = \frac{\sqrt{2}-1}{\sqrt{2}+1},$$

it follows from Poncelet's theorem, that for all values of x intermediate between 0 and 1, ψx will differ from ϕx by less than $\epsilon\phi x$.

Now it will easily be found by ordinary integration that

$$\int_0^1 dx\,\psi x = \frac{f}{2c}\log\frac{1+c}{1-c} + \frac{g}{c}\tan^{-1}\frac{b}{c}.$$

Hence $\int_0^1 dx\,\phi x$ must be always less than

$$\frac{f}{2(1-\epsilon)c}\log\frac{1+c}{1-c} + \frac{g}{(1-\epsilon)c}\tan^{-1}\frac{b}{c},$$

that is,

$$< \frac{1}{2c}\log\frac{1+c}{1-c} + \frac{1}{c}\tan^{-1}\frac{b}{c},$$

when c becomes indefinitely near to unity; that is, when b becomes indefinitely small, this approaches indefinitely near to $\log\frac{2}{b} + \frac{\pi}{2}$. But we know, by a theorem of Legendre, that the approximate value for the integral in such case is $\log\frac{4}{b}$; so that the superior limiting value of $\int_0^1 dx\,\phi x$, found by the application of Poncelet's method, approaches in this instance indefinitely near to the value itself. The explanation of this is easy. As c approximates to unity, the only important values of x in the integral

$$\int_0^1 \frac{dx}{\sqrt{\{(1-x^2)(1-c^2x^2)\}}},$$

are those which lie in the *immediate* vicinity of 1; and for all such values the relative error is at a *negative maximum*.

The method, however, of Mr Merrifield in working out this conception is, I believe, entirely different from that here indicated: how the many mathematicians of a practical stamp, English and foreign, who have worked with

It is not a little remarkable that so rude an application of Poncelet's method should serve to indicate almost with the force of rigorous demonstration the approximate formula

$$F(c) = \log\frac{1}{b} + \text{constant},$$

when c approaches indefinitely near to unity, the constant left undetermined being known to be less than $\log 2 + \frac{\pi}{2}$.

Nay, the demonstration may be made absolutely rigid if we set about to find an inferior limit. To this end make

$$\psi x = \frac{1}{\{f\sqrt{(1-x^2)} + gbx\}\sqrt{(1-x^2)}},$$

we shall find without difficulty

$$\int_0^1 \psi dx = \frac{1}{f\gamma} \log \frac{1+\gamma-b}{\gamma-1+b}\,\frac{\gamma+b}{\gamma-b}, \text{ where } \gamma = \sqrt{(1+b^2)},$$

and consequently we shall obtain as an *inferior* limit to $F(c)$ the expression

$$\frac{1}{\gamma}\log\frac{1+\gamma-b}{\gamma-1+b}\,\frac{\gamma+b}{\gamma-b},$$

which approaches indefinitely near to $\log\frac{2}{b}$ as c approaches indefinitely near to unity. It is thus seen that Legendre's $F(c)$, when c is indefinitely near to 1, lies between $\log\frac{2}{b}$ and $\log\frac{2}{b}+\frac{\pi}{2}$; the arithmetical mean between these limits is $\log\frac{2}{b}+\frac{\pi}{4}$, that is, $\log\frac{1}{b}+1{\cdot}4785$, differing by only $\cdot 0923$ from the true value $\log\frac{1}{b}+\log 4$. Of course, when the form of $F(c)$ in the case supposed is known, namely, $\log\frac{1}{b}+C$, there is no difficulty in determining C (as may be seen in Verhulst's *Traité des Fonctions Elliptiques*); but the process above given of throwing the general value of $F(c)$ between limits, is, I believe, by far the easiest and most natural method of obtaining this form. The limits themselves, it should be noticed, have virtually been found by the method, simple to *naïveté*, of writing $\sqrt{(1-c^2x^2)} = \sqrt{(p^2+q^2)}$, where $p=\sqrt{(1-x^2)}$ and $q=bx$, and then substituting for $\frac{1}{\sqrt{(p^2+q^2)}}$, $\frac{1}{p+q}$ as an inferior, and $\frac{p+q}{p^2+q^2}$ as a superior limit in the quantity to be integrated. *Closer* and *calculable* limits *ad libitum* to the integral may be arrived at by substituting for $\frac{1}{\sqrt{(p^2+q^2)}}$ one or the other of the two following rational functions of p, q, according as we wish to obtain an inferior or superior limit to the integral, namely,

$$\frac{\{p+q+\sqrt{(p^2+q^2)}\}^i - \{p+q-\sqrt{(p^2+q^2)}\}^i}{\{p+q+\sqrt{(p^2+q^2)}\}^i + \{p+q-\sqrt{(p^2+q^2)}\}^i}\,\frac{1}{\sqrt{(p^2+q^2)}},$$

or

$$\frac{\{p+q+\sqrt{(p^2+q^2)}\}^i + \{p+q-\sqrt{(p^2+q^2)}\}^i}{\{p+q+\sqrt{(p^2+q^2)}\}^i - \{p+q-\sqrt{(p^2+q^2)}\}^i}\,\frac{1}{\sqrt{(p^2+q^2)}},$$

in which formulæ the greater i is taken the closer will be the approximation. I am not aware that any of these limits to $F(c)$ (even the simplest of which, namely, those given above, may have some value for computational purposes, and have fallen thus very incidentally in my way) have ever before been noticed.

It is not unworthy of notice that the second superior limit to $\frac{1}{\sqrt{(p^2+q^2)}}$, namely, $\frac{p^2+pq+q^2}{(p+q)(p^2+q^2)}$, is an arithmetic mean between the first superior and first inferior limits, and

Poncelet's method during the last quarter of a century, should have managed to overlook so obvious and important an extension of the principle and its applications, I find hard to realize; and my wonder is even greater that I should not have been anticipated twenty years ago, than that I should have been anticipated so recently. But the algebraical theory to which this extension points the way is replete with interest of a far higher order than its applications to practice; for plainly the derived approximate fractions, however sufficient for the purposes of computation, are not, nor ever can be the best and closest of their respective kinds*. To fix the ideas, let us

consequently our second superior limit to the integral when b is indefinitely small becomes $\log\frac{2}{b}+\frac{\pi}{4}$, which brings the constant much nearer to its true value than did the use of the first limit; and as this approximation will evidently not stop at the second step of the process, we may safely infer that the integral derived from either formula when $i=\infty$ (for all values of b, whether finite or indefinitely small), not merely bears to $F(c)$ a ratio differing infinitely little from that of equality, but is absolutely equal to, and may for all analytical purposes be employed to represent $F(c)$.

I have been at the trouble of calculating the inferior limit afforded by the second approximation, and find that for b indefinitely small it is $\log\frac{2}{b}+\frac{\pi}{3\sqrt{3}}$, that is, $\log\frac{1}{b}+1{\cdot}2977$; the superior limit has been shown to be $\log\frac{1}{b}+1{\cdot}4785$, the mean is therefore $\log\frac{1}{b}+1{\cdot}3881$, differing by only ·0018 from the true value! As the constant continues for all values of i to be a multiple of π, the ith approximations *à suprà* and *ab infrà*, which are always effectible, will give (on making $i=\infty$) two new expansions for π, one infinitesimally in excess, the other infinitesimally in defect of its true value expressed as a multiple of log 2, which it might well repay the trouble of some young analyst to develope.

* That the fractional forms derived from the linear substitutive form are not the best of their respective kinds, appears immediately, so far as the derivatives of the odd order (subsequent to the first) are concerned, from the consideration that the limits of error in excess and in defect will be actually attained for values of x lying within the prescribed limits; but these errors, ϵ_i and η_i (when $\epsilon=\eta$, which is true by hypothesis), are never equal, the former (the extreme error in defect) being always the greater of the two; but if any such derivative were the best of its kind, the absolute values of the extreme errors of excess and defect ought to be *equal* to each other. But more generally, if possible, let the ith derivative to $L(x)$ (where $L(x)$ represents the radical linear approximant $fx+b$ to $\sqrt{(a+bx+cx^2)}$, say $Q(x)$), namely,

$$\frac{(Lx+Qx)^i+(Lx-Qx)^i}{(Lx+Qx)^i-(Lx-Qx)^i}\,Q(x),$$

be supposed the best of its kind: then the relative error is $\frac{2\,(Lx-Qx)^i}{(Lx+Qx)^i-(Lx-Qx)^i}$, and the maximum value of this must be equal (to the sign *près*) to the value which it has when we give to x either of its extreme connecting values. Now obviously the above is a maximum only when $\frac{Lx+Qx}{Lx-Qx}$ is a minimum, and therefore when $\frac{Lx}{Qx}$ is a maximum; but by hypothesis, the value of x, say m, which makes this a maximum, gives to $\frac{Lx}{Qx}-1$ the same value with the opposite sign to that which it would have in writing for x either of its limiting values, say k or k'.

Thus we have two equations for determining $\frac{Lk}{Qk}$, $\frac{Lm}{Qm}$, namely,

$$\frac{Lm}{Qm}-1=1-\frac{Lk}{Qk},$$

confine ourselves to the *second* Ponceletic approximation to $\sqrt{(a+bx+cx^2)}$, namely, that which has the form $\dfrac{\lambda+\mu x+\nu x^2}{1+qx}$, where λ, μ, ν are to be determined. The problem to be solved is the following.

Let
$$\lambda+\mu x+\nu x^2 = V,$$
$$(1+qx)\sqrt{(a+bx+cx^2)} = U;$$

it is required to assign the four constants λ, μ, ν, q, so that the maximum value of $\left(\dfrac{V}{U}-1\right)^2$ for values of x intermediate between a and b shall be the least possible. Some little way, but only a little way, into the solution of this problem we can look in advance. In the first place, if we seek for the maximum values of $\left(\dfrac{V}{U}-1\right)^2$, we obtain the rational equation

$$\sqrt{(a+bx+cx^2)}\left(U\frac{dV}{dx}-V\frac{dU}{dx}\right)=0,$$

which will easily be seen to be a *cubic* (not a *biquadratic*) equation in x. Call $\left(\dfrac{V}{U}-1\right)=\phi(x)$; then the three roots of this equation being named x_1, x_2, x_3, the law of equality explained in my preceding paper would seem to show* that we must be able to satisfy the following equations,

$$(\phi x_1)^2=(\phi x_2)^2=(\phi x_3)^2=(\phi a)^2=(\phi b)^2,$$

which amount to four independent equations, the precise number of constants λ, μ, ν, q to be determined. So in like manner the ith rational *approximant* will contain $2i$ disposable constants; the differentiation of the quantity analogous to $\dfrac{V}{U}$ will give rise to an equation of the $(2i-1)$th degree; and

and
$$\left(\frac{Lm}{Qm}+1\right)^i-\left(\frac{Lm}{Qm}-1\right)^i=(-)^{i-1}\left\{\left(\frac{Lk}{Qk}+1\right)^i-\left(\frac{Lk}{Qk}-1\right)^i\right\}.$$

Thus, suppose $i=2$, we should obtain from the second equation $\dfrac{Lm}{Qm}=-\dfrac{Lk}{Qk}$, which is inconsistent with the first; so if $i=3$, we should obtain $\left(\dfrac{Lm}{Qm}\right)^2=\left(\dfrac{Lk}{Qk}\right)^2$, and therefore, on account of the first equation, $\dfrac{L(m)}{Q(m)}=1$; and so in like manner for any value of i, we should derive one or more *numerical* values for $\dfrac{Lm}{Qm}$, which is absurd, since this quantity is a function of k, k', the two connecting values of x.

* Is it not, however, somewhat uncertain whether the equalities

$$(\phi x_1)^2=(\phi x_2)^2=(\phi x_3)^2$$

must all, in all cases (that is to say, for all given values of the limits) subsist? since the law of equality will not apply to such values of x as lie without the prescribed limits, and *non constat à priori* that the roots of the cubic do all lie within these limits. The subject at the very threshold is beset with doubts and difficulties of a peculiar kind, which we can hardly hope to overcome without calling in geometrical imagination to our aid.

there will be $2i-1+2$, that is, $2i+1$ functions of these $2i$ quantities to be equated, which furnish precisely the required number of equations to make the problem definite. It is, however, apparent that in solving these equations we shall find a *multiplicity* of systems, by which I mean a *definite* number of systems of values of the disposable constants which will equally well satisfy the equations. For instance, in the theory of the second approximation, the equalities

$$(\phi x_1)^2 = (\phi x_2)^2 = (\phi x_3)^2$$

will be satisfied by supposing $x_1 = x_2 = x_3$*. But it is by no means evident *à priori* that this system of equalities will correspond to the absolute minimum of which we are in quest: nay, though even we had $\phi x_1 = \phi x_2 = \phi x_3$, those equations do not necessarily imply $x_1 = x_2 = x_3$. Of the multiplicity of solutions referred to, one only gives the true minimum; but to assign *à priori* the distinguishing marks of this truest and best, *hic labor, hoc opus est.* It will be delightful to find, if it turn out to be true, that for the best form, $\frac{P}{Q}$ representing $\sqrt{X}$ (P being a rational function of the ith degree, and Q of the $(i-1)$th in x), the rational quantity

$$XQ\frac{dP}{dx} - \sqrt{X}\,P\frac{d}{dx}(Q\sqrt{X})$$

must be a perfect $(2i-1)$th power of a linear function of x; but in the present state of my ignorance I dare not do more than affirm that there is a bare probability in favour of this being true: whoever shall first succeed in discovering the true form of the expression will have established a remarkable theorem. Here for the moment I break off, contented with having pointed to a theory as yet, if the expression may be allowed, sleeping in its cradle, but destined, I am persuaded, at no distant day to set in motion as large a mass of algebraical thought as has been set in motion by the never-to-be-forgotten Hessian discussion of the flexures of the cubic curve,—the turning-point between the old algebra and the new.

Henceforward Poncelet's theorem figures no longer as a detached method, a mere stroke of art in aid of the computer, but becomes integrally attached to the grand and progressive body of doctrine of the modern algebra.

* If this is so, we shall have for determining the four constants the following equations:

$$x_1 = x_2 = x_3, \quad \phi a = \phi b = -\phi x_1.$$

But more probable than this seems the conjecture, that, supposing x_1, x_2, x_3 to be arranged in the order of their relative magnitudes, the determining equations might be

$$x_1 = x_3, \quad \phi a = \phi b = \phi x_2 = -\phi x_1.$$

Or is it possible that the *character* of the solution may be discontinuous, and may depend upon the magnitudes, relative or absolute, of the given limits a and b? Probably Dr Tchebitcheff would be able better than any other living analyst to answer these queries. But what an endless vista of future research does the prosecution of the Ponceletic method open out to us!

33.

NOTES TO THE MEDITATION ON PONCELET'S THEOREM, INCLUDING A VALUATION OF THE TWO NEW DEFINITE INTEGRALS

$$\int_0^{\frac{\pi}{2}} \frac{\log \cos \phi \, d\phi}{\sqrt{\{1 - b^2 (\cos \phi)^2\}}}, \quad \int_0^{\frac{\pi}{2}} \frac{\log [1 + \sqrt{\{1 - b^2 (\cos \phi)^2\}}] \, d\phi}{\sqrt{\{1 - b^2 (\cos \phi)^2\}}}.$$

[*Philosophical Magazine,* xx. (1860), pp. 525—533.]

NOTE A.

THE method given in the October Number of the *Magazine* for approximately representing a quadratic surd by a rational fraction is equally applicable to a surd of any degree. To fix the ideas, suppose we wish to approximate in this manner to $\sqrt[3]{R}$.

If we assume P as the first approximation, and make

$$L = P + \sqrt[3]{R}, \quad M = P + \rho \sqrt[3]{R}, \quad N = P + \rho^2 \sqrt[3]{R},$$

where $\rho^3 = 1$, and write

$$\begin{aligned} F_1 &= L^i + M^i + N^i, \\ F_2 &= L^i + \rho^2 M^i + \rho N^i, \\ F_3 &= L^i + \rho\, M^i + \rho^2 N^i, \end{aligned}$$

$$U_1 = \frac{F_1}{F_2} R^{\frac{1}{3}}, \quad U_2 = \frac{F_2}{F_3} R^{\frac{1}{3}}, \quad U_3 = \frac{F_3}{F_1} R^{\frac{1}{3}},$$

$$V_1 = \frac{F_2}{F_1} R^{\frac{2}{3}}, \quad V_2 = \frac{F_3}{F_2} R^{\frac{2}{3}}, \quad V_3 = \frac{F_1}{F_3} R^{\frac{2}{3}},$$

we may easily establish the following propositions, which indeed are almost self-evident :—

(1) Each U and V is a rational fraction.

(2) When $i = \infty$, each $U = R^{\frac{1}{3}}$, each $V = R^{\frac{2}{3}}$.

(3) For all finite values of i, $R^{\frac{1}{3}}$ is intermediate between the least and greatest U, and $R^{\frac{2}{3}}$ between the least and greatest V.

So in general if k is any prime number, we may form $(k-1)$ cycles, each cycle containing k fractions possessing precisely analogous properties as regards representing approximately and limiting the successive powers of $R^{\frac{1}{k}}$. By means of these formulæ [the theory of which might be extended to algebraic quantities of every order (in Abel's sense of the word)], we obtain a complete command over the integration of surd quantities in general as they may appear in any physical problem, being thereby enabled to represent the integrals, not merely arithmetically, but analytically (which is of much higher importance) by logarithmic and circular functions to any degree of accuracy that may be required, and with known assignable numerical limits of error.

Note B.

This note relates to the concluding paragraph of the long note at page 313 in the October Number of the *Magazine* [203, above]. I find that the ith inferior limit to $F(c) - \log\frac{2}{b}$, when c differs indefinitely little from unity given by the method therein explained, is

$$\log\frac{2}{b} + \frac{2}{i}\sum_{k=E\frac{i}{2}}^{k=1} \frac{\cos\frac{2k-1}{2k}\pi}{\sqrt{\left\{1+\left(\sin\frac{2k-1}{2i}\pi\right)^2\right\}}}\cos^{-1}\left(\sin\frac{2k-1}{2i}\pi\right)^2,$$

and that the superior limit is

$$\log\frac{2}{b} + \frac{\pi}{2i} + \frac{2}{i}\sum_{k=E\frac{i}{2}}^{k=1} \frac{\cos\frac{k\pi}{i}}{\sqrt{\left\{1+\left(\sin\frac{k\pi}{i}\right)^2\right\}}}\cos^{-1}\left(\sin\frac{k\pi}{i}\right)^2.$$

When $i=\infty$ these limits of course come together, and the finite sums resolve themselves into the definite integral

$$\frac{2}{\pi}\int_0^{\frac{\pi}{2}} d\tau\,\frac{\cos\tau}{\sqrt{\{1+(\sin\tau)^2\}}}\cos^{-1}(\sin\tau)^2,$$

of which, therefore, the value must be log 2. Hence, writing $(\sin\tau)^2 = \cos 2\theta$, we obtain

$$\int_0^{\frac{\pi}{4}} dt\,\frac{\theta\sin\theta}{\sqrt{(\cos 2\theta)}} = \frac{\log 2}{\sqrt{2}}\frac{\pi}{4}.$$

Note C.

It may be shown that any of the expressions for $N^{\frac{1}{k}}$ derived from making $i=\infty$ in the general formulæ given in Note A, are in fact tantamount to its representation as a definite integral of a very simple kind. I shall not go into

the proof of this here; it may be sufficient to indicate that it depends upon the fact that the equation of infinite degree $(\phi x)^i + (\psi x)^i + (\vartheta x)^i +$ &c., may be resolved into sets of factors of a known form. In the question before us, the function to be so resolved is the denominator of any one of the quantities analogous to U or V in Note A; and ϕx, ψx, ϑx ... become linear functions of x with imaginary coefficients. Its resolution into factors is rendered possible by the circumstance that only two of the quantities ϕ, ψ, ϑ ... can bear a finite ratio to each other for any given value of x, and consequently all the roots of the equation

$$(\phi x)^i + (\psi x)^i + (\vartheta x)^i \ldots = 0\text{*}$$

are contained *among* the roots of several binary equations

$$(\phi x)^i = (\psi x)^i, \quad (\phi x)^i = (\vartheta x)^i, \text{ \&c.}:$$

which are the roots of any one of these equations (as for example of the first) that belong to the *given* equation will be determined by the condition that they must make the norms of all the other functions (for example of ϑx) indefinitely small as compared with the norms of those two which appear in it (for example ϕx, ψx). In this manner, if the total number of the functions is k, supposing ϕ, ψ, ϑ ... to be all linear functions of x, each binomial equation out of its entire stock of i roots will contribute $\dfrac{i}{k\dfrac{k-1}{2}}$ roots available towards the solution of the given equation. Mr Cayley has remarked to me the analogy between this determination and Newton's method of finding the form of the several parabolic equations $y = cx^\lambda$ which represent the branches of a given algebraical curve at its origin. In the equation to the given curve cx^λ is to be substituted for y; the terms will then all become powers of x (an infinitesimal) whose indices will be linear functions of λ; every pair of them in turn is equated to zero, and of all the values of λ thus obtained only those will be preserved which cause the two equated linear functions of λ belonging to any given pair of terms to be less than all the others, and consequently the terms themselves (whose indices the linear functions are) infinitely greater than all the other terms.

Linear functions of a variable figure in both investigations, namely, in Newton's as indices of the same infinitesimal quantity, in mine as quantities whose infinite index is the same†; but the logic and mode of procedure (utterly unlike as are the questions in their origin and subject matter) is the same in either case.

* My friend, M. Jordan, of the École des Mines (author of a remarkable thesis on *groups*), has developed some interesting geometrical consequences arising out of the study of this equation, which I hope he may be induced to publish.

† In a word, Newton's equation is an exponential one made up of nothings, mine an algebraical one made up of infinities.

Note D.

The remark contained in the preceding note, as to the effect of representing $N^{\frac{1}{k}}$ by an infinite rational fraction being identical with that of expressing it as a definite integral, combined with a consideration of the cause of the success of the particular method referred to in Note B, has led me to the investigation following, of the value of the complete elliptic function of the first species. As usual denoting it by $F(c)$, we have

$$F(c) = \int_0^{\frac{\pi}{2}} d\theta \frac{1}{\sqrt{\{1 - c^2 (\sin\theta)^2\}}}$$
$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} d\theta \int_0^{\infty} dx \frac{\cos\theta}{1 - c^2(\sin\theta)^2 + (\cos\theta)^2 x^2}$$
$$= \frac{2}{\pi}\int_0^{\infty} dx I,$$

where

$$I = \int_0^{\frac{\pi}{2}} d\theta \frac{\cos\theta}{(1+x^2) - (c^2 + x^2)(\sin\theta)^2}$$
$$= \frac{1}{\sqrt{\{(1+x^2)(c^2+x^2)\}}}\left[\log\{\sqrt{(1+x^2)} + \sqrt{(c^2+x^2)}\} - \log\sqrt{(1-c^2)}\right].$$

Let $x = \tan\phi$, $b = \sqrt{(1-c^2)}$; then

$$F(c) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{\{1 - b^2(\cos\phi)^2\}}}\{\log[\sec\phi + \sqrt{\{(\sec\phi)^2 - b^2\}}] - \log b\}$$
$$= \frac{2}{\pi}\log\frac{1}{b} F(b) + \frac{2}{\pi} R,$$

where

$$R = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{\{1 - b^2(\cos\phi)^2\}}} \log[\sec\phi + \sqrt{\{(\sec\phi)^2 - b^2\}}]$$
$$= \int_{\frac{\pi}{2}}^0 d\phi \left\{\frac{\log(\cos\phi)}{\sqrt{\{1 - b^2(\cos\phi)^2\}}}\right\} + \int_0^{\frac{\pi}{2}} d\phi \frac{\log[1 + \sqrt{\{1 - b^2(\cos\phi)^2\}}]}{\sqrt{\{1 - b^2(\cos\phi)^2\}}}.$$

It will presently appear that these two definite integrals are equal to one another!

Let $$V_{2r} = \int_{\frac{\pi}{2}}^0 (\cos\phi)^{2r} \log(\cos\phi)\, d\phi.$$

Then we may easily establish the formula of reduction,

$$V_{2r} = \frac{2r-1}{2r} V_{2r-2} - \frac{1.3.5..(2r-3)}{2.4.6..(2r-2)2r}\frac{\pi}{2};$$

and since (as is well known) $V_0 = \frac{\pi}{2}\log 2$, we have

$$V_2 = \frac{1}{2}\frac{\pi}{2}\left(\log 2 - \frac{1}{1.2}\right),$$

$$V_4 = \frac{1.3}{2.4}\frac{\pi}{2}\left(\log 2 - \frac{1}{1.2} - \frac{1}{3.4}\right),$$

$$V_6 = \frac{1.3.5}{2.4.6}\frac{\pi}{2}\left(\log 2 - \frac{1}{1.2} - \frac{1}{3.4} - \frac{1}{5.6}\right),$$

$$\&c. = \&c.$$

Hence, by expanding the denominator in a series proceeding according to powers of $(\cos\phi)^2$, it is readily seen that the first integral becomes

$$\frac{\pi}{2}\left\{\log 2 + \left(\frac{1}{2}\right)^2\left(\log 2 - \frac{1}{1.2}\right)b^2 + \left(\frac{1.3}{2.4}\right)^2\left(\log 2 - \frac{1}{1.2} - \frac{1}{3.4}\right)b^4 + \&c.\right\}.$$

To find the second integral, we must obtain the general term in the expansion in a series of powers of t of

$$\frac{\log\{1 + \sqrt{(1-t^2)}\}}{\sqrt{(1-t^2)}}$$

(where t stands for $b\cos\phi$), that is, of

$$\frac{1}{\sqrt{(1-t^2)}}\int dt\left(\frac{1}{t} - \frac{1}{t\sqrt{(1-t^2)}}\right),$$

say of $\phi t = \frac{1}{\sqrt{(1-t^2)}}\psi t$. Now

$$\left(\frac{d}{dt}\right)^2\{(1-t^2)\phi t + \int dt\,(t\phi t)\} = \left(\frac{d}{dt}\right)^2\left(\sqrt{(1-t^2)}\,\psi t + \int dt\,\frac{t\psi t}{\sqrt{(1-t^2)}}\right)$$

$$= \psi t\left\{\left(\frac{d}{dt}\right)^2\sqrt{(1-t^2)} + \frac{d}{dt}\frac{t}{\sqrt{(1-t^2)}}\right\} + \psi' t\left(\frac{-2t}{\sqrt{(1-t^2)}} + \frac{t}{\sqrt{(1-t^2)}}\right) + \sqrt{(1-t^2)}\,\psi'' t$$

$$= -\frac{t}{\sqrt{(1-t^2)}}\psi' t + \sqrt{(1-t^2)}\,\psi'' t$$

$$= -\frac{t}{\sqrt{(1-t^2)}}\left\{\frac{1}{t} - \frac{1}{t\sqrt{(1-t^2)}}\right\} - \sqrt{(1-t^2)}\left\{\frac{1}{t^2} + \frac{2t^2-1}{t^2(1-t^2)^{\frac{3}{2}}}\right\}$$

$$= -\frac{1}{\sqrt{(1-t^2)}} - \frac{\sqrt{(1-t^2)}}{t^2} + \frac{1}{1-t^2} - \frac{2t^2-1}{t^2(1-t^2)}$$

$$= \frac{-1}{t^2\sqrt{(1-t^2)}} + \frac{2}{t^2}$$

$$= \frac{1}{t^2} - \frac{1}{2} - \frac{1.3}{2.4}t^2 - \frac{1.3.5}{2.4.6}t^4, \ \&c.$$

Hence, writing

$$\frac{\log\{1 + \sqrt{(1-t^2)}\}}{\sqrt{(1-t^2)}} = \log 2 + K_2 t^2 + \ldots + K_{2i-2}t^{2i-2} + K_{2i}t^{2i} + \&c.,$$

and equating the coefficients of t^{2i}, we obtain

$$2i(2i-1)(K_{2i}-K_{2i-2})+(2i-1)K_{2i-2}=-\frac{1.3.5\ldots(2i-1)}{2.4.6\ldots 2i},$$

that is, $$K_{2i}=\frac{2i-1}{2i}K_{2i-2}-\frac{1.3.5\ldots 2i-3}{2.4.6\ldots(2i-2)(2i)^2}.$$

Thus $K_0=\log 2,\quad K_2=\frac{1}{2}\left(\log 2-\frac{1}{1.2}\right),\quad K_4=\frac{1.3}{2.4}\left(\log 2-\frac{1}{1.2}-\frac{1}{3.4}\right),$

&c. = &c.,

and consequently $$\int_0^{\frac{\pi}{2}} d\phi\,\frac{\log[1+\sqrt{\{1-b^2(\cos\phi)^2\}}]}{\sqrt{\{1-b^2(\cos\phi)^2\}}}$$

$$=\frac{\pi}{2}\left\{\log 2+\left(\frac{1}{2}\right)^2\left(\log 2-\frac{1}{1.2}\right)b^2+\left(\frac{1.3}{2.4}\right)^2\left(\log 2-\frac{1}{1.2}-\frac{1}{3.4}\right)b^4+\text{\&c.}\right\}.$$

Thus, then, we obtain the following remarkable equalities*:

$$\frac{\pi}{2}F(c)=\log\frac{1}{b}F(b)+2\int_{\frac{\pi}{2}}^{0} d\phi\,\frac{\log(\cos\phi)}{\sqrt{\{1-b^2(\cos\phi)^2\}}}$$

$$=\log\frac{1}{b}F(b)+2\int_0^{\frac{\pi}{2}} d\phi\,\frac{\log[1+\sqrt{\{1-b^2(\cos\phi)^2\}}]}{\sqrt{\{1-b^2(\cos\phi)^2\}}},$$

or $$\int_0^{\frac{\pi}{2}} d\phi\,\frac{\log[1+\sqrt{\{1-b^2(\cos\phi)^2\}}]}{\sqrt{\{1-b^2(\cos\phi)^2\}}}=\int_{\frac{\pi}{2}}^{0} d\phi\,\frac{\log(\cos\phi)}{\sqrt{\{1-b^2(\cos\phi)^2\}}}$$

$$\left[=\int_0^{\frac{\pi}{2}} d\phi F(b,\phi)\cot\phi\right]=\frac{\pi}{4}F(c)+\frac{1}{2}\log bF(b).$$

When b is indefinitely small, it is obvious from either of these equations that

$$F(c)=-\frac{2}{\pi}\log b\,\frac{\pi}{2}+2\log 2=\log\frac{4}{b},$$

Legendre's well-known formula previously referred to.

The equality of the first two definite integrals in the *sorites* above given, is, as we have seen, a consequence of the equality

$$\frac{\log\{1+\sqrt{(1-t^2)}\}}{\sqrt{(1-t^2)}}=\frac{2}{\pi}\int_{\frac{\pi}{2}}^{0} d\theta\,\{\log\cos\theta+\log\cos\theta\,(\cos\theta)^2t^2$$
$$+\log\cos\theta\,(\cos\theta)^4t^4+\text{\&c.}\}.$$

[* The reader may be glad to have the references: *Schlömilch Zeitschrift*, II. (1857), pp. 49, 414; *Tortolini Annali*, III. (1860), p. 254. See also below, p. 298.]

Hence we have

$$\int_{\frac{\pi}{2}}^{0} \frac{\log \cos \theta}{1 - b^2 (\cos \theta)^2} d\theta = \frac{\pi}{2} \frac{\log \{1 + \sqrt{(1 - b^2)}\}}{\sqrt{(1 - b^2)}} *.$$

The extreme facility and brevity with which the method in the text gives the value of $F(c)$ for b indefinitely small is worthy of notice, as in the usual text-books it is obtained by a very indirect and circuitous process. We may obtain in like manner the value of

$$\int_0^{\frac{\pi}{2}} d\theta \frac{1}{(1 - e \sin \theta)} \cdot \frac{1}{\sqrt{\{1 - c^2 (\sin \theta)^2\}}}$$

on the same supposition as to c, whether $1 - e$ vanishes with $1 - c$ or remains finite when $c = 1$. On the latter supposition, the definite integral in question has for its value

$$\frac{1}{1 - e} \log \frac{2}{b} + \frac{1}{1 - e^2} \log \frac{2}{(1 + e)^e}.$$

When $e = 1$, this becomes infinite; when $e = -1$, the second term becomes $\frac{1}{4} + \frac{1}{2} \log 2$, and the entire integral is $\frac{1}{2} \log \frac{4}{b} + \frac{1}{4}$; when $e = 0$, it is $\log \frac{4}{b}$. Subtracting the half of the latter integral from the former, we shall obtain

$$\int_0^{\frac{\pi}{2}} d\theta \frac{(1 - \sin \theta)^2}{(\cos \theta)^3} = \frac{1}{2},$$

which is easily verified.

By taking successively $e = \sqrt{(-n)}$, $e = -\sqrt{(-n)}$, and adding together the halves of the two integrals corresponding to these suppositions, we obtain the *ultimate* value of the complete elliptic integral of the third kind, namely,

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{1 + n (\sin \theta)^2} \cdot \frac{1}{\sqrt{\{1 - c^2 (\sin \theta)^2\}}},$$

from the general formula above given, always of course subject to the condition that c is supposed indefinitely near to 1†.

* From this it will readily be seen that when n is any integer we may obtain

$$\int_0^{\frac{\pi}{2}} \frac{\log \cos \theta}{\{1 - b^2 (\cos \theta)^2\}^n}$$

by processes of differentiation in a form involving only algebraical and logarithmic quantities, and so, from what precedes, when n is any half-integer, in terms of such quantities and of complete elliptic functions.

† It seems to be expected of every pilgrim up the slopes of the mathematical Parnassus, that he will at some point or other of his journey sit down and invent a definite integral or two towards the increase of the common stock. The author of these notes has been somewhat late in acquitting himself of this debt of honour, but ventures to hope that the principal results contained in the text above may be thought not unworthy of a place in some future edition of that noble and sumptuous monument of Dutch learning, industry, and fine taste, the invaluable collection of definite integrals by M. Bierens de Haan.

34.

ON THE PRESSURE OF EARTH ON REVETMENT WALLS.

[*Philosophical Magazine*, XX. (1860), pp. 489—499.]

PART I.

Critique of the Hypothesis of Parallel "Planes of Rupture."

THE ensuing investigation deals with the pressure of *Mathematical* earth. By mathematical earth, I mean earth treated according to the idea of Coulomb, namely, as a *continuous** mass separable by planes in all directions, but whose separating surfaces exert upon one another forces consisting of two parts, one of the nature of ordinary friction, the other of so-called cohesion. Of the latter, for greater simplicity, I shall commence with taking no account, so that the matter with which we have to deal becomes, so to say, "a frictional fluid." If we isolate in idea any element of this fluid—suppose, to fix the ideas, a molecule bounded by plane faces, this molecule will be kept at rest by its own weight, the pressures on the several faces, and the forces of friction acting along these faces: these last-named forces are limited not to exceed the product of the corresponding pressures by a certain coefficient, termed the coefficient of friction.

In order to render the inquiry before us quite definite, let us begin with supposing two vertical side walls and a back of solid immoveable masonry, between which the earth is piled up in a determinate form, fronted by a pier of given specific gravity, whose *minimum* thickness is to be determined by the condition that it may just suffice to prevent the pier from being either forced forward or turned round over its further edge. The earth is thus of course supposed to have only one free face, being entirely supported at the sides and the back by the masonry just spoken of. The problem then that we have to solve is evidently the following:—"Of all the possible states of equilibrium of the earth consistent with the assigned conditions, to determine that one which shall make the greater of two quantities to be

* The only *essential* quality of our mathematical earth which differentiates it from actual vulgar earth is this of *continuity*.

named the least possible,"—one of these quantities being the thickness of the wall determined by the condition that its friction with the ground shall be just equal to the sum of the horizontal pressures on the wall, the other by the condition that its moment about the edge most remote from the earth shall be just equal to the sum of the moments of the entire thrust upon the wall at each several element thereof in respect to the same edge.

Whenever Coulomb's method leads to a right solution of the problem of revetments, the thrusts on the several elements of the wall will be all parallel; and it may easily be seen that, in solving the problem for this case, we are solving the problem of making the statical sum of the thrusts a minimum; and the result will be the same, whether the pier can only be pushed bodily on its base, or can only turn over an edge, or can do both one and the other. But it must obviously be erroneous to assume as a universal principle, that in the state bordering upon motion, or what is going still further, in a state antecedent to this, the statical sum of the pressures will be a minimum; and if Mr Moseley's "principle of least resistance," quoted by Professor Rankine, means this, I have no scruple in proclaiming my entire dissent from such an assumption. I do not here enter at all into the question of determining pressure, except in the state of equilibrium bordering upon motion; and in that state common sense points out that it is not the pressure or sum of pressures, but the effect of such pressure or pressures in inducing motion in a certain possible manner, or in any one out of a choice of possible manners, that governs the determination of the minimum. This principle of least resistance is one of the shoals upon which Mr Rankine's investigation appears to me to have split.

Be it observed that the only physical assumption which I propose is this, that if *equilibrium can be preserved consistently with the imposed conditions, equilibrium will be preserved.* Without such a supposition the question would be incapable of treatment without further laws regulating the interior forces than we suppose given. The legitimacy of such an assumption cannot, I think, be seriously called into question, and once made, the problem of determining the wall's thickness becomes a purely mathematical question; one undoubtedly of great difficulty, but perfectly determinate, and falling under the dominion of the Calculus of Variations, as will easily be recognized from the circumstance that the integration of the general equations of equilibrium, if it could be performed, would necessarily contain arbitrary functions, whose form would have to be assigned so as to make a certain quantity or the greatest of a set of quantities a minimum; but the peculiar manner in which the internal forces are defined as subject to satisfy not an equation or system of equations, but a law of inequality, must render it a task exceeding the present powers, at all events, of the writer of this paper, to arrive at a result by the direct application of the Calculus referred to. In order to pave the

way to the discussion of the more general inquiry, I shall commence with examining whether under any and what circumstances the forced solution of Coulomb and his followers, founded upon the notion of what have been (it seems to me incautiously) termed planes of fracture or rupture (but which really mean no more than planes for which friction at each point thereof is acting with its utmost energy, that is, if we please so to say, planes of greatest frictional energy*), is the true solution; that is to say, I shall investigate under what conditions the surfaces of "rupture" or "of greatest energy of friction" are or can be planes; and I shall easily be able to ascertain these conditions, and to prove that when they are satisfied (but not otherwise) the results of the received theory are exact.

Professor Rankine, in the light in which he appears in a paper published in the *Transactions* of the Royal Society, is not to be ranked among those whom I have called the followers of Coulomb. He is entitled to the merit of

* It is obvious that the notion of the planes in question being the planes in which the earth would begin with crumbling, if the equilibrium were disturbed by the wall giving way (for such is the idea intended to be conveyed by their being called planes of rupture), is quite irrelevant to the determination of their position, and to the solution of the question of the thrust in the wall. But such a notion in itself is objectionable, as assuming a physical fact for which there is no just ground. The idea, or rather I may say the metaphysical process, which unconsciously has swayed Coulomb and his followers to give them this name, appears to me to be the following. "Since it is only along these planes that friction is acting at its full energy, and since, when motion ensues, friction must be acting at its full energy, therefore a change must have taken place in the friction of any other plane before motion can take place along it, which change does not take place along the planes in question. Now every change must operate in time, therefore the motion must have begun along the planes of greatest friction before it can have taken place along any other." But it is a most dangerous proceeding, and fraught with errors familiar to mathematicians, to attempt to reason from the conditions of equilibrium to those of incipient motion; and that dynamical considerations, and not statical, must decide the incipient directions of the motion in the case before us, will be obvious when we reflect that the friction might be supposed to become *nil*, and then we should be treating of a perfect fluid, in which case the planes of rupture disappear, but none the less would motion take place in determinate directions on any wall of the reservoir containing the fluid giving way. A notable example of the important distinction between rest and equilibrium is afforded by the question (which, I am informed, originated in Caius College, Cambridge) of finding the tension of a rope by which a bucket full of water, with a cork tied to its bottom, is fastened to a fixed point, at the moment when the fastening is cut or gives way. At that moment the vertical pressure in the bottom of the bucket, supposing the specific gravity of the cork to be one-fourth that of water, if it could be estimated on statical principles, that is with reference to the elevation of the surface of the fluid [and some non-mathematical physicists might easily suppose it could be so estimated, since motion has not *yet taken place*, but is only *imminent*], would be the weight of the bucket together with that of the water, together with four times that of the cork, and so it would appear as if the tension would be increased by the cutting of the string, whereas, in fact, precisely the contrary effect will take place; for since downward momentum must result from the impending motion of the cork upwards and the water downwards, part of the weight of the water and cork is spent as downward moving force, and consequently only a portion remains to act as vertical pressure upon the bucket, just as an air-cushion will press with less force than its weight on the seat which bears it, when, in consequence of the air being let out, part of the weight is being expended in lowering the top of the cushion.

having perceived that the received hypothesis rested on no solid foundation, and of having been the first (publicly at least) to assert that the equations of internal equilibrium must be resorted to for the satisfactory discussion of the question; but, notwithstanding the sincere esteem in which I hold the great abilities of this gentleman, I have been compelled to come to the conclusion, and trust to be able to satisfy himself, that the use he has made of these equations is illusory, and that his results bear upon their very face a demonstrable character of error.

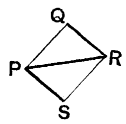
Under the supposed *data*, it is, if not obvious, at all events assumed by all writers on the subject, that the equilibrium of every vertical section of the earth, parallel to the side walls, may be determined *per se*, and that we may treat the question as one regarding space of only two dimensions. I shall therefore, with a view to clearness, treat of the equilibrium of any one such section; the molecules, whose equilibrium is to be considered, will be spoken of as bounded by lines instead of planes, and so we shall speak of lines instead of planes of "rupture," and we may thus conform our language to the relations of the figure actually represented upon the paper.

For the benefit of those to whom the conditions of molecular equilibrium are new, it may be well to indicate briefly how they may be obtained, still keeping within our prescribed framework of two-dimensioned space (although the reader will not experience the slightest difficulty in extending them to space of three dimensions)*. Through any point in the interior of the plane-mass at rest, imagine a small rectilinear element to be drawn. The entire molecular force exerted on this actual element might be termed the thrust; but by the thrust I shall understand *the unit of thrust*, corresponding to the well-known conception of *unit of pressure* for the particular case of a fluid mass. This thrust (or unit of thrust) may be imagined separated into two parts, one perpendicular to the element, which may be

* I have purposely begun with the beginning, because I wish to give perfect precision to the terms Thrust, Pressure, and Stress, as I shall use them. Some recent authors on mechanics have wished to distinguish force measured statically from force measured by acceleration, by giving to the former the name of pressure. But surely unnecessary confusion is introduced into mechanical language when we are thereby reduced to speak of the pressure of friction, and ought to enunciate the cardinal law of friction by stating that the *pressure of friction* bears to the *pressure of pressure* a certain limiting relation. I acknowledge an objection scarcely less valid (except that it has antiquity to plead in excuse) to the use of the term accelerating force; as we may be thereby reduced to speak of the accelerating force of a retarding influence, as friction, or of an influence which does not necessarily either accelerate or retard, as in the case of a centripetal pull upon a body moving uniformly in a circle. I think this difficulty in language may be met to some extent by giving to force, usually called accelerative, the designation of alternative, and to force measured by weight or momentum that of quantitative force. There is no magic in names, however well selected, but there may be a great deal of mischief arising out of a confused and uncertain nomenclature.

termed the *pressure*, the other parallel to it, which may be termed *face-force* [for the case we shall have more especially to consider, the face-force receives the name of *friction*, and is limited to be less than the *pressure* multiplied by the so-called coefficient of friction]. As the element acted upon turns round, the thrust changes in magnitude and direction, and to the totality of the thrusts going forth in all directions from a given point we may give the name of *stress**. We shall now be able to obtain two sorts of conditions—one giving the necessary law connecting the various *thrusts* of the same *stress*, the other expressing the law of the variation of the pressure and facial force (together constituting the thrust) upon an element given in direction in passing from one stress to another; we may call these respectively the equations of distribution and the equations of variation.

Let $PQRS$ be any infinitely small molecule bounded by lines at right angles to one another. Since this is kept at rest by its own weight, by the lines of pressures perpendicular to QR and PS, and the other pair perpendicular to PQ and RS, and by the facial forces acting along PQ, QR, RS, SP respectively, if we call f the face-force [that is, the unit of face-force] on PS, it is obvious that the corresponding quantity for QR will differ from it by an infinitely small quantity; in like manner f' may be taken as the face-force on QR, PS respectively. Hence the couples whose moments $(f\,.\,QR) \times QP$ and $(f'\,.\,PQ) \times QR$ respectively must be equal and opposite, or in other words, f being understood to act to or from Q, according as f' acts to or from Q, we must have $f = f'$. To fix the ideas, conceive the face-forces to tend towards Q. Let us now consider the equilibrium of the triangular molecule PQR. Call the pressure on PQ (R), the pressure on RQ (P), the face-force on PQ or QR (Q). In comparison with the thrusts on the faces of our triangular molecule, gravity or other impressed forces may be neglected as giving rise to quantities of an inferior order of smallness.



Let QP, QR be regarded as two fixed rectangular axes, and let $QPR = \theta$. Let the pressure and face-force on PR (always understanding thereby the units of such forces) be called N and F respectively (F, to fix the ideas, being taken to act from P to R). Then resolving the forces perpendicular to PR, we obtain

$$N\,.\,PR = R\,.\,PQ\,.\cos QPR + P\,.\,QR\cos QRP$$
$$+ Q\,.\,PQ\sin QPR + Q\,.\,QR\sin QRP,$$

or

$$N = R\,(\cos\theta)^2 + 2Q\cos\theta\sin\theta + P\,(\sin\theta)^2;$$

* Thus, stress stands in somewhat the same relation to its component thrusts, as a radiant point to the luminous rays which it emits.

and resolving parallel to PR, we have

$$F \times PR = R \,.\, PQ \sin QPR - P \,.\, QR \sin QRP$$
$$+ Q \,.\, PQ \cos QPR - Q \,.\, QR \cos QRP,$$

or

$$F = (R - P) \sin\theta \cos\theta.$$

Imagine now QPR to be represented by a single point O. R, P are respectively the pressures, and N the face-force ("units of pressures and of face-force") on elements drawn in the orthogonal directions OX, OY; N the pressure, and F the face-force on an element drawn in the direction OP, making an angle θ with OX. Obviously, therefore, if we draw in all directions from O lines whose lengths are as the inverse square roots of the pressure-part of the thrust acting on those lines, calling the length of line corresponding to θ, r, we have

$$\frac{1}{r^2} = R (\cos\theta)^2 + 2Q \sin\theta \cos\theta + P (\sin\theta)^2,$$

R, Q, P being constant quantities.

Consequently the locus of the extremities of these lines is a conic; and taking new axes of coordinates in the directions of the principal axes of this conic, and understanding by R and P the pressures perpendicular to those axes respectively, the equations obtained assume the form

$$N = R (\cos\theta)^2 + P (\sin\theta)^2, \qquad (1)$$
$$F = (R - P) \sin\theta \,.\, \cos\theta; \qquad (2)$$

showing that in elements in the directions of the principal axes the face-forces vanish, and the thrusts become purely pressures, that is, forces perpendicular to the surfaces upon which they act. R and P are of course essentially positive, as otherwise the molecules would be subject to a force of separation instead of compression, and consequently the conic in question is an ellipse. The total value of the thrust $= \sqrt{(N^2 + F^2)}$

$$= \sqrt{\{R^2 (\cos\theta)^2 + P^2 (\sin\theta)^2\}}. \qquad (\alpha)$$

R and P will evidently be in the directions in which, for a given point, the entire thrust, as well as the pressure-part of it, is the least and greatest. These directions may be said to be those of "principal thrust." If we start from any point and proceed from that point always in the direction of a line of principal thrust so as to form a continuous curve, two such curves cutting each other at right angles will intersect every point of the mass at rest, of which, in the case of mathematical earth, I may state, by way of anticipation, that only one can cut the free surface when that surface is supposed to form part of a horizontal plane.

These lines may also be termed the principal lines of pressure, or simply the lines of pressure; and this name may be considered indifferently to have

reference either to the fact that the thrust *in the direction of the tangent* at any point in any such curve is the thrust acting upon the normal, or to the fact that the thrust *upon the tangent* at any point is in the direction of the normal; as either one of such conditions implies the other.

The cosine of the angle between the pressure and the thrust will be

$$\frac{R(\cos\theta)^2 + P(\sin\theta)^2}{\sqrt{\{R^2(\cos\theta)^2 + P^2(\sin\theta)^2\}}};$$

which, calling the principal semiaxes of the ellipse referred to a and b respectively, and the rectangular coordinates of any point therein x and y, becomes

$$\frac{\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}}{\sqrt{\left(\dfrac{x^2}{a^4} + \dfrac{y^2}{b^4}\right)}} = \frac{1}{\sqrt{\left(\dfrac{x^2}{a^4} + \dfrac{y^2}{b^4}\right)}}$$

which is equal to the perpendicular from the centre on the tangent divided by the radius vector, showing that the direction of the thrust on any radius of the ellipse in question is in the direction of the conjugate diameter, whereby it is seen that the line of thrust, and the line thrust upon, stand in a reciprocal relation to each other.

I may add the cursory remark as regards the value of the total thrusts in the case more immediately before us, that [as is apparent from the equation (α)] they will be represented in relative magnitude by the radius vector drawn in the direction of the line thrust upon, to meet, not the ellipse of pressures just described, but another ellipse whose major and minor axes are to one another in the duplicate ratio of the other two.

If we wish, however, to present the above results in a form more immediately translateable into the actual case of nature, I mean that of space with three dimensions, it becomes expedient to use a different ellipse, or rather the same ellipse in another position, to represent the stress at any point.

In the equations above found, connecting N and F with P, Q, R, θ is the angle made with a fixed axis, not by the line of pressure R, but by the element on which this pressure is exerted. Let ϕ be the angle made by the pressure itself, so that $\phi = \theta + \frac{\pi}{2}$, then we have

$$N = P(\cos\phi)^2 - 2Q\sin\phi\cos\phi + R(\sin\phi)^2$$
$$F = (P - R)\sin\phi\cos\phi.$$

And the same process as has been already employed will serve to show that we may construct an ellipse such that the inverse square of the radius vector in every direction may represent the magnitude of the *pressure* in that

direction (that is the magnitude of the normal part of the thrust upon the element perpendicular to that direction), and in this ellipse the radius vector and perpendicular to the tangent at each point will represent the corresponding directions of pressure and thrust, which obviously will coincide for the directions of greatest and least pressure.

If, now, we go out into space of three dimensions, it will readily be anticipated, and may easily be proved, that an ellipsoid whose radii vectores represent the relative magnitudes of the inverse square roots of the pressures takes the place of the ellipse, the thrusts and pressures correspond respectively (in direction) to the normal and radius vector at each point, and in three directions, at right angles to each other, these latter come together.

It is desirable that the reader should bear in mind that the ellipse of which I have spoken is in fact only a principal section of this ellipsoid. The assumption which (following in the track of my predecessors) I shall make, that the greatest energy of friction exerted at any point will be exerted in some direction in a vertical plane parallel to the revetment wall, will be seen from what follows a little further on, to imply that every such plane contains the radius vector which makes the greatest angle with the normal, and consequently the section of the ellipsoid of stress with which we are dealing will be the plane of greatest and least thrust, or greatest and least pressure. By way of aid to the imagination in seizing this subtle conception of stress (a real conquest in physical ideology due to the last quarter of the present century, although its first germ may be recognized in the much earlier molecular view of the circumambient pressures round about each internal point of a perfect fluid), I have gone thus briefly into the generation of the ellipse and ellipsoid above described; but I shall have very little occasion, except for occasional facility of reference, to have resort to them, as the equations (1) and (2) will suffice for my purpose in the present inquiry.

These are the equations which govern the distribution of stress; and it may be convenient to confer upon the ellipse whose radii vectores are in length inversely as the square roots of the pressures acting upon them, the name of the ellipse of *pressures*, in order to obviate any possibility of the position of this ellipse being confounded with that of the one which would, I believe, more ordinarily go by the name of the ellipse of *stress*. Every point in the mass is the centre of such an ellipse; and those ellipses, if properly drawn, will represent completely, and on the same scale, the magnitude and distribution of the pressures round about any point. It is almost needless to add that for a perfect fluid these ellipses would become circles.

Let us now proceed to establish the law of the variation of the stresses, or, to speak more accurately, of the thrusts acting on planes drawn in

any given directions, on passing from one point of the mass to another. Returning to our little rectangular element $PQRS$, and considering the lines PQ, PS to be given in direction, so that we may consider $PQ = dx$ and $PS = dy$, and calling the units of pressure on PQ and RQ L and N, the unit of face-force M, the impressed forces of acceleration X in the direction of x, and Y in the direction of y, and the unit of mass ρ, by simple estimation of the forces in the directions of x and y respectively we obviously obtain, due attention being paid to the mode of fixing the positive directions of X and Y,

$$\frac{dL}{dy} + \frac{dM}{dx} = \rho Y,$$

$$\frac{dN}{dx} + \frac{dM}{dy} = \rho X.$$

If, as in the case with which we shall have to deal, the sole impressed force is that of gravity, and if we treat the weight of a unit of the mass as unity, and make the axis of x horizontal and that of y vertical, the equations become

$$\frac{dL}{dy} + \frac{dM}{dx} = 1,$$

$$\frac{dN}{dx} + \frac{dM}{dy} = 0.$$

These, being the equations which control the law of the variation of the thrusts estimated in given directions in passing from one stress to another, I call the equations of variation of stress.

I now proceed to the application of the principles above set forth to the treatment of the particular question in hand.

Let μ be the coefficient of friction of the earth upon itself, and $\mu = \tan \lambda$, so that λ is the angle of repose; by this is to be understood that the thrust on any element can never make, with the perpendicular to that element, an angle greater than λ. Now the general law of the distribution of stress proves that the actual angle between the perpendicular to the element and its thrust will in two directions be zero. Hence at any given point it will pass through all gradations, from zero up to a certain limit. Here presents itself the question, Is that limit λ, or can it be λ for every point in the mass? As we have no right to assume *à priori* that this limiting angle in that state of equilibrium which we wish to determine must be equal to λ throughout the mass, and obviously it will not be so for actual cases of equilibrium which arise, we want a name to distinguish the maximum ratio which friction bears to pressure in any specified stress from the absolute maximum which this ratio is capable of attaining. We may name the former the coefficient of frictional energy; and for every point where this is equal to the absolute coefficient of friction, we may say the friction of the stress is at its maximum

energy. Let (μ) be the coefficient of frictional energy for any given stress, and $(\lambda) = \tan^{-1}(\mu)$ the corresponding angle of repose. [We may also, if we please, term (μ) and (λ) the relative coefficient and relative angle of repose respectively, that is, relative to any assigned stress.] Let the ratio between the maximum and minimum thrust of any stress be called γ^2: a simple relation connects γ and (λ)*.

For calling, as before, L the pressure, and M the face-force (now the friction), we have by equation (1),

$$L = P(\cos\theta)^2 + R(\sin\theta)^2,$$
$$M = (P - R)\sin\theta\cos\theta,$$
$$R = P\gamma^2,$$
$$\tan(\lambda) = (\mu) = \text{maximum value of } \frac{M}{L}.$$

To find this maximum, we have

$$\delta\{\cot\theta + \gamma^2\tan\theta\} = 0.$$

Hence $$\gamma\tan\theta = 1,$$

and therefore $$\cot(\lambda) = \frac{2\gamma}{1-\gamma^2},$$

therefore $$(1-\gamma^2) - 2\gamma\tan\lambda = 0,$$

or $$\gamma = \sec(\lambda) - \tan(\lambda) = \frac{1-\sin(\lambda)}{\cos(\lambda)} = \tan\left(45^\circ - \frac{(\lambda)}{2}\right).$$

This equation expresses the universal relation between the form of the ellipse of pressures for any stress and the relative angle of repose for such stress.

The problem we have just solved may be presented advantageously, in order to make the impression of it more vivid (as it is of cardinal importance), under a geometrical point of view. Taking any radius vector of the ellipse of pressures, the angle between it and its conjugate radius is 90° at any vertex; at some point therefore it will be at a minimum, and this minimum will be the complement of the relative angle of repose.

From the preceding investigation, it will easily be seen that, to find the ray-directions which give this minimum, we have only to construct a rectangle circumscribing the ellipse, and either of its two diagonals will be in the direction required, and the angle between either such ray and the principal axes *plus* or *minus* half the angle between it and the normal (which angle is the relative angle of repose) will be half a right angle†.

* This relation and its importance are well known to Professor Rankine.

† In fact the diameters which coincide with the directions of these diagonals are conjugate diameters, equally inclined to the principal axes; and these, as I suppose must be well known, are the conjugate diameters whose inclination to each other is a minimum.

35.

ON THE EQUATION

$$P(m)+E\left(\frac{m}{m-1}\right)P(m-1)+E\left(\frac{m}{m-2}\right)P(m-2)+\ldots+E(m)=m\frac{m+1}{2}.$$

[*Quarterly Journal of Mathematics*, III. (1860), pp. 186—190.]

$P(m)$ I use to denote the number of integers less than m and prime to it except when $m=1$, in which case $P(m)=1$. $E\left(\frac{m}{r}\right)$ I use to denote the integer part of $\frac{m}{r}$, or the whole of $\frac{m}{r}$ if $\frac{m}{r}$ is an integer.

Then evidently if we use $\frac{m}{r;}$ to denote *unity* when m contains r and *zero* in all other cases

$$E\left(\frac{m}{r}\right)-E\left(\frac{m-1}{r}\right)=\frac{m}{r;}.$$

Again, it is well known that the factors of any binomial function, as for instance $x^{12}-1$, are made up of the prime factors of all the binomial factors of $x^{12}-1$ as x^2-1, x^3-1, x^4-1, x^6-1, $x^{12}-1$, and consequently that

$$m=\frac{m}{1;}P(1)+\frac{m}{2;}P(2)+\frac{m}{3;}P(3)+\ldots+\frac{m}{m;}P(m),$$

which equation may also be easily proved independently (*vide* note at end).

Let now

$$E\left(\frac{m}{m}\right)P(m)+E\left(\frac{m}{m-1}\right)P(m-1)+E\left(\frac{m}{m-2}\right)P(m-2)$$
$$+\ldots+E\left(\frac{m}{1}\right)P(1)=u_m.$$

Then
$$E\left(\frac{m-1}{m-1}\right)P(m-1)+E\left(\frac{m-1}{m-2}\right)P(m-2)$$
$$+\ldots+E\left(\frac{m-1}{1}\right)P(1)=u_{m-1}.$$

Hence $u_m - u_{m-1} = \frac{m}{m;} P(m) + \frac{m}{m-1;} P(m-1) + \ldots + \frac{m}{1;} P(1) = m.$

Hence $$u_m = m\frac{m+1}{2} + C,$$

and since $u_1 = 1$ we must make $C = 0$, and

$$u_m = m\frac{m+1}{2},$$

as was to be shown.

NOTE.—Proof of the equation

$$P(m) + \frac{m}{(m-1);} P(m-1) + \frac{m}{(m-2);} P(m-2) + \ldots + 1 = m.$$

Let $a, b, c \ldots$ be the prime factors of m, so that

$$m = a^\alpha . b^\beta . c^\gamma \ldots,$$

and, for example, suppose

$$m = a^\alpha . b^\beta . c^\gamma.$$

Then the numbers contained in m may be divided into groups as follows: one group in which a, b, c all appear, another in which only two of the letters a, b, c appear, a third in which only one of them appears, and finally *unity* in which none of them appears.

The sum of the numbers of integers prime to m and less than it for the factors in the first group

$$\begin{aligned} &= (a^{\alpha-1} + a^{\alpha-2} + \ldots + 1)(b^{\beta-1} + b^{\beta-2} + \ldots + 1)(c^{\gamma-1} + c^{\gamma-2} + \ldots + 1) \\ &\qquad\qquad \times (a-1)(b-1)(c-1) \\ &= (a^\alpha - 1)(b^\beta - 1)(c^\gamma - 1). \end{aligned}$$

In like manner the sum of the numbers of such integers for the factors in the second group

$$= (a^\alpha - 1)(b^\beta - 1) + (a^\alpha - 1)(c^\gamma - 1) + (b^\beta - 1)(c^\gamma - 1),$$

for the third group

$$= (a^\alpha - 1) + (b^\beta - 1) + (c^\gamma - 1),$$

and for *unity*

$$= 1.$$

Hence the total sum of such factors

$$= a^\alpha . b^\beta . c^\gamma$$
$$= m,$$

as was to be shown, and so in the like manner whatever may be the number of prime constituents $a, b, c \ldots$ in m.

Q. E. D.

P.S. 1. By successive integration the theorem first established may be generalized, and preserving the same notations as before, it emerges into the following proposition: [cf. the form below]

$$\Sigma_{\infty}^{0} {}_i P(i^r)\times\left[\frac{\left(E\frac{m}{i}\right)\left(E\frac{m}{i}+1\right)\ldots\left\{E\frac{m}{i}+(r-1)\right\}}{1.2\ldots r}\right]$$

$$=\Sigma_m^1{}_m(m^r).$$

Thus let $$r=2.$$

Then $$\Sigma_{\infty}^{0}{}_i\, P(i^2)\left\{\frac{\left(E\frac{m}{i}\right)\left(E\frac{m}{i}+1\right)}{2}\right\}$$

$$=\Sigma m^2=\frac{m(m+1)(2m+1)}{2.3},$$

or observing that

$$P(i^q)=i^{q-1}.P(i),$$

$$\Sigma_{\infty}^{0}{}_i\, iP(i)\left\{E\left(\frac{m}{i}\right)E\left(\frac{m}{i}+1\right)\right\}$$

$$=\frac{m(m+1)(2m+1)}{3}.$$

Example, let $$m=5,$$

$$5P(5)=20,$$

$$4P(4)=8,\quad E(\tfrac{5}{4})=1,$$

$$3P(3)=6,\quad E(\tfrac{5}{3})=1,$$

$$2P(2)=2,\quad E(\tfrac{5}{2})=2,$$

$$E(\tfrac{5}{1})=5,$$

$$20\times 2+8\times 2+6\times 2+2\times 6+5\times 6$$

$$=110,$$

$$\frac{5\times 6\times 11}{3}=110.$$

Or we may use the theorem under the form following:

$$\Sigma_n^1{}_i\left[P(i^r)\times S\left\{E\left(\frac{n}{i}\right)\right\}^{r-1}\right]=S(n^r),$$

where it is to be observed that

$$Sq^r \text{ means } 1^r+2^r+\ldots+q^r.$$

Example, let $$r = 3,$$

then $$S\left(E\frac{n}{i}\right)^2 = \frac{\left(E\frac{n}{i}\right)\left(E\frac{n}{i}+1\right)\left(2E\frac{n}{i}+1\right)}{2\,.\,3},$$

$$Sn^3 = \left\{n\left(\frac{n+1}{2}\right)\right\}^2,$$

accordingly $$\Sigma_n^1 i\left\{P(i^3)\times\frac{\left(E\frac{n}{i}\right)\left(E\frac{n}{i}+1\right)\left(2E\frac{n}{i}+1\right)}{6}\right\}$$

$$= \left(n\,\frac{n+1}{2}\right)^2.$$

Thus let $$n = 4,$$

then $$E(\tfrac{4}{4}) = 1,\ \frac{1\,.\,2\,.\,3}{2\,.\,3} = 1,\quad P(4^3) = 16\times 2 = 32,$$

$$E(\tfrac{4}{3}) = 1,\ \frac{1\,.\,2\,.\,3}{2\,.\,3} = 1,\quad P(3^3) = 9\times 2 = 18,$$

$$E(\tfrac{4}{2}) = 2,\ \frac{2\,.\,3\,.\,5}{2\,.\,3} = 5,\quad P(2^3) = 4\times 1 = 4,$$

$$E(\tfrac{4}{1}) = 4,\ \frac{4\,.\,5\,.\,9}{2\,.\,3} = 30,\quad P(1^3) = 1,$$

$$32 + 18 + 20 + 30 = 100,$$

$$\left(\frac{4\,.\,5}{2}\right)^2 = 100.$$

P.S. 2. The fundamental theorem in its simplest terms is as follows: If $i_1, i_2 \ldots i_r$ be any arbitrary positive integers

$$n^r = (\Sigma)^r\left[P\{(i_1)^{r-1}\}\,P\{(i_2)^{r-2}\}\ldots P(i_{r-1})\times\frac{n}{i_1 i_2\ldots i_r}\right];$$

the $(\Sigma)^r$ meaning merely the sign of summation r times repeated.

Example, let $$r = 2,\quad n = 4,$$

4 is divisible by $$1\times 1,\ 2\times 1,\ 4\times 1,$$

$$1\times 2,\ 2\times 2,$$

$$1\times 4,$$

$$P(1) = 1,\ P(2) = 1,\ P(4) = 2,$$

$$1\times 1\times 1 + 1\times 1\times 1 + 1\times 1\times 2$$

$$+ 2\times 1\times 1 + 2\times 1\times 1$$

$$+ 4\times 2\times 1$$

$$= 4 + 4 + 8 = 16 = 4^2.$$

It is obvious that this theorem must be capable of being reduced to an algebraical identity by writing $n = a^\alpha . b^\beta . c^\gamma \ldots$ as I have shown in the *note* above for the case $r = 1$.

The proof is left to the ingenuity of the reader.

36.

SUR UNE PROPRIÉTÉ DES NOMBRES PREMIERS QUI SE RATTACHE AU THÉORÈME DE FERMAT.

[*Comptes Rendus de l'Académie des Sciences*, LII. (1861), pp. 161—163.]

En étudiant les propriétés arithmétiques des nombres de Bernoulli et des autres nombres qui leur sont analogues, je suis tombé tout récemment sur une représentation du résidu par rapport au module p^2 de la même fonction exponentielle r^{p-1} dont le théorème de Fermat enseigne que le résidu par rapport à p est l'unité. Nommons le nombre entier $\frac{r^{p-1}-1}{p}$ le quotient de Fermat, dont p sera dit le module et r la base. En supposant que la base est un nombre premier, je trouve qu'on peut exprimer son résidu par rapport au module au moyen d'une série de fractions dont les dénominateurs seront tous les nombres inférieurs au module p, et les numérateurs des nombres périodiques qui ne dépendent que de la base r.

En effet, si le module est un nombre premier impair, les fractions qui expriment ce résidu auront pour dénominateurs successifs $p-1$, $p-2$, $p-3, \ldots, 2, 1$, et pour numérateurs* le cycle toujours répété $1, 2, 3, \ldots, r-1, r$, sauf à entendre que le cycle des numérateurs commence avec le terme qui est congru à $\frac{1}{p}$ par rapport à r. Par exemple, soit $r=5$, nous aurons d'après cette règle

$$\frac{5^{p-1}-1}{p} \equiv \frac{1}{p-1} + \frac{2}{p-2} + \frac{3}{p-3} + \frac{4}{p-4} + \frac{5}{p-5} + \frac{1}{p-6} + \frac{2}{p-7} + \ldots,$$

quand p est de la forme $10k+1$, mais* [à cause de $2 \times 3 \equiv 1 \pmod 5$]

$$\equiv \frac{3}{p-1} + \frac{4}{p-2} + \frac{5}{p-3} + \frac{1}{p-4} + \frac{2}{p-5} + \ldots,$$

quand p est de la forme $10k+2$. Il est bon de remarquer que la somme des réciproques des dénominateurs étant congrue à zéro pour le module p, on peut augmenter ou diminuer simultanément (à volonté) tous les termes

[* See correction below, p. 241.]

du cycle d'un même nombre quelconque, et conséquemment pour le cycle 1, 2, 3, ..., r, on peut substituer un cycle plus symétrique dans lequel le terme au milieu sera zéro. Ainsi on trouve en prenant $r=3$ (suivant le module p)

$$\frac{3^{p-1}-1}{p} \equiv -\frac{1}{p-1}+\frac{1}{p-3}-\frac{1}{p-4}+\frac{1}{p-6}-\frac{1}{p-7}\cdots,$$

ou

$$\equiv -\frac{1}{p-2}+\frac{1}{p-3}-\frac{1}{p-5}+\frac{1}{p-6}\cdots,$$

selon que p est de la forme $6n+1$ ou $6n-1$ respectivement.

Par exemple, faisons $p=7$, alors

$$-\frac{1}{6}+\frac{1}{4}-\frac{1}{3}+\frac{1}{1} \equiv -6+2-5+1 \equiv 6 \equiv \frac{3^6-1}{7}$$

c'est-à-dire $\equiv 104$ (mod 7).

Prenons encore $p=11$, alors

$$\frac{1}{9}-\frac{1}{8}+\frac{1}{6}-\frac{1}{5}+\frac{1}{3}-\frac{1}{2} \equiv 5-7+2-9+4-6 \equiv 0 \equiv \frac{3^{10}-1}{11}$$

c'est-à-dire $\equiv 22 \times (3^5+1)$ (mod 11).

Reste à donner la série pour le cas où la base du quotient de Fermat est le nombre 2 [cf. p. 235 below]. Par ce cas on trouve

$$\frac{2^{p-1}-1}{p} \equiv \frac{2}{p-3}+\frac{2}{p-4}+\frac{2}{p-7}+\frac{2}{p-8}+\frac{2}{p-11}+\cdots,$$

ou*

$$\equiv \frac{2}{p-2}+\frac{2}{p-3}+\frac{2}{p-6}+\frac{2}{p-7}+\frac{2}{p-10}+\cdots,$$

selon que p est de la forme $4k+1$ ou $4k-1$ respectivement. On peut énoncer des théorèmes plus généraux en substituant pour p et $p-1$ un nombre quelconque et un indicateur maximum respectivement. Pour le moment je me borne à faire une remarque sur la constitution arithmétique des nombres de Bernoulli et des nombres analogues qui entrent dans le développement des sécantes, dont l'étude m'a conduit à la loi donnée plus haut. Quant aux nombres de Bernoulli, on sait déjà par le théorème publié presque simultanément par MM. Clausen et Staudt, que le dénominateur de B_n est un produit de puissances simples de nombres premiers, étant composé du produit de tous les nombres premiers qui, diminués par l'unité, sont diviseurs de $2n$. Mais on paraît ne pas avoir fait la remarque importante que le numérateur de B_n contiendra tous les facteurs de n qui ne sont pas puissances des facteurs du dénominateur, de telle sorte que, si n contient

[* The sign of every term in the second expression should be changed. Stern, *Crelle*, Bd c. (1887), p. 188.]

p^i, mais ne contient pas $p-1$, le numérateur de B_n contiendra p^i; comme corollaire, on peut remarquer que, p étant un nombre premier quelconque, le numérateur de B_p contiendra toujours p. Quant aux nombres de la série pour la sécante qu'on peut nommer les nombres d'Euler qui le premier en a fait le calcul, et qui sont tous, comme on sait, des nombres entiers et positifs, et que je propose de dénoter par le symbole E, voici une propriété dont ils jouissent.

Désignons par p un nombre premier tel que $p-1$ ou plus généralement $(p-1)p^i$ soit un facteur de $2n$; alors, dans le cas où p est de la forme $4h+1$, p^{i+1} sera un facteur de E_n, mais dans le cas où p est de la forme $4h-1$, p^{i+1} sera un facteur de $2(-1)^{n-1}+E_n$. On comprend que E_n exprime le coefficient de $\dfrac{x^{2n}}{1.2\ldots 2n}$ dans le développement de sécante de x. Par parenthèse il sera bon de remarquer qu'en combinant les deux règles pour B_n et E_n on voit que le dénominateur de leur produit ne peut les contenir comme facteurs aucuns nombres premiers de la forme $4k+1$.

Euler a fait le calcul des E jusqu'à E_9, mais a donné une valeur erronée de cette dernière qui a été corrigée par M. Rothe, dans le *Journal de Crelle*, dans un Mémoire communiqué par M. Ohm*. Selon ma règle E_9+2 doit contenir les trois facteurs 3, 7, 19, ce qui s'accorde avec la valeur donnée par Rothe, mais non pas avec celle d'Euler. C'est à propos de ma nouvelle théorie des partitions des nombres que je me suis intéressé spécialement aux nombres de Bernoulli et d'Euler, qui tous les deux font une partie des développements qu'elle exige; en effet, on a besoin dans cette théorie de toutes les espèces de nombres dont la fonction génératrice est $\Sigma \dfrac{\rho^g}{e^t-\rho}$ (g étant un entier quelconque donné, et ρ une racine de l'unité d'un degré quelconque). Selon le degré de l'équation dont ρ est une racine primitive, on peut les nommer des nombres bernoulliens (ou si l'on veut sous-bernoulliens) d'un tel ou tel ordre. Jusqu'à présent on paraît n'avoir tenu compte que des nombres bernoulliens du premier et du second ordre (qui sont liés entre eux par le facteur exponentiel si bien connu) et de ceux du quatrième ordre auquel appartiennent en effet les nombres dits d'Euler. Mais ces nombres pour tous les ordres possèdent des propriétés arithmétiques très-dignes d'être étudiées; j'espère pouvoir y revenir et avoir l'honneur d'en faire le sujet d'une nouvelle communication à l'Académie.

[* Bd xx.]

37.

ADDITION À LA NOTE INSÉRÉE DANS LE PRÉCÉDENT COMPTE RENDU.

[*Comptes Rendus de l'Académie des Sciences*, LII. (1861), pp. 212—214.]

DANS la Note que j'ai eu l'honneur de présenter lundi dernier à l'Académie et qui a été insérée au *Compte rendu*, j'ai fait connaître [p. 231 above] le résidu du nombre E_n par rapport au module p^{i+1}, pour le cas où n contient le facteur $(p-1)p^i$, p étant un nombre premier impair. Il restait à exprimer ce même résidu dans le cas de $p=2$, c'est-à-dire dans le cas où n contient le facteur 2^i. Je trouve alors que E_n est congru à 1 suivant le module 2^{i+1}.

Mais j'ai obtenu en même temps un autre théorème très-général et très-utile pour ce genre de calculs; voici en quoi il consiste. Si n et n' sont des nombres entiers différents de zéro, et que $2n$ et $2n'$ soient congrus suivant le module $(p-1)p^i$, on aura

1° $$(-)^n E_n \equiv (-)^{n'} E_{n'} \pmod{p^{i+1}},$$

lorsque p sera un nombre premier impair,

2° $$E_n = E_{n'} \pmod{2^i},$$

lorsque l'on aura $p=2$, c'est-à-dire lorsque n et n' seront congrus par rapport à 2^{i-1} *.

Si l'on se rappelle que $E_1 = 1$ et que l'on combine la dernière partie de ce théorème avec celui qui se trouve énoncé plus haut, on arrive immédiatement à cette conséquence remarquable, que: Tout nombre d'Euler est de la forme $4k+1$. Cette loi si simple paraît avoir échappé à l'illustre inventeur de ces nombres puisque la valeur qu'il a donnée pour E_9 est de la forme $4k-1$. En se reportant aux théorèmes que j'ai obtenus, on ne peut guère commettre

* Un théorème tout à fait analogue doit avoir lieu pour les nombres de Bernoulli du 2me ordre, c'est-à-dire pour les nombres qui multiplient $\frac{x^{2n-1}}{1.2\ldots 2n}$ dans le développement de $\frac{1}{e^x+1}$ en série.

d'erreurs, sans les reconnaître, dans le calcul des nombres E. Par exemple, en partant des quatre valeurs

$$E_1 = 1, \quad E_2 = 5, \quad E_3 = 61, \quad E_4 = 1385,$$

on peut affirmer à priori que E_9 appartient à toutes les formes linéaires

$$5k+1, \quad 11k+1, \quad 13k+9, \quad 16k+1, \quad 17k+1;$$

en outre, à cause de la forme du double 18 de l'indice 9, lequel contient les facteurs 6, 2×3, 18, on sait que E_2 appartient encore aux formes linéaires

$$7k-2, \quad 9k-2, \quad 19k-2.$$

La valeur 24048 79661 671 obtenue par Euler ne satisfait à aucune de ces huit conditions; celles-ci, au contraire, sont toutes vérifiées par la valeur 24048 79675 441 donnée par M. Rothe. Ainsi on peut non-seulement affirmer que la première valeur est erronée, mais encore on a tout lieu de croire à l'exactitude de la seconde, quoique M. Rothe ne l'ait pas justifiée en présentant les détails de ses calculs.

Je remarquerai, en terminant, que le théorème énoncé plus haut offre le moyen de reconnaître si un nombre premier donné p peut figurer comme facteur dans quelqu'un des termes de la suite indéfinie E, ou dans quelqu'un des termes de la suite $E \pm \alpha$, tirée de la première en augmentant ou en diminuant ses termes d'un même nombre donné α. Car si cette circonstance se présente, à l'égard de l'une de ces suites, p sera nécessairement facteur d'un au moins des $\frac{p-1}{2}$ premiers termes de cette suite, et même de l'un des $\frac{p-3}{2}$ premiers termes dans le cas de $\alpha = 0$. On reconnaît l'exactitude de ce dernier point en se rappelant que tous les nombres premiers $4k+1$ étant facteurs des nombres d'Euler, il n'y a lieu de considérer que les diviseurs $4k-1$; or, d'après le théorème de la Note précédente, $E_{\frac{p-1}{2}}$ ne peut être divisible par p quand ce nombre est de la forme $4k-1$. Par exemple, l'inspection des quatre premiers nombres d'Euler 1, 5, 61, 5×277, suffit pour démontrer qu'aucun des nombres de la suite indéfinie E n'est divisible par 3, 7 ou 11.

Des considérations analogues s'appliquent sans difficulté aux diviseurs p^i.

38.

NOTE RELATIVE AUX COMMUNICATIONS FAITES DANS LES SÉANCES DES 28 JANVIER ET 4 FÉVRIER 1861.

(*Extrait d'une Lettre adressée à* M. SERRET *par* M. SYLVESTER.)

[*Comptes Rendus de l'Académie des Sciences*, LII. (1861), pp. 307, 308.]

DANS la Note que j'ai eu l'honneur de présenter récemment à l'Académie et qui a été insérée au *Compte rendu* de la séance du 4 février dernier, j'ai fait connaître un théorème qui lie entre elles deux congruences, dont l'une se rapporte aux *indices* des nombres d'Euler et l'autre à ces nombres eux-mêmes; et, en même temps, j'ai avancé* qu'un théorème analogue doit avoir lieu pour les nombres de Bernoulli. Voici en quoi consiste ce théorème:

Soient p un nombre premier, n et n' deux nombres entiers dont les doubles $2n$, $2n'$ ne contiennent aucun des facteurs $p, p-1$, et soient congrus suivant le module $(p-1)\,p^i$ (i étant un entier quelconque positif ou nul); les nombres de Bernoulli B_n et $B_{n'}$ seront liés entre eux par la congruence

$$(-)^n \frac{B_n}{n} \equiv (-)^{n'} \frac{B_{n'}}{n'} \pmod{p^{i+1}}.$$

On doit remarquer que, d'après les conditions de l'énoncé, p ne peut être égal ni à 2, ni à 3.

Pour donner un exemple de ce théorème, prenons $n=7$, $n'=17$; les nombres $2n$ et $2n'$ seront congrus par rapport à $11-1$ et aussi par rapport à $(5-1)\,5$; d'ailleurs

$$\frac{B_7}{7} = \frac{1}{6}, \quad \frac{B_{17}}{17} = \frac{2\ 577\ 687\ 858\ 367}{17 \times 6};$$

par conséquent, on aura

$$\frac{B_7}{7} - \frac{B_{17}}{17} = -\frac{2\ 577\ 687\ 858\ 350}{102} \equiv 0 \pmod{11 \times 25},$$

ce que l'on peut vérifier immédiatement.

* Voir à la page [232] de ce volume.

Je profite de cette occasion pour présenter une remarque importante au sujet de la formule par laquelle j'ai exprimé [p. 230 above] le résidu de $\frac{r^{p-1}-1}{p}$ suivant le module p, dans le cas où l'on a $r=2$. Cette formule peut être remplacée avec avantage par la suivante:

$$\frac{2^{p-1}-1}{p} \equiv -\frac{\frac{1}{2}}{p-1} + \frac{\frac{1}{2}}{p-2} - \frac{\frac{1}{2}}{p-3} + \dots \pmod{p},$$

qui est tout à fait semblable aux formules relatives au cas où r est un nombre premier impair, et qui n'exige pas, comme celle que j'avais trouvée d'abord, que l'on distingue les formes $4k+1$ et $4k-1$ du module premier p.

Pour ce qui concerne le cas où la base r du quotient de Fermat $\frac{r^{p-1}-1}{p}$ est un nombre composé, il n'y a aucune difficulté à exprimer le résidu de ce quotient suivant le module p, par des suites de fractions dont les dénominateurs sont les nombres inférieurs à p, et dont les numérateurs constituent des cycles exactement comme dans le cas où r est un nombre premier. Pour obtenir, en effet, les suites dont je viens de parler, il suffit de faire usage de la congruence évidente

$$\frac{(abc\dots k)^{p-1}-1}{p} \equiv \frac{(a^{p-1}-1)+(b^{p-1}-1)+(c^{p-1}-1)+\dots+(k^{p-1}-1)}{p} \pmod{p},$$

dans laquelle $a, b, c, \dots, k$, désignent des entiers quelconques égaux ou inégaux. Au moyen de cette congruence, on ramène immédiatement, par de simples additions, le cas où r est un nombre composé au cas où cette base est un nombre premier.

39.

SUR L'INVOLUTION DES LIGNES DROITES DANS L'ESPACE CONSIDÉRÉES COMME DES AXES DE ROTATION.

[*Comptes Rendus de l'Académie des Sciences,* LII. (1861), pp. 741—745.]

Note présentée par M. CHASLES.

ON sait qu'on peut représenter un déplacement infiniment petit quelconque d'un corps rigide au moyen des rotations du corps autour de six axes. En effet, la méthode usuelle de représenter ce déplacement au moyen de trois mouvements de rotation et de trois de translation rentre, comme un cas particulier, dans la méthode dont je parle, en prenant trois axes sur les six à une distance infiniment éloignée du corps. Cependant il n'est pas vrai que la disposition des six axes soit arbitraire dans un sens absolu. Car si les six axes sont choisis de telle façon qu'on peut trouver des forces qui, agissant dans leurs directions sur un corps rigide, feront équilibre entre elles, les rotations autour de ces axes ne restent plus indépendantes, c'est-à-dire une rotation autour d'un de ces axes peut être décomposée dans ses rotations autour des autres, et conséquemment les six axes n'équivaudront en réalité qu'à cinq tout au plus. Dans ce cas, on peut dire que les six axes forment un système en *involution*; et l'objet de cette Note est de préciser les caractères géométriques par lesquels on peut reconnaître une pareille involution et, de plus, de fournir les moyens de construire un tel système, et, en supposant cinq des axes donnés, de trouver le lieu le plus général du sixième.

L'auteur traite d'abord les cas où les droites données sont en nombre inférieur à cinq, et où il s'agit d'en déterminer une de plus qui fasse avec les droites données un système de droites pouvant représenter les directions d'un système de forces (ou de rotations, ce qui revient au même) se faisant équilibre. Il a occasion de citer la Statique de M. Mœbius (*Lehrbuch der Statik*; Lepzig, 1837), et surtout un Mémoire dans lequel ce savant géomètre a traité ces mêmes questions (*Ueber die Zusammensetzung unendlich kleiner Drehungen*; voir *Journal de Crelle,* 1, XVIII, p. 189—212).

Il continue ainsi :

Je passe à la question (objet principal de cette Note) de l'involution du nombre maximum de six lignes. Je suppose que ces lignes soient données, à l'exception d'une seule dont il s'agit de déterminer le lieu géométrique. Je combine les cinq lignes données quatre à quatre ; et, quand cela peut se faire, je mène deux transversales rencontrant les quatre droites de chaque combinaison. L'on aura ainsi, en général, cinq paires de transversales.

Dans ces circonstances, je suis à même d'énoncer la proposition géométrique remarquable qui suit : En choisissant arbitrairement un point dans l'espace, et en menant par ce point une transversale à chacune des paires de transversales nommées plus haut, toutes ces transversales ainsi menées (en général au nombre de cinq) se trouveront dans le même plan ; et corrélativement, en coupant les paires de transversales par un plan quelconque, les droites (généralement cinq en nombre) qui joignent les deux points d'intersection de la même paire, se croiseront toutes dans le même point. Je nomme un plan et un point ainsi déterminés réciproquement, *pôle* et *plan polaire.*

Je prends arbitrairement une droite qui coupe une paire quelconque de transversales, et je choisis à volonté deux points O et O' sur cette ligne ; je trouve les plans polaires respectifs de O et O' (ce qu'il est toujours possible de faire, parce qu'il y a deux paires de transversales au moins, outre la paire coupée par la ligne OO'), disons P et P'. Dans le plan P, je prends à volonté deux points E et F, et par E et F je mène deux lignes qui coupent respectivement les deux lignes d'une quelconque des paires de transversales dont j'ai parlé et qui rencontrent le plan P' en E' et F' ; je construis deux *faisceaux homographiques* situés dans P et P', pour lesquels les rayons OO', OE', OF' correspondent respectivement à $O'O$, $O'E$, $O'F$, et je dis que toute droite qui coupe deux rayons correspondants quelconques de ces deux faisceaux sera en involution avec les cinq lignes données, et *vice versâ,* chaque ligne en involution avec les cinq lignes données coupera deux rayons correspondants de ces deux faisceaux.

Jusqu'ici j'ai supposé que la ligne commune aux deux faisceaux a été choisie dans une direction qui traverse les deux droites d'une des paires de transversales connues. Cette restriction peut maintenant être abandonnée, car on pourra choisir pour la ligne des centres des faisceaux une droite quelconque qui coupe deux rayons correspondants ; c'est-à-dire une sixième ligne quelconque qui se trouve en involution avec cinq lignes données, pourra servir de rayon commun à deux faisceaux plans homographiques ainsi disposés que chaque ligne coupant deux rayons correspondants dans deux faisceaux sera elle-même en involution avec les cinq lignes données.

J'ajoute, comme étant compris virtuellement dans ce qui précède, que le lieu de toutes les lignes qui sont en involution avec les lignes données et

passent par un point donné, est le plan polaire de ce point (selon la définition expliquée ci-dessus du pôle et du plan polaire). M. Mœbius avait déjà démontré que ce lieu doit être un plan; mais il avait omis de donner le moyen de la construire.

On peut aussi remarquer que chacune des cinq lignes données passe par deux rayons correspondants dans chaque couple de faisceaux construit selon la méthode fournie plus haut; la même chose aura lieu pour chaque ligne droite qui se trouve dans l'hyperboloïde dont trois quelconques des lignes données sont des génératrices; et j'ajoute que six lignes quelconques, chacune desquelles passe par deux rayons correspondants dans un couple de faisceaux, seront en involution entre elles.

On peut donner le nom d'*axes conjugués* à chaque paire de lignes dont toutes les transversales sont en involution avec un système donné de cinq droites. Ces systèmes d'axes possèdent entre eux des propriétés remarquables dont, pour le moment, je veux seulement indiquer la suivante: *On peut toujours mener un hyperboloïde par deux paires quelconques d'axes conjugués.*

Voici les propriétés métriques les plus frappantes des couples de faisceaux homographiques dont il est question. Les deux droites* perpendiculaires à la ligne des centres dans les deux plans de l'homographie seront des rayons correspondants; en conséquence, si l'on fait tourner l'un des faisceaux autour de la ligne des centres jusqu'à ce qu'il se trouve dans le même plan avec l'autre faisceau, les rayons correspondants s'entrecouperont dans une ligne droite perpendiculaire à la ligne des centres*, et je trouve que le point où cette perpendiculaire coupe la ligne des centres sera le *pôle* du plan qui, passant par cette ligne, divise en deux parties égales l'angle dièdre formé par les deux plans homographiques. Nommons ce point le *pivot* de la ligne des centres: j'aurai tout à l'heure l'occasion d'y revenir.

Considérons l'ensemble de tous les axes conjugués, c'est-à-dire de toutes les paires de rayons correspondants de tous les couples de faisceaux appartenant à un système donné de cinq lignes, je dis qu'on peut appliquer dans les directions de ces deux axes deux forces dont le rapport de grandeur sera absolument constant pour le système donné, de façon qu'elles seront statiquement équivalentes à deux forces de grandeurs convenablement choisies dans les directions de deux autres axes conjugués quelconques. En considérant une ligne quelconque coupant ces deux axes comme la ligne des centres d'un couple homographique contenant ces deux axes pour rayons correspondants, les deux forces qui doivent agir dans leur direction pour balancer les deux forces fixes auront des *moments* égaux par rapport an *pivot* de cette ligne. Par conséquent, si l'on connaît le pivot d'une seule ligne de centres qui rencontre deux axes conjugués fixes porteurs des lignes en involution avec un système de cinq lignes données, on peut construire tous les couples de

[* See the correction below, p. 244.]

faisceaux homographiques dont les lignes et centres rencontrent ces mêmes axes. Car non-seulement les plans d'homographie de chaque couple seront connus, mais le rapport anharmonique de ses deux faisceaux le sera de même, et cela parce que la position des pivots devient déterminée. On peut ajouter que, puisque tous les pivots appartenant aux mêmes axes conjugués doivent être très-éloignés de ces deux axes par des distances perpendiculaires qui sont dans un rapport constant entre elles, le lieu géométrique qui les contient tous sera une surface du second degré et évidemment un hyperboloïde.

Puisque tous les axes conjugués appartenant à un système de cinq droites données peuvent être considérés comme les directions de deux forces qui équivalent statiquement à deux forces données en grandeur et en position, on voit par ce qui a été dit plus haut que l'ensemble infini de toutes les paires de forces équivalentes entre elles possède cette propriété remarquable, déjà donnée par M. Mœbius (*Journal de Crelle*, t. x. p. 317), que les transversales tirées du même point quelconque dans l'espace de manière à rencontrer les directions des forces dans chaque paire, seront situées dans le même plan, qu'on peut nommer le plan polaire au point donné. C'est une polarité réciproque tout aussi nettement définie que la polarité plus ordinaire qui se rattache à une surface donnée du second degré. On voit que la polarité dont il est ici question peut être considérée comme se rattachant à deux paires de lignes droites qui sont les génératrices du même hyperboloïde.

Dans une communication subséquente, j'ajouterai brièvement les caractères algébriques de tous les cas d'involution, et je ferai connaître un déterminant (composé de déterminants obtenus par la combinaison des coefficients des équations de six ou d'un moindre nombre de lignes droites, mises sous leurs formes les plus générales) au moyen duquel on peut s'assurer si ces droites sont en involution ou non, et, de plus, distinguer entre les diverses espèces d'involution, et même reconnaître d'autres dispositions singulières de ces lignes qui constituent une espèce d'involution imparfaite. Toute cette théorie découle, selon ma méthode de la traiter, des notions les plus élémentaires de la statique des corps rigides.

40.

NOTE SUR L'INVOLUTION DE SIX LIGNES DANS L'ESPACE.

[*Comptes Rendus de l'Académie des Sciences*, LII. (1861), pp. 815—817.]

DÉSIGNONS six droites par les chiffres 1, 2, 3, 4, 5, 6. Prenons les équations de chacune de ces lignes sous la forme la plus générale (en nous servant de coordonnées tétraédrales). Ainsi, soit la ligne i définie par les équations

$$a_i x + b_i y + c_i z + d_i u = 0, \quad \alpha_i x + \beta_i y + \gamma_i z + \delta_i u = 0\,;$$

et de même la ligne j par les équations

$$a_j x + b_j y + c_j z + d_j u = 0, \quad \alpha_j x + \beta_j y + \gamma_j z + \delta_j u = 0,$$

et sous-entendons par i, j le déterminant

$$\begin{vmatrix} a_i, & b_i, & c_i, & d_i \\ \alpha_i, & \beta_i, & \gamma_i, & \delta_i \\ a_j, & b_j, & c_j, & d_j \\ \alpha_j, & \beta_j, & \gamma_j, & \delta_j \end{vmatrix}.$$

Cela étant fait, formons le déterminant (que je nommerai Δ_6):

$$\begin{vmatrix} & 1,2 & 1,3 & 1,4 & 1,5 & 1,6 \\ 2,1 & & 2,3 & 2,4 & 2,5 & 2,6 \\ 3,1 & 3,2 & & 3,4 & 3,5 & 3,6 \\ 4,1 & 4,2 & 4,3 & & 4,5 & 4,6 \\ 5,1 & 5,2 & 5,3 & 5,4 & & 5,6 \\ 6,1 & 6,2 & 6,3 & 6,4 & 6,5 & \end{vmatrix}.$$

Si les six droites 1, 2, 3, 4, 5, 6 sont en involution, on aura

$$\Delta_6 = 0\,;$$

et réciproquement si $\Delta_6 = 0$, les six lignes seront en involution.

En nous bornant aux cinq chiffres 1, 2, 3, 4, 5, on peut former un déterminant analogue à Δ_6 (disons Δ_5) qui ne contiendra que cinq lignes et cinq colonnes, et qui sera un déterminant mineur du premier ordre du grand déterminant Δ_6. De même pour Δ_4, etc.

Si $\Delta_6 = 0$ et $\Delta_5 = 0$ (sans que *tous* les déterminants mineurs du premier ordre de Δ_6 soient zéro), les cinq lignes 1, 2, 3, 4, 5 formeront un système en involution entre elles. Si tous les déterminants mineurs du premier ordre sont zéro (ce qui ne suppose qu'une seule condition de plus, c'est-à-dire trois conditions en tout), les six lignes de 1 à 6 seront toutes rencontrées par la même droite.

Si $\Delta_6 = 0$, $\Delta_5 = 0$, $\Delta_4 = 0$, alors en général les quatre lignes de 1 à 4 seront en involution entre elles; je n'insisterai pas ici sur les cas possibles d'exception; j'ajouterai seulement que si $\Delta_5 = 0$ sans autre condition, les cinq lignes de 1 à 5 seront toutes rencontrées par la même droite. Si $\Delta_4 = 0$, sans autre condition, les quatre lignes de 1 à 4 n'admettront qu'*une seule transversale* qui les rencontre toutes quatre, au lieu des deux qui existent ordinairement pour quatre droites dans l'espace. C'est M. Cayley qui le premier a fait cette dernière remarque. De plus il a trouvé indépendamment un déterminant qui est égal à la racine carrée de Δ_6 et qui conséquemment sert tout aussi bien que Δ_6 pour définir l'involution.

L'espace me manque pour produire ici cet autre déterminant, mais je dois ajouter que c'est d'une grande utilité dans l'étude analytique de la théorie d'involution.

Je prie qu'il me soit permis de profiter de cette occasion pour rectifier une erreur qui s'est glissée dans l'énoncé d'un théorème donné dans les *Comptes rendus* (26 janvier 1861). Dans le second paragraphe de la Note [p. 229 above], au lieu de "pour numérateurs le cycle toujours répété...par rapport à r," lisez "pour numérateurs le cycle toujours répété des nombres entiers congrus* à $\frac{1}{p}, \frac{2}{p}, \frac{3}{p}, \ldots, \frac{r}{p}$ et compris parmi les nombres $1, 2, 3, \ldots, r$." Et plus bas au lieu de "mais à cause... $10k+2$" lisez: "mais, à cause de $7 \times 3 \equiv 1 \pmod 5$,

$$\equiv \frac{3}{p-1} + \frac{1}{p-2} + \frac{4}{p-3} + \frac{2}{p-4} + \frac{5}{p-5} + \ldots$$

quand p est de la forme $10k+7$."

[* mod. r.]

41.

NOTE SUR LES 27 DROITES D'UNE SURFACE DU 3^e DEGRÉ.

[*Comptes Rendus de l'Académie des Sciences*, LII. (1861), pp. 977—980.]

MES recherches sur l'involution d'axes de rotation m'a forcément conduit à étudier les propriétés géométriques des 27 lignes droites qui sont situées sur chaque surface générale du 3^e degré, et j'ai trouvé un théorème pour représenter ces lignes d'une manière à ôter toute difficulté en approfondissant leurs rapports mutuels. En se servant de ce théorème que je lui ai communiqué, M. Cayley est parvenu avant moi à donner une construction géométrique de ces 27 lignes; mais sa construction exige la connaissance de 8 droites données, c'est-à-dire d'une ligne droite prise comme base, coupée par 3 paires de droites qui se croisent (et dont les traces sur la base forment un système de 6 points en *involution*), et coupée aussi par une 7^e droite. C'est une conséquence de la théorie connue de ces 27 lignes (comme l'a bien montré mon ami distingué), qu'une surface du 3^e degré peut être construite, qui contiendra ces 8 droites (la base et les 7 autres lignes qui la coupent).

En me prévalant d'une autre façon de mon théorème, je suis parvenu à donner une construction d'une nature semblable, mais plus symétrique et plus simple que celle de M. Cayley, au moins dans des données qui pour moi sont une ligne droite coupée par 5 autres droites sans autre condition.

C'est le système de droites qui s'offre tout naturellement dans la théorie de mécanique dont je m'occupais et dont je me fus proposé de prime abord de me servir pour résoudre la question au temps même que j'ai reçu de la part de M. Cayley la solution avec le nouveau système de données dont j'ai parlé plus haut. Voici une première observation qui sera utile dans la suite. En prenant 5 lignes droites tout à fait arbitraires, disons a, b, c, d, e, en les joignant quatre à quatre, on peut construire 5 systèmes de paires de transversales; mais si les 5 données rencontrent la même droite, disons x, il est évident que ces 5 paires se réduiront à cette droite et 5 autres transversales; or il est facile de démontrer que ces 5 dernières seront toutes rencontrées elles-mêmes par une autre droite, disons ξ; elles peuvent être convenablement nommées $\alpha, \beta, \gamma, \delta, \epsilon$; où α est la seconde transversale à b, c, d, e; β à a, c, d, e, etc.

Je fais une seconde observation très-importante, voir que 6 droites dont 5 sont coupées par la 6^e, sont situées sur la même surface du 3^e degré, et réciproquement tout système de 5 droites sur une surface du 3^e degré qui ne se coupent pas entre elles sont coupées par la même droit. Je dois ajouter que si 5 droites sont toutes coupées par les mêmes 2 lignes droites, on peut faire passer un nombre infini de surfaces du 3^e degré par ces 7 lignes, parmi lesquelles se trouveront comprises 2 surfaces réglées, et le théorème réciproque aura aussi lieu.

Écrivons les 12 lignes

$$\begin{array}{ccccc} & & x & & \\ a, & b, & c, & d, & e \\ \alpha, & \beta, & \gamma, & \delta, & \epsilon \\ & & \xi & & \end{array}$$

où on suppose que a, b, c, d, e sont rencontrées par x, mais non par aucune autre droite, et que $\alpha, \beta, \gamma, \delta, \epsilon$ sont les 5 transversales à a, b, c, d, e prises quatre à quatre, et ξ la transversale commune à $\alpha\beta\gamma\delta\epsilon$.

Formons encore le système $ABCDE$, où A est la transversale à $xa\alpha\xi$, B à $xb\beta\xi$, C à $xc\gamma\xi$, D à $xd\delta\xi$, E à $xe\epsilon\xi$; c'est-à-dire A est l'intersection des plans qui passent respectivement par xa, $\xi\alpha$ et de même pour B, C, D, E.

Finalement menons les 10 transversales désignées par la combinaison des symboles des 4 lignes qu'elles rencontrent respectivement, c'est-à-dire $a\alpha b\beta$, $a\alpha c\gamma$, $a\alpha d\delta$, $a\alpha e\epsilon$, $b\beta c\gamma$, $b\beta d\delta$, $b\beta e\epsilon$, $c\gamma d\delta$, $c\gamma e\epsilon$, $d\delta e\epsilon$. Il est bon de remarquer que les deux droites a, β se croisent, comme aussi b, α, et que $a\alpha b\beta$ signifie l'intersection des deux plans de $a\beta$, $b\alpha$. Une remarque semblable a lieu pour les autres droites de cette série de 10. On voit qu'on a obtenu

$$1 + 5 + 5 + 1 + 5 + 10 = 27 \text{ droites.}$$

Il est facile de démontrer géométriquement que toutes ces droites sont situées sur la même surface du 3^e degré, et que cette surface ne contiendra pas aucune autre ligne droite sur elle. Je dois ajouter, pour rendre plus complète l'image de ce système de 27 droites, que les 10 dernières couperont chacune 6 autres au-dessus des 4 exprimées par la notation quaternaire même, c'est-à-dire $a\alpha b\beta$ ne rencontrera pas seulement a, α, b, β, mais aussi C, D, E et $c\gamma d\delta$, $c\gamma e\epsilon$, $d\delta e\epsilon$ et ainsi pour les autres, de sorte qu'on trouvera facilement que chaque droite des 27 sera rencontrée par 10 autres, chaque combinaison de 3 qui ne se rencontrent pas par 3 autres qui ne se rencontrent pas, chaque combinaison de 4 qui ne se rencontrent pas par 2 autres sur la surface, etc.; conformément aux beaux résultats de MM. Salmon et Cayley, déjà, il y a longtemps, donnés dans le *Cambridge and Dublin Mathematical Journal.*

On peut résumer en peu de mots la construction précédente.

5 droites rencontrées par une 6e étant données, on construit 5 autres rencontrées par une nouvelle 6e, telles que chaque droite d'un des groupes de 5 rencontre 4 de l'autre groupe. Les 12 droites ainsi liées s'entrecoupent (par construction) en $2 \times 5 + 5 \times 4$, c'est-à-dire en 30 points, et conséquemment sont situées deux à deux en 30 plans dont chacun joint d'un rapport de réciprocité avec quelque autre. Les intersections de ces paires des plans réciproques donnent naissance à 15 nouvelles droites, lesquelles, combinées avec les 12 déjà nommées, constituent un système (le plus général qui peut exister) de 27 droites *réelles* appartenant à une surface du 3e degré. Il va sans dire qu'il existe des surfaces de ce degré pour lesquelles les 27 droites ne sont pas toutes *réelles.*

Je me propose de faire construire en fil de fer ou d'archal un système de 27 droites par la méthode donnée en haut, et d'en faire des copies stéréographiques, de sorte qu'on pourra éprouver le plaisir inattendu de voir avec les yeux du corps toutes les droites (le squelette pour ainsi dire) d'une surface du 3e degré avec leurs 135 points d'intersection, les 45 triangles les hexagones situés sur le même hyperboloïde et des autres non pas ainsi situés, et les autres merveilles de cette involution si compliquée, mais en même temps si symétrique.

Je prie qu'il me soit permis de profiter de cette occasion pour rectifier une erreur dans ma communication donnée dans les *Comptes rendus* (15 avril 1861): Dans le 4e paragraphe* les mots "les deux droites perpendiculaires ....correspondants; en conséquence" doivent être rayés. Plus bas dans le même paragraphe les mots "perpendiculaire à la ligne des centres" doivent être rayés, et dans la ligne suivante pour "perpendiculaire" on doit lire "droite."

La belle observation de M. Chasles dans le même numéro des *Comptes rendus,* sur une méthode de trouver un système de 6 droites en involution au moyen des perpendiculaires aux trajectoires de 6 points dans le déplacement infiniment petit d'un corps rigide, se trouve confirmée par une application assez simple de la méthode des vélocités virtuelles.

Car en donnant à un corps rigide sollicité par 6 forces agissant suivant des lignes droites données 6 déplacements arbitraires, on obtiendra 6 équations indépendantes et homogènes auxquelles les valeurs des 6 forces doivent satisfaire pour qu'elles fassent équilibre entre elles; ce qui en général ne sera pas possible; mais en supposant qu'un des déplacements peut être effectué d'une telle manière, que toutes les vélocités virtuelles des 6 points d'application seront nulles, une des six équations disparaîtra, c'est-à-dire deviendra une identité, et le système de cinq équations linéaires qui restent admettra une solution."

[* p. 238 above.]

42.

GÉNÉRALISATION D'UN THÉORÈME DE M. CAUCHY*.

[*Comptes Rendus de l'Académie des Sciences*, LIII. (1861), pp. 644, 645.]

DANS son Mémoire sur les *arrangements*, 1844, M. Cauchy a établi le théorème suivant:

Soit n un nombre entier donné,

$$\alpha a + \beta b + \gamma c + \ldots + \lambda l = n\,;$$

en supposant $a, b, c, \ldots, l$ des nombres entiers et inégaux, $\alpha, \beta, \gamma, \ldots, \lambda$ des nombres entiers, et en faisant varier de toutes les manières possibles les valeurs du système $a, b, c, \ldots, l$, on aura

$$\Sigma \frac{1}{\pi\alpha \,.\, \pi\beta \ldots \pi\lambda a^{\alpha} b^{\beta} \ldots l^{\lambda}} = 1,$$

où πx signifie le produit $1 \,.\, 2 \,.\, 3 \ldots x$.

Je vais démontrer qu'on peut exprimer d'une manière très-simple la valeur générale de $\Sigma \dfrac{\omega^{\alpha+\beta+\ldots+\lambda}}{\pi\alpha . \pi\beta \ldots \pi\lambda a^{\alpha} b^{\beta} \ldots l^{\lambda}}$ pour une valeur quelconque d'une constante ω.

En effet, il est très-facile de voir qu'en posant l'équation en nombres positifs et entiers

$$x_1 + x_2 + x_3 + \ldots + x_r = n,$$

et en attribuant à $x_1, x_2, \ldots, x_r$ toutes les valeurs possibles qui satisfont à cette équation (en regardant comme distinctes les solutions qui diffèrent dans les valeurs de x, quoique contenant le même système de valeurs), on peut représenter la série (nommée fonction de n et ω) sous la forme

$$\sum_{r=\infty}^{r=1} \Sigma \frac{1}{x_1 x_2 \ldots x_r} \frac{\omega^r}{\pi(r)},$$

c'est-à-dire

$$\sum_{r=\infty}^{r=1} F(r, n) \frac{\omega^r}{\pi(r)}.$$

[* See below, p. 290.]

Or on voit immédiatement que $F(r, n)$ n'est autre chose que le coefficient de t^n dans le développement de la fonction génératrice $\left(t+\frac{t^2}{2}+\frac{t^3}{3}+\ldots\right)^r$, c'est-à-dire dans le développement de $[\log(1-t)^{-1}]^r$. Donc évidemment la série totale sera le coefficient de t^n dans le développement de $e^{\omega \log[(1-t)^{-1}]}$, c'est-à-dire de t^n dans $\left(\frac{1}{1-t}\right)^\omega$.

En prenant $\omega=1$, on voit que la valeur est toujours l'unité pour toute valeur de n, ce qui est le théorème de Cauchy. En prenant $\omega=-i$, i étant un nombre entier quelconque plus petit que n, on trouve la valeur zéro. Pour le cas de $\omega=-1$, cette remarque avait déjà été faite par M. Cayley, dans le *Philosophical Magazine* (mars 1861). En prenant $\omega=\frac{1}{2}$, on trouve pour la valeur de la série $\frac{1.3.5\ldots 2n-1}{2.4.6\ldots 2n}$, ce qui peut se déduire aussi par la méthode des arrangements, en se servant du théorème que le nombre des *substitutions* de $2n$ lettres qui peuvent être représentées par des égales d'un rang exclusivement pair est $[1.3.5\ldots(2n-1)]^2$, théorème que je crois être nouveau, mais qui est intimement lié au théorème célèbre de M. Cayley sur la valeur des déterminants dits *gauches*.

Voici une dernière observation que je fais sur le théorème général. On remarquera que l'exposant de $\omega^{\alpha+\beta+\ldots+\lambda}$ est le nombre des parties dans la partition de n, représentée par α répétitions de a, β de b, …, λ de l: je nommerai donc $\alpha+\beta+\gamma+\ldots+\lambda$ l'*indice* de cette partition, et je dis qu'étant donné le *nombre* de ces indices, disons ν (nombre qu'on peut trouver pour une valeur quelconque de n par le théorème très-bien connu d'Euler sur les partitions indéfinies), on peut faire dépendre les valeurs de ces ν indices de la solution d'un système de 2μ équations algébriques à 2μ inconnues. Car pour une valeur quelconque de ω on connaîtra par le théorème du texte la valeur de $\frac{\omega^{i_1}}{q_1}+\frac{\omega^{i_2}}{q_2}+\ldots+\frac{\omega^{i_\mu}}{q_\mu}$, où $i_1, i_2, \ldots, i_\mu$ seront les indices cherchés, et $q_1, q_2, \ldots, q_\mu$ des quantités inconnues, mais indépendantes de ω. En substituant pour ω successivement $\omega, \omega^2, \omega^3, \ldots, \omega^{2\mu}$ et en écrivant $\omega^{i_r}=I_r$, on aura 2μ équations de la forme

$$\frac{I_1{}^k}{q_1}+\frac{I_2{}^k}{q_2}+\ldots+\frac{I_\mu{}^k}{q_\mu}=C,$$

k prenant toutes les valeurs de 1 jusqu'à 2μ. On peut donc former une équation dont dépendra la valeur de chacune des quantités $I_1, I_2, \ldots, I_\mu$, et conséquemment de leurs logarithmes $i_1, i_2, \ldots, i_\mu$, les μ indices de la partition indéfinie de n.

43.

ADDITION À LA NOTE INTITULÉE: "GÉNÉRALISATION D'UN THÉORÈME DE M. CAUCHY," ET INSÉRÉE DANS LE "COMPTE RENDU" DE LA SÉANCE DU 7 OCTOBRE DERNIER.

[*Comptes Rendus de l'Académie des Sciences*, LIII. (1861), pp. 722—725.]

En suivant la même marche que dans la Note dont il s'agit [p. 245 above], on parvient très-facilement à résoudre une question un peu plus compliquée de la théorie des *arrangements*, savoir : *Trouver le nombre de substitutions de n lettres qu'on peut représenter par le moyen d'un nombre donné r de substitutions cycliques d'ordres impairs.*

Pour que ce nombre ne soit pas zéro, il faut que $n-r$ soit un nombre pair $2i$; alors le nombre demandé sera la somme suivante,

$$\Sigma\,[(\nu_1^2+\nu_1)(\nu_2^2+\nu_2)\ldots(\nu_e^2+\nu_e)\ldots(\nu_i^2+\nu_i)],$$

où le signe Σ se rapporte à tous les systèmes possibles de nombres entiers $\nu_1, \nu_2, \ldots, \nu_e, \ldots, \nu_i$ qui satisfont aux inégalités

$$\nu_e > \nu_{e-1}+1, \quad \nu_e < n-1.$$

Désignons par $[n, r]$ le nombre des substitutions exprimé par la somme précédente, et par (n, r) le nombre correspondant pour le cas où les r substitutions cycliques sont chacune indifféremment d'ordre pair ou d'ordre impair. On a déjà trouvé que (n, r) est la somme des produits de $n-r$ quelconques des nombres $1, 2, 3, \ldots, (n-1)$, et l'on voit à présent que $[n, r]$ n'est autre chose que la somme des produits de $\frac{n-r}{2}$ facteurs dont chacun est le produit d'un terme de la même suite de nombres par le terme suivant. Et de même que (n, r) satisfait à l'équation fonctionnelle

$$\frac{(n+1, r+1)-(n, r)}{n} = (n-1, r),$$

la fonction $[n, r]$ satisfait à l'équation analogue

$$\frac{[n+2, r+2]-[n, r]}{n}=(n+1)[n-2, r]+(n-1)[n-3, r-1],$$

comme il est facile de s'en assurer.

On peut ajouter que (n,r) (pour $n-r$ positif) et $[n,r]$ $\left(\text{pour } \frac{n-r}{2} \text{ positif}\right)$ sont tous deux divisibles par n quand n est un nombre premier. Ce théorème est bien connu en ce qui concerne (n, r), mais il me paraît nouveau à l'égard de $[n, r]$. Au reste, on peut appliquer aux deux cas la même démonstration fondée sur ce que le nombre de produits de cycles correspondant à chaque partition de n est évidemment un multiple de n.

Voici un exemple du théorème énoncé au commencement de cette Note: Le nombre des substitutions de 6 lettres qu'on peut représenter par le produit de deux cycles d'ordres impairs sera, d'après notre formule générale,

$$2\times 12+2\times 20+6\times 20=184,$$

ce que l'on peut vérifier bien facilement en remarquant que ce nombre doit être

$$6\times 24+10\times 4=184.$$

On démontre encore très-facilement que le nombre total des substitutions de n lettres représentées par le produit de substitutions cycliques d'ordres impairs est

$$[1.3.5\ldots(n-1)]^2$$

quand n est pair (c'est le même nombre que nous avons déjà obtenu pour les substitutions cycliques d'ordre pair), et

$$n[1.3.5\ldots(n-2)]^2$$

quand n est impair.

On peut donner une extension* très-considérable aux théorèmes énoncés précédemment, en considérant le nombre des substitutions de n lettres formées avec les produits de r substitutions cycliques où l'ordre de chaque cycle est congru à un nombre ρ par rapport à un module donné μ.

La solution dépend toujours des combinaisons des nombres de la série $1, 2, 3, \ldots, (n-1)$. Je me bornerai ici au cas de $\rho=1$ qui est le plus simple, en exceptant celui de $\rho=0$, dont la solution est évidente. Dans le cas de $\rho=1$, le nombre cherché est exprimé par la somme

$$\Sigma\frac{\Pi(\nu_1+\mu-1)\,\Pi(\nu_2+\mu-1)\ldots\Pi(\nu_i+\mu-1)}{\Pi(\nu_1-1)\,\Pi(\nu_2-1)\ldots\Pi(\nu_i-1)},$$

[* Cf. below, p. 293.]

où l'on fait $i = \dfrac{n-r}{\mu}$ et où les nombres ν sont assujettis aux conditions

$$\nu_e > \nu_{e-1} + \mu - 1, \quad \nu_e < n - 1,$$

et, en conséquence, on peut énoncer le théorème suivant:

Si n est un nombre premier, r et μ deux nombres quelconques donnés, la somme des produits de r groupes de μ termes consécutifs de la série $1.2.3\ldots(n-1)$ (en supposant que chaque groupe contient des nombres distincts de ceux qui sont contenus dans chacun des autres groupes) sera divisible par n, pourvu que μr soit inférieur à n.

Dans le cas de $\mu = 1$, on retombe sur le théorème si connu, associé au théorème de Wilson.

Comme exemple du nouveau théorème, prenons $n = 11$, $\mu = 3$, $r = 3$. On doit trouver et l'on trouvera effectivement que la somme

$$\begin{aligned} &1.2.3.4.5.6.7.8.\,9 \\ +&1.2.3.4.5.6.8.9.10 \\ +&1.2.3.5.6.7.8.9.10 \\ +&2.3.4.5.6.7.8.9.10 \end{aligned}$$

est divisible par 11. En effet cette somme est le nombre de substitutions de 11 lettres formées par les produits de deux substitutions cycliques assujetties à ne contenir que. 1, 4, 7 ou 10 lettres. Les quatre produits qui figurent dans cette somme font partie des cinquante-cinq produits qu'on devrait prendre dans le cas correspondant du théorème ordinaire associé à celui de Wilson.

44.

DÉMONSTRATION DIRECTE DU THÉORÈME DE LAGRANGE, SUR LES VALEURS NUMÉRIQUES MINIMA D'UNE FONCTION LINÉAIRE À COEFFICIENTS ENTIERS D'UNE QUANTITÉ IRRATIONNELLE*.

[*Comptes Rendus de l'Académie des Sciences,* LIII. (1861), pp. 1267—1272.]

APRÈS Euler, je me servirai du symbole $(a, b, c, \ldots, l)$ pour représenter le dénominateur de la fraction convergente dont $a, b, c, \ldots, l$ sont les quotients partiels, de sorte que $(b, c, \ldots, l)$ représentera le numérateur de la même fraction. Soit ν une quantité quelconque incommensurable à l'unité,

$$\frac{(b, \ldots, h, k)}{(a, b, \ldots, h, k)}, \quad \frac{(b, \ldots, h, k, l)}{(a, b, \ldots, h, k, l)},$$

deux réduites consécutives de ν. Comme à l'ordinaire, je nommerai ces convergentes $\frac{p}{q}, \frac{p'}{q'}$; on aura

$$\nu = \frac{[b, \ldots, h, (k+\theta)]}{[a, b, \ldots, h, (k+\theta)]} = \frac{N}{D}, \text{ où } \theta < \frac{1}{l};$$

on en conclut

$$\begin{aligned} p - \nu q &= \frac{(b, \ldots, h, k)(a, b, \ldots, h, k+\theta) - (a, b, \ldots, h, k+\theta)(b, \ldots, h)}{D} \\ &= \theta \frac{(b, \ldots, h, k)(a, b, \ldots, h) - (a, b, \ldots, h, k)(b, \ldots, h)}{D} \\ &= (-1)^i \frac{\theta}{D}, \end{aligned}$$

i désignant le nombre des quantités $a, b, \ldots, h$.

Faisons $$p - \nu q = \Delta,$$
on aura $$D\Delta = (-1)^i \theta. \qquad (1)$$

[* Cf. p. 306, below.]

Prenons $$(p-\lambda)-\nu(q-\mu)=\Delta',$$
λ et μ étant des nombres entiers quelconques, tels que $\Delta'^2<\Delta^2$, avec exclusion du cas où $p-\lambda=0$, $q-\mu=0$; alors

$$\Delta'=\Delta+\frac{\mu(b,\ldots,h,k+\theta)-\lambda(a,b,\ldots,h,k+\theta)}{D}=[(-1)^i\theta+A\theta+B]\div D$$

où
$$\left.\begin{aligned}A&=(b,\ldots,h)\mu-(a,b,\ldots,h)\lambda,\\ B&=(b,\ldots,h,k)\mu-(a,b,\ldots,h,k)\lambda.\end{aligned}\right\}\tag{2}$$

Donc, pour que Δ'^2 soit moindre que Δ^2, A et B doivent être de signes contraires, à moins que A ou B soit zéro.

Si $A=0$,
$$\lambda=r(b,\ldots,h),\quad \mu=r(a,b,\ldots,h),$$
$$B=r[(a,b,\ldots,h)(b,\ldots,h,k)-(b,\ldots,h)(a,b,\ldots,h,k)]=(-1)^i r,$$
et
$$D\Delta'=(-)^i(\theta+r),$$
ce qui serait contraire à l'hypothèse.

De même si $B=0$,
$$\lambda=r(b,\ldots,h,k),\quad \mu=r(a,b,\ldots,h,k),$$
et $D\Delta'$ devient
$$(-)^i\theta(1-r),$$
de sorte que Δ'^2 ne peut pas être au-dessous de Δ^2, à moins que $r=1$, ce qui donnerait
$$p-\lambda=0,\quad q-\mu=0,$$
cas dont on a fait exclusion.

Donc, puisque A et B doivent avoir des signes contraires, $\frac{\lambda}{\mu}$ sera intermédiaire entre $\frac{(b,\ldots,h,k)}{(a,b,\ldots,h,k)}$ et $\frac{(b,\ldots,h)}{(a,b,\ldots,h)}$, c'est-à-dire $\frac{(b,\ldots,h,\infty)}{(a,b,\ldots,h,\infty)}$, et conséquemment, comme il est très-facile de le voir, $\frac{\lambda}{\mu}$ sera de la forme

$$\frac{\left(b,\ldots,h,\frac{\rho}{\sigma}\right)}{\left(a,b,\ldots,h,\frac{\rho}{\sigma}\right)}.$$

Or on peut supposer $\frac{\rho}{\sigma}$ ou un nombre entier ou une fraction irréductible plus grande que k; de plus, comme il est facile de démontrer que $\sigma.\left(b,\ldots,h,\frac{\rho}{\sigma}\right)$, $\sigma.\left(a,b,\ldots,h,\frac{\rho}{\sigma}\right)$ seront premiers entre eux, on aura nécessairement

$$\lambda=r(b,\ldots,g,h)+s(b,\ldots,g),\quad \mu=r(a,b,\ldots,g,h)+s(a,b,\ldots,g),$$
avec la condition $r>ks$.

Donc, en substituant ces valeurs en (2), $D\Delta'$ devient égal à

$$(-)^i\,\theta + rP + sQ,$$

$$P = (b, \ldots, h, k)(a, b, \ldots, h) - (a, b, \ldots, h, k)(b, \ldots, h) = (-1)^i,$$

$$\begin{aligned} Q = \theta\,[(b, \ldots, g, h)\,(a, b, \ldots, g) - (a, b, \ldots, h)\,(b, \ldots, g)] \\ + (b, \ldots, g, h, k)(a, b, \ldots, g) - (a, b, \ldots, g, h, k)(b, \ldots, g) \\ = -\,\theta\,(-1)^i + (-)^\omega k, \end{aligned}$$

ω étant le nombre des lettres $(a, b, \ldots, h, k)$, c'est-à-dire $i+1$.

Donc $$D\Delta' = (-1)^i\,(\theta - s\theta + r - sk). \qquad (3)$$

Maintenant, imposons à volonté sur λ la limite $\lambda < p + p'$, ou bien sur μ la limite $\mu < q + q'$; pour fixer les idées, disons $\lambda < p + p'$:

$$p' = (b, \ldots, h, k, l) = (kl+1)\,(b, \ldots, g, h) + l\,(b, \ldots, g),$$

$$p = (b, \ldots, h, k) = k\,(b, \ldots, g, h) + (b, \ldots, g);$$

donc $$p' + p = (kl + k + 1)\,(b, \ldots, g, h) + (l+1)\,(b, \ldots, g).$$

Mais $$\lambda = r\,(b, \ldots, g, h) + s\,(b, \ldots, g).$$

Donc je dis que s ne peut pas excéder l.

Car si $$s \geqslant l + 1,$$

r, qui est au moins $ks+1$, sera $\geqslant kl + k + 1$, et λ ne sera pas moindre que $p' + p$, ce qui est contraire à l'hypothèse. Donc

$$s\theta \leqslant l\theta < 1\,;$$

mais $$r - sk > 1,$$

donc $$(-)^i\,D\Delta' > \theta,$$

c'est-à-dire $$> (-)^i\,D\Delta,$$

et l'on peut, de la même manière, démontrer que, si $\mu < q + q'$,

$$(-)^i\,D\Delta' > (-)^i\,D\Delta.$$

Donc il est évident que $(p - q\nu)^2$ sera moindre que $(x - y\nu)^2$ si $x < p'$ ou si $y < q'$. Toujours excluant le cas, on a en même temps

$$x = 0, \quad y = 0.$$

Je nomme ce résultat la *conclusion A*.

J'ajoute une *observation* importante pour ce qui sort immédiatement de la forme de l'équation (2): c'est que $(D\Delta')^2$ sera plus grand que $(D\Delta)^2$ si $\lambda = 0$ pour toute valeur de $\mu > 0$, et de même si $\mu = 0$ pour toute valeur de $\lambda > 0$. Je nomme cette observation *conclusion B*.

En vertu de ces deux conclusions, on peut démontrer très-facilement ce qui est le but du théorème Lagrange donné dans les Additions de l'Algèbre d'Euler, c'est-à-dire que la condition *nécessaire* et *suffisante* que $\frac{p}{q}$ soit une convergente de ν sera que la valeur $(p - q\nu)$ sera toujours augmentée en diminuant ou p ou q, ou tous les deux.

La nécessité de cette condition découle immédiatement et avec surabondance de la conclusion A, qui affirme qu'un changement quelconque de p qui ne le rend pas égal à p', ou de q qui ne le rend pas égal à q', aura l'effet d'augmenter $p - q\nu$.

Pour prouver que la condition est suffisante, il faut montrer que si a et b ne sont pas simultanément de la forme p, q, $a - b\nu$ peut être diminué en diminuant ou a ou b, ou tous les deux.

Si $\frac{p_e}{q_e}$ est une convergente de ν du rang e,

$\frac{p_i}{q_i}$ une autre convergente de ν du rang i,

1°. Si $a = p_e$, $b = q_i$, si $i > e$, il découle de la conclusion B, que $(p_e - q_e\nu)^2$ sera plus petit que $(p_e - q_i\nu)^2$, et de même si $e > i$, $(p_i - q_i\nu)^2$ sera plus petit que $(p_e - q_i\nu)^2$, et conséquemment $p_e - q_i\nu$ diminue en diminuant ou p_e ou q_i.

2°. Si l'une au moins des suppositions faites en 1° n'a pas lieu, par exemple si a tombe entre p_e et p_{e+1}, en vertu de la conclusion A, $(p_e - b\nu)$ sera plus petit que $a - b\nu$, et de même si b tombe entre q_i et q_{i+1}, $(a - q_i\nu)$ sera plus petit que $a - b\nu$.

Donc, à moins que $a = p_e$, $b = q_e$, $(a - b\nu)$ ne sera pas un minimum.

La conclusion A, quoiqu'elle n'ait pas été formellement énoncée par M. Hermite, était contenue implicitement, je dois le dire, dans une belle démonstration du théorème de Lagrange fondée sur d'autres principes et que M. Hermite a bien voulu me communiquer il y a un an ou deux.

45.

NOTE ON THE NUMBERS OF BERNOULLI AND EULER, AND A NEW THEOREM CONCERNING PRIME NUMBERS.

[*Philosophical Magazine*, XXI. (1861), pp. 127—136.]

FOLLOWING the accepted *Continental* notation, I denote by B_n* the positive value of the coefficient of t^{2n} in $\dfrac{t}{1-e^t}$ multiplied by the continual product $1.2.3 \ldots 2n$.

The law which governs the fractional part of B_n was first given in Schumacher's *Nachrichten*, by Thomas Clausen in 1840; and almost immediately afterwards a demonstration was furnished by Professor Staudt in *Crelle's Journal*, with a reclamation of priority, supported by a statement of his having many years previously communicated the theorem to Gauss.

The law is this, that the positive or negative fractional residue of B_n (according as n is odd or even) is made up of the simple sum of the reciprocals of all the prime numbers which, respectively diminished by unity, are contained in $2n$. The proof, which is of an inductive kind, is virtually as follows: Suppose the law holds good up to $(n-1)$ inclusive; if we expand† Σx^{2n} under the form $\dfrac{1}{e^{\frac{d}{dx}}-1} x^{2n}$, we shall evidently obtain $\dfrac{\Sigma x^{2n}}{x} \pm B_n$ under the form of a finite series, of which the terms are numerical multiples of the products of powers of x by the Bernoullian numbers of an order inferior to the nth. If, now, we make x equal to the product of all the primes which, diminished by unity, are contained in $2n$, it will at once be

* Were it not for the general usage being as stated in the text, I certainly think it would be far more convenient to use a notation agreeing with the Continental method as to sign, and nearly, but not quite, with Mr De Morgan's as to quantity, namely, to understand by B_n the coefficient of t^n in $\frac{1}{2}t\dfrac{e^t+1}{e^t-1}$ taken positively, so that B_n should be equal to zero for all the odd values of n, not excepting $n=1$.

[† Σx^{2n} denotes $1^{2n}+2^{2n}+\ldots+(x-1)^{2n}$. Cf. p. 227.]

seen (on inspection of the series) that all its terms become integer numbers, and consequently $\frac{\Sigma x^{2n}}{x} \pm B_n$ becomes an integer; and therefore the law will hold good up to n, since it may easily be shown, by an application of Fermat's theorem and elementary arithmetical considerations, that if N be the product of any prime numbers whatever, and if p is the general name of such of them as diminished by unity are factors of μ, then $\frac{\Sigma N^{\mu}}{N} + \Sigma \frac{1}{p}$ is an integer.

Hence, since the law holds good for $n = 1$, it is universally true. This theorem, then, of Staudt and Clausen, *inter alia*, gives a rule for determining what primes alone enter into the denominators of the Bernoullian numbers when expressed as fractions in their lowest terms; it enables us to affirm that only simple powers of primes enter into those denominators, and to know *à priori* what those prime factors are. This note is intended to supply a law concerning the *numerators* of the Bernoullian numbers, which I have not seen stated anywhere, and which admits of an instantaneous demonstration, *to wit*, that the whole of n will appear in the numerator of B_n, save and except such primes, or the powers of such primes, as we know by the Staudt-Clausen law must appear in the denominator.

I am inclined to believe that this law of mine was not known, at all events, in 1840, from the circumstance that in Rothe's Table, published by Ohm in *Crelle's Journal* in that year, which gives the values of B_n up to $n = 31$, the numerators are, with one exception (about to be named), all exhibited in such a form as to show such low factors as readily offer themselves, but for B_{23} the fact of the divisibility of the numerator by 23 is not indicated. This numerator is 596451111593912163277961, which in fact $= 23 \times 25932657025822267968607$. It is obvious, indeed, under my law, that whenever p is a prime number other than 2 and 3, the numerator of B_p must contain p, because in such case $p - 1$ cannot be a factor of $2p$. When $p = 3$ or $p = 2$, $2p$ always contains $(p - 1)$, so that 2 and 3 are necessarily constant factors of the Bernoullian denominators, and can therefore never appear in the numerators. In Schumacher the law of the denominator is given as "a passing" (or *chance*?) "specimen" of a promised memoir by Clausen on the Bernoullian numbers, as to which I shall feel obliged if any of the readers of this *Magazine* will inform me whether it has appeared anywhere, and if so, where. Now for my demonstration of the law of the numerators.

By definition, $B_n = \Pi(2n) \times$ coefficient of t^{2n-1} in $\frac{1}{e^t - 1}$. Let μ be any integer number; then $\pm(\mu^{2n} - 1) B_n = \Pi(2n) \times$ coefficient of t^{2n-1} in

$$\frac{\mu}{e^{\mu t} - 1} - \frac{1}{e^t - 1},$$

or in
$$\frac{(\mu-1)-(e^{(\mu-1)t}+e^{(\mu-2)t}+\ldots+e^{t})}{e^{\mu t}-1},$$

or in
$$-\frac{e^{(\mu-2)t}+2e^{(\mu-3)t}+\ldots+(\mu-2)e^{t}+(\mu-1)}{e^{(\mu-1)t}+e^{(\mu-2)t}+\ldots+e^{t}+1}.$$

But obviously, by Maclaurin's theorem, the coefficient of t^{2n-1} in the expansion of this last generating function will be of the form $\pm\frac{1}{\Pi(2n-1)}\cdot\frac{I}{\mu^{2n-1}}$, where I is an integer, and therefore B_n will be of the form $\frac{2nI}{\mu^{2n-1}(\mu^{2n}-1)}$.

Suppose now, when $\frac{2nI}{\mu^{2n-1}(\mu^{2n}-1)}$ is reduced to its lowest terms, that p (a prime contained in $2n$) does not appear in the numerator, this can only happen by virtue of p being contained in $\mu^{2n-1}(\mu^{2n}-1)$; let now μ be taken successively 2, 3, 4, ... $(p-1)$, then $\mu^{2n}-1$ in all these cases is divisible by p; and therefore, by an obvious inverse of Fermat's theorem, $(p-1)$ must be contained in $2n$, that is, p must be a factor of the denominator of B_n under the Staudt-Clausen law, which proves my theorem.

As a corollary to the foregoing, using Herschel's transformation, we see that if μ be taken any integer whatever,

$$\pm B_n=\frac{2n}{\mu^{2n}-1}\cdot\frac{(1+\Delta)^{\mu-2}+2(1+\Delta)^{\mu-3}+\ldots+(\mu-1)}{\Delta^{\mu-1}+\mu\Delta^{\mu-2}+\mu\frac{\mu-1}{2}\Delta^{\mu-3}+\ldots+\mu}0^{2n}$$

$$=\frac{2n}{\mu^{2n}-1}\frac{\Delta^{\mu-2}+\mu\Delta^{\mu-3}+\mu\frac{\mu-1}{2}\Delta^{\mu-4}+\ldots+\mu\frac{\mu-1}{2}}{\Delta^{\mu-1}+\mu\Delta^{\mu-2}+\mu\frac{\mu-1}{2}\Delta^{\mu-3}+\ldots+\mu\frac{\mu-1}{2}\Delta+\mu}0^{2n};$$

and if we write 0^{2n+1} instead of 0^{2n}, the result vanishes. For the case of $\mu=2$, this theorem accords with one well known. As this subject is so intimately related to that of the Herschelian differences of zero, I may take this occasion of stating a proposition concerning the latter, which (simple as it is) appears to have escaped observation, namely, that $\frac{\Delta^r 0^{n+r}}{\Pi(r)}$ is in fact the expression for the sum of the homogeneous products of the natural numbers from 1 to r, taken n together. For

$$\frac{1}{(x-r)(x-r+1)\ldots(x-1)x}$$
$$=\frac{1}{\Pi(r)}\left\{\frac{1}{x-r}-\frac{r}{x-r+1}+\frac{r\cdot\frac{r-1}{2}}{x-r+2}\ldots\pm\frac{1}{x}\right\}.$$

Hence obviously

$$\frac{1}{\Pi(r)}\left\{r^n - r(r-1)^n + r\,.\,\frac{r-1}{2}(r-2)^n \mp \&c.\right\},$$

that is

$$\frac{\Delta^r 0^n}{\Pi(r)} = \text{coefficient of } \frac{1}{x^n} \text{ in } \frac{1}{(x-r)(x-r+1)\dots(x-1)}$$

$$= \text{the sum of the } (n-r)\text{ary homogeneous products of } 1, 2, 3, \dots r.$$

Thus, then, we are able to affirm, from what is known concerning $\dfrac{\Delta^r 0^{r+n}}{\Pi r}$ (see Prof. De Morgan's *Calculus*), that the r-ary homogeneous product-sum of 1, 2, 3, ... n (which is of the degree $2r$ in n) always contains the algebraic factor $n(n+1)\dots(n+r)$.

Addendum.—Since sending the above to press, I have given some further and successful thought to the Staudt-Clausen theorem. Staudt's demonstration labours under the twofold defect of indirectness and of presupposing a knowledge of the law to be established. In it the Bernoullian numbers are not made the subject of a direct contemplation, but are regarded through the medium of an alien function, one out of an infinite number, in which they are as it were latently embodied; and the proof, like all other inductive ones, whilst it convinces the judgment, leaves the philosophic faculty unsatisfied, inasmuch as it fails to disclose the reason (the title, so to say, to existence) of the truth which it establishes. I present below an immediate and a direct proof of this beautiful and important proposition, founded upon the same principle as gives the law of the necessary factor in the numerators (namely, the arbitrary decomposition of the generating function of Bernoulli's numbers into partial fractions), and resting upon a simple but important conception, that of *relative* as distinguished from absolute integers.

I generalize this notion, and define a quantity to be an integer relative to r (or, for brevity's sake, to be an rth integer) when it may be represented by a fraction of which the denominator does not contain r.

The lemma* upon which my demonstration rests is the following, which

* This lemma is the converse of a self-evident fact, and it virtually embodies a principle respecting an arithmetical fraction strikingly analogous to a familiar one respecting an algebraical one; namely, in the same way as a rational algebraical function of x can be expressed *in one, and only one, way* as an integral function augmented by a sum of negative powers of linear functions of x, so a rational arithmetical quantity can be expressed in one, and only one, way as an integer augmented by the sum of negative powers of simple prime numbers multiplied respectively by numbers less than such primes. In drawing this parallel, the arithmetical quantity $\frac{c}{p^i}$, where $c<p$, is regarded as the analogue of the algebraical one $\frac{1}{(ax+b)^i}$, as is quite

is itself an immediate corollary from the arithmetical theorem that if $a, b, c, \ldots l$, with or without repetitions, are the distinct prime factors of the denominator of a fraction, the fraction itself may be resolved into the sum of simple fractions,

$$\frac{A}{a^\alpha}+\frac{B}{b^\beta}+\frac{C}{c^\gamma}+\&\text{c.}+\frac{L}{l^\lambda}$$

(itself a direct inference from the familiar theorem that if p, q be any two relative primes, the equation $px-qy=c$ is soluble in integers for all values of c). The lemma in question is as follows: If the quantity above described is representable under the several forms,

$$\frac{a'}{a^f}+\text{an } (a\text{th}) \text{ integer},\quad \frac{b'}{b^g}+\text{a } (b\text{th}) \text{ integer},\ \ldots \frac{l'}{l^k}+\text{a } (k\text{th}) \text{ integer},$$

then it is equal to

$$\frac{a'}{a^f}+\frac{b'}{b^g}+\ldots+\frac{l'}{l^k}+\text{an } \textit{absolute} \text{ integer}.$$

From what has been already shown, it is obvious that μ being any prime number, the highest power of μ which *can* enter into the denominator of $(\mu^{2n}-1)B_n$ is μ^{2n}, and consequently $\mu^{2n}B_n$ is an integer relative to μ. Also it is clear that only those values of μ can appear in the denominator of B_n which, diminished by unity, are factors of $2n$. We have, moreover,

$$(-)^{n-1}(\mu^{2n}-1)B_n=\Pi(2n)\times\text{coefficient of } t^{2n-1} \text{ in } \frac{\mu}{e^{\mu t}-1}-\frac{1}{e^t-1},$$

that is, coefficient of t^{2n-1} in $\dfrac{-N}{e^{\mu t}-1}$, where

$$N=\Pi(2n)\{e^{(\mu-1)t}+e^{(\mu-2)t}+\ldots+e^t-(\mu-1)\}$$
$$=\nu_1 t+\nu_2 t^2+\ldots\ldots+\nu_{2n}t^{2n}+\&\text{c.},$$

where obviously $\nu_1, \nu_2, \ldots \nu_{2n}$ are all integers, and the last of them

$$=(\mu-1)^{2n}+(\mu-2)^{2n}+\ldots+2^{2n}+1^{2n}.$$

proper, for both of them are fractions in their simplest forms, which would not be the case for the former were c equal to or greater than p, since in such case $\frac{c}{p^i}$ could be more simply expressed under the form $\frac{\gamma}{p^{i-1}}+\frac{\gamma'}{p^i}$.

This principle amounts to an affirmation that the equation in positive integers,

$$(b\ldots kl)\,x+(ab\ldots l)\,y+\ldots+(ab\ldots k)\,t-(ab\ldots kl)\,u=N,$$

where $a, b, \ldots k, l$ are relative primes, and $N<(ab\ldots kl)$, always admits of a solution, which may be termed the primitive one, and which will be unique, that namely in which $x, y, \ldots z, t$ are respectively less than $a, b, \ldots k, l$.

Suppose now that $2n$ contains $(\mu-1)$, then by Fermat's theorem

$$\nu_{2n} \equiv (\mu - 1) \quad [\text{mod } \mu].$$

Again, a very slight consideration* will serve to show that when μ is any prime other than 2, $e^{\mu t} - 1$ is of the form

$$\mu\,(t + \mu\delta_1 t^2 + \mu\delta_2 t^3 + \ldots + \mu\delta_{2n-1} t^{2n} + \&\text{c.}),$$

where $\delta_1, \delta_2, \ldots \delta_{2n-1} \ldots$ are all integers *relative* to μ. Now suppose

$$\frac{\mu N}{e^{\mu t}-1} = q_0 + q_1 t + q_2 t^2 + \ldots + q_{2n-1} t^{2n-1} + \&\text{c.};$$

then by multiplication and comparison of coefficients we obtain the identities following:

$$q_0 = \nu_1, \quad q_1 + \mu q_0 \delta_1 = \nu_2, \quad q_2 + \mu q_1 \delta_1 + \mu q_0 \delta_2 = \nu_3, \ \ldots$$
$$q_{2n-1} + \mu q_{2n-1} \delta_1 + \ldots + \mu q_0 \delta_{2n-1} = \nu_{2n};$$

obviously therefore $q_{2n-1} = \mu \times (\text{an integer relative to } \mu) + \nu_{2n}$. Hence

$$(-1)^n B_n = (\text{an integer relative to } \mu) - \frac{\nu_{2n}}{\mu}$$
$$= (\text{an integer relative to } \mu) + \frac{1}{\mu}.$$

And this relation obtains for any value of μ other than 2, which (or a power of which) *could* be contained in $2n$. When $\mu = 2$, the δ series will *not* all of them be the doubles of relative integers to 2; but the ν series, on account of the factor $\Pi(2n)$, will obviously, up to ν_{2n-1} inclusive, all contain 2 and $\nu_{2n} = 1$; consequently q_{2n} will be twice (an integer *quâ* 2) + 1, and B_n will

* For μ being a prime number greater than 2, if we put $\frac{\mu^r}{\Pi(r)}$ (the coefficient of t^r in $e^{\mu t} - 1$) under the form of (an integer *quâ* μ) $\times \mu^i$, we have

$$i = r - E\left(\frac{r}{\mu}\right) - E\left(\frac{r}{\mu^2}\right) - E\left(\frac{r}{\mu^3}\right) - \&\text{c.}$$
$$= \text{ or } > r - \frac{r}{\mu - 1} = \text{ or } > \frac{r}{2} > 1 \text{ when } r > 2;\ \text{also when } r = 2,\ i = 2 - E\left(\frac{2}{\mu}\right) = 2.$$

When $\mu = 2$, this would be no longer true; and in fact it is easily seen that in this case, whenever r is a power of 2, i will be only equal to 1.

For the benefit of my younger readers, I may notice that the *direct* proof of the theorem that the product of any r consecutive numbers must contain the product of the natural numbers up to r, or, in other words, that the trinomial coefficient $\frac{\Pi n}{\Pi\nu\,\Pi\nu'}$, where $\nu + \nu' = n$, is an integer, is drawn from the fact that this fraction may be represented as an integer *quâ* μ (any prime) multiplied by μ^i, where

$$i = \left[E\left(\frac{n}{\mu}\right) - E\left(\frac{\nu}{\mu}\right) - E\left(\frac{\nu'}{\mu}\right)\right] + \left[E\left(\frac{n}{\mu^2}\right) - E\left(\frac{\nu}{\mu^2}\right) - E\left(\frac{\nu'}{\mu^2}\right)\right] + \&\text{c.}$$

($E(x)$ meaning the integer part of x), so that i is necessarily either zero or positive, because the value of each triad of terms within the same parenthesis is essentially zero or positive. This is the natural and only direct procedure for establishing the proposition in question.

still be (an integer relative to μ) $+\frac{1}{\mu}$ as before. Hence it follows from the lemma that $(-1)^n B_n =$ an absolute integer $+ \Sigma \frac{1}{\mu}$, or

$$B_n = \text{an integer} + (-)^n \Sigma \frac{1}{\mu},$$

which is the equation expressed by the Staudt-Clausen theorem*.

My researches in the theory of partitions have naturally invested with a new and special interest (at least for myself) everything relating to the Bernoullian numbers. I am not aware whether the following expression for a Bernoullian of any order as a quadratic function of those of an inferior order happens to have been noticed or not. It may be obtained by a simple process of multiplication, and gives a means (not very expeditious, it is true) for calculating these numbers from one another without having recourse to the calculus of differences or Maclaurin's theorem, namely

$$-\frac{B_n}{\Pi(2n)} = (2^2-1)\frac{B_1}{\Pi(2)}\cdot\frac{B_{n-1}}{\Pi(2n-2)} + (2^4-1)\frac{B_2}{\Pi(4)}\cdot\frac{B_{n-2}}{\Pi(2n-4)}$$
$$+ \text{\&c.} \ldots + (2^{2n-4}-1)\frac{B_{n-2}}{\Pi(2n-4)}\cdot\frac{B_2}{\Pi(4)}$$
$$+ (2^{2n-2}-1)\frac{B_{n-1}}{\Pi(2n-2)}\cdot\frac{B_1}{\Pi(2)},$$

in which formula the terms admit of being coupled together from end to end, excepting (when n is even) one term in the middle.

To illustrate my law respecting the numerators of the numbers of Bernoulli, and its connexion with the known law for the denominators, suppose twice the index of any one of these numbers to contain the factor $(p-1)p^i$, where p is any prime; then this number will contain the first power of p in its denominator; but if the factor p^i is contained in double the index in question, but $(p-1)$ *not*, then p^i will appear bodily as a factor of the numerator.

* I ought to observe that in all that has preceded I have used the word *integer* in the sense of positive or negative integer, and the demonstration I have given holds good without assuming B_n to be positive. That this is the case, or, in other words, that the signs of the successive powers in $\frac{e^t-1}{e^t+1}$ are alternately positive and negative, may be seen at a glance by putting $t=2\sqrt{(-1)}\,\theta$, and remembering that all the coefficients in the series for $\tan\theta$ in terms of θ are necessarily positive, because $\left(\frac{d}{d\theta}\right)^i \tan\theta$ obviously only involves positive multiples of powers of $\tan\theta$ and $\sec\theta$.

It has occurred to me that it might be desirable to adhere to the common definition of "*Bernoulli's numbers*," but at the same time to use the term Bernoulli's *coefficients* to denote the actual coefficients in $\frac{e^t+1}{2(e^t-1)}$; so that if the former be denoted in general by B_n and the latter by β_n, we shall have

$$\beta_{2n} = (-)^{n-1} B_n,$$

$$\beta_{2n+1} = 0.$$

In the absence of some such term as I propose, many theorems which are really single when affirmed of the *coefficients*, become duplex or even multifarious when we are restrained to the use of the *numbers* only.

Postscript.—The results obtained concerning Bernoulli's numbers in what precedes, admit of being deduced still more succinctly; and this simplification is by no means of small importance, as it leads the way to the discovery of analogous and unsuspected properties of Euler's numbers (namely the coefficients of $\frac{\theta^{2n}}{\Pi(2n)}$ in the expansion of $\sec\theta$), and to some very remarkable theorems concerning prime numbers in general.

In fact, to obtain the laws which govern the denominators and numerators of Bernoulli's numbers, we need only to use the following principles:—(1) That μ being a prime*, $\Sigma\mu^n \equiv 0$, or $\equiv -1$ to the modulus μ, according as $\mu-1$ is not, or is, a factor of n,—the second part of this statement being a direct consequence of Fermat's theorem, the first part a simple inference from its inverse. (2) That $e^{\mu t}-1$ is of the form $\mu t + \mu^2 t^2 T$, where T is a series of powers of t, all of whose coefficients are integers relative to μ, except for the case of $\mu = 2$, when $e^{\mu t}-1$ is of the form $2t + 2t^2 T$. We have then $(\mu^{2n}-1)(-1)^n B_n = \Pi(2n)$. coefficient of t^{2n-1} in $\frac{e^{(\mu-1)t}+e^{(\mu-2)t}+\ldots+e^t-(\mu-1)}{e^{\mu t}-1}$; this by actual division (in virtue of principle (2)) $= I + \frac{R}{\mu}$, where I is an integer relative to μ, containing n, and $R = 1^{2n} + 2^{2n} + \ldots + (\mu-1)^{2n}$. Hence $(-)^n B_n =$ an integer relative to μ, or to such integer $+\frac{1}{\mu}$, according as $2n$ does not or does contain $(\mu-1)$, which proves the law for the numerators; and so if μ^i is a factor of n, but $(\mu-1)$ not a factor of $2n$, $\frac{R}{\mu}$ will vanish, and $\mu^{2n}-1$ will not contain μ; hence $(\mu^{2n}-1)B_n$, and consequently B_n will be the product of μ^i by an integer relative to μ, which proves my numerator law.

[* $\Sigma\mu^n$ denotes $1^n + 2^n + \ldots + (\mu-1)^n$.]

So by extending the same method to the generating function $\frac{1}{e^t + \sqrt{(-1)}}$, it may very easily be proved that if we write

$$\sec\theta = E_0 + E_1 \frac{\theta^2}{1.2} + E_2 \frac{\theta^4}{1.2.3.4} + \ldots + E_n \frac{\theta^{2n}}{1.2.3\ldots 2n} + \&c.,$$

every prime number μ of the form $4n+1$, such that $(\mu - 1)$ is a factor of $2n$, will be contained in E_n; and every such prime, when of the form $4n-1$, will be contained in $E_n + 2(-)^{n-1}$.

I call the numbers $E_1, E_2, \ldots E_n$ Euler's 1st, 2nd, ... nth numbers, as Euler was apparently the first to bring them into notice. In the *Institutiones Calculi Diff.* he has calculated their values up to E_9 inclusive: in this last there is an error, which is specified by Rothe in Ohm's paper above referred to; had Euler been possessed of my law this mistake could not have occurred, as we know that $E_9 + 2$ ought to contain the factors 19 and 7, neither of which will be found to be such factors if we adopt Euler's value of E_9, but both will be such if we accept Rothe's corrected value. But in still following out the same method, I have been led, through the study of Bernoulli's and the allied numbers, and with the express aid of the former, to a perfectly general theorem concerning prime numbers, in which Bernoulli's numbers no longer take any part. Fermat's theorem teaches us the residue of $r^{\mu-1}$ in respect to μ, namely, that it is unity; but I am not aware of any theorem being in existence which teaches anything concerning the relation of $\frac{r^{\mu-1}-1}{\mu}$ to μ (or, which is the same thing, of the relation of $r^{\mu-1}$ to the modulus μ^2). I have obtained remarkable results relative to the above quotient, which I will state for the simplest case only, namely, that where r as well as μ is a prime number. I find that when r is any odd prime,

$$\frac{r^{\mu-1}-1}{\mu} \equiv \frac{c_1}{\mu-1} + \frac{c_2}{\mu-2} + \frac{c_3}{\mu-3} + \ldots + \frac{c_{\mu-1}}{1}, \text{ (to mod. } \mu),$$

where $c_1, c_2, c_3, \ldots c_{\mu-1}$ are continually recurring cycles of the numbers $1, 2, 3, \ldots r$, the cycle beginning with that number r' which satisfies the congruence $\mu r' \equiv 1$ (mod. r). Since we know that

$$\frac{1}{\mu-1} + \frac{1}{\mu-2} + \frac{1}{\mu-3} + \ldots + \frac{1}{1} \equiv 0 \text{ (to mod. } \mu)$$

in place of the cycle $1, 2, 3, \ldots r$, we may obviously substitute the reduced cycle

$$-\frac{r-1}{2}, \quad -\frac{r-3}{2}, \ldots -1, 0, 1, \ldots \frac{r-3}{2}, \frac{r-1}{2}.$$

Thus*, for example, $\dfrac{3^{\mu-1}-1}{\mu}$, when μ is of the form $6n+1$,

$$\equiv \frac{1}{\mu-1} - \frac{1}{\mu-3} + \frac{1}{\mu-5} - \frac{1}{\mu-7} \dots + 1, \text{ (to mod. } \mu),$$

and when μ is of the form $6n-1$,

$$\equiv \frac{-1}{\mu-2} + \frac{1}{\mu-3} - \frac{1}{\mu-4} + \frac{1}{\mu-5} \dots - 1, \text{ (to mod. } \mu).$$

When r is 2, the theorem which replaces the preceding is as follows†:

$$\frac{2^{\mu-1}-1}{\mu},$$

when μ is of the form $4n+1$,

$$\equiv \frac{1}{\mu-1} + \frac{1}{\mu-2} - \frac{1}{\mu-3} - \frac{1}{\mu-4} + \frac{1}{\mu-5} + \frac{1}{\mu-6}$$

$$- \frac{1}{\mu-7} - \frac{1}{\mu-8} + \frac{1}{\mu-9} \pm \text{\&c., (to mod. } \mu),$$

and when μ is of the form $4n-1$,

$$\equiv -\frac{1}{\mu-1} + \frac{1}{\mu-2} + \frac{1}{\mu-3} - \frac{1}{\mu-4} - \frac{1}{\mu-5}$$

$$+ \frac{1}{\mu-6} + \frac{1}{\mu-7} \mp \text{\&c., (to mod. } \mu).$$

When r is not a prime, a similar theorem may be obtained by the very same method, but its expression will be less simple. The above theorems would, I think, be very noticeable were it only for the circumstance of their involving (as a condition) the primeness as well of the base as of the augmented index of the familiar Fermatian expression $r^{\mu-1}$,—a condition which here makes its appearance in the theory of numbers (as I believe) for the first time.

[* Cf. the formulae at the top of p. 230 above. The second of these had originally a wrong sign throughout, but has been corrected, after a sentence inserted by the author at the end of the paper 40 above (p. 241), not reproduced here.]

[† The sign of every term in the two following expressions should be changed.]

46.

NOTE ON THE HISTORICAL ORIGIN OF THE UNSYMMETRICAL SIX-VALUED FUNCTION OF SIX LETTERS.

[*Philosophical Magazine*, XXI. (1861), pp. 369—377.]

THE discovery and first announcement of the existence of the celebrated function of six letters having six values, and not symmetrical in respect to all the letters, is usually assigned to my illustrious friend M. Hermite, to whom M. Cauchy expressly ascribes it in a memoir inserted in the *Comptes Rendus* of the Institut for December 8, 1845, p. 1247, and again, January 5, 1846, p. 30.

M. Cauchy adds that the conversation he held with M. Hermite on this subject excited in himself a lively desire to sound to its depths the question of permutations, and to develope the consequences to be deduced from the application of the principles relative thereto, which he had himself long previously laid down.

I was not at that date in the habit of consulting the *Comptes Rendus*, or I should at once have made the reclamation of priority which I now do, not from any unworthy motive of self-love in so small a matter, but out of regard to historic truth. It is a year or two since I first learnt that the origin of this function was usually referred to M. Cauchy or M. Hermite; but although aware that its existence was known to myself long previous to the dates quoted, I did not recollect that I had ever communicated it to the world through the medium of the press, and I therefore kept silence on the subject.

Turning over, a few days ago, for another purpose, the pages of a back volume of this *Magazine*, my eye chanced to alight on a footnote to a paper of my own inserted therein*, under date of April 1844, "On the Principles of Combinatorial Aggregation," which I will take the liberty of quoting at length, as it proves incontestably the priority which I lay claim to.

[* p. 92 of Vol. I. of this Reprint.]

"When the modulus is four, there is only one synthematic arrangement possible, and there is no indeterminateness of any kind; from this we can infer, *à priori*, the reductibility of a biquadratic equation; for using ϕ, f, F to denote rational symmetrical forms of function, it follows that

$$F\begin{Bmatrix} f\{\phi(a, b), \phi(c, d)\} \\ f\{\phi(a, c), \phi(b, d)\} \\ f\{\phi(a, d), \phi(b, c)\} \end{Bmatrix} \text{ is itself a rational symmetric function of } a, b, c, d.$$

Whence it follows that if a, b, c, d be the roots of a biquadratic equation, $f\{\phi(a, b), \phi(c, d)\}$ can be found by the solution of a cubic: for instance, $(a+b)\times(c+d)$ can be thus determined, whence immediately the sum of any two of the roots comes out from a quadratic equation.

"To the modulus 6 there are fifteen different synthemes capable of being constructed. At first sight it might be supposed that these could be classed in natural families of three or of five each, on which supposition the equation of the sixth degree could be depressed; but on inquiry this hope will prove to be futile, not but what natural affinities do exist between the totals; but in order to separate them into families, each will have to be taken twice over; or in other words, the fifteen synthemes to modulus 6 being reduplicated, subdivide into six natural families of five each."

The six families above referred to (in which it is to be understood that $p \,.\, q$ and $q \,.\, p$ are identical in effect) are the following:—

$$\begin{array}{ccccccccccc}
a.b & c.d & e.f & & a.c & d.e & f.b & & a.d & e.f & b.c \\
a.c & b.e & d.f & & a.d & c.f & e.b & & a.e & d.b & f.c \\
a.d & b.f & c.e & & a.e & c.b & d.f & & a.f & d.c & e.b \\
a.e & b.d & c.f & & a.f & c.e & d.b & & a.b & d.f & e.c \\
a.f & b.c & d.e & & a.b & c.d & e.f & & a.c & d.e & f.b \\
\\
a.e & f.b & c.d & & a.f & b.c & d.e & & a.b & c.d & e.f \\
a.f & e.c & b.d & & a.b & f.d & c.e & & a.c & b.e & d.f \\
a.b & e.d & f.c & & a.c & f.e & b.d & & a.d & b.f & c.e \\
a.c & e.b & f.d & & a.d & f.c & b.e & & a.e & b.d & c.f \\
a.d & e.f & b.c & & a.e & f.b & c.d & & a.f & b.c & d.e
\end{array}$$

And it will be observed that every two families have one, and only one, syntheme in common between them; and precisely in the same way as in the note above quoted it is especially shown that the one single natural family

$$\begin{vmatrix} a.b & c.d \\ a.c & b.d \\ a.d & b.c \end{vmatrix}$$

gives rise to a function of four letters with only one value, so the six functions analogously formed with these six families obviously give rise to six func-

tions, which change into one another when any interchange is effected between the letters which enter into them; so that any one of these is a function of six letters having only six values. I conceive that, after this reference, no writer on the subject wishing to specify the function in question would hesitate to call it after my name.

I may also take occasion to observe that, in connexion with my researches in combinatorial aggregation, long before the publication of my unfinished paper in the *Magazine*, I had fallen upon the question of forming a heptatic aggregate of triadic synthemes comprising all the duads to the base 15, which has since become so well known, and fluttered so many a gentle bosom, under the title of the fifteen school-girls' problem; and it is not improbable that the question, under its existing form, may have originated through channels which can no longer be traced in the oral communications made by myself to my fellow-undergraduates at the University of Cambridge long years before its first appearance, which I believe was in the *Ladies' Diary* for some year which my memory is unable to furnish.

In order to relieve this notice from the mere personal character which it may thus far appear to bear, I will state another question concerning the combinatorial aggregation of fifteen things which may serve as a pendant to the famous school-girl problem.

The number of triads to the base 15 is $\dfrac{15 \times 14 \times 13}{3\,.\,2\,.\,1} = 5 \times 91$. Let it be required to arrange these into 91 synthemes, in other words, to set out the walks of 15 girls for 91 days (say a quarter of the year) in such a manner that the same three shall never *all* come together more than once in the quarter. Of the various ways in which it is probable this problem may be solved, the following deserves notice. Let 15 letters be arbitrarily divided into 5 sets, namely,

$$a_1b_1c_1;\quad a_2b_2c_2;\quad a_3b_3c_3;\quad a_4b_4c_4;\quad a_5b_5c_5.$$

The sets as they stand will represent one of the 91 arrangements sought for, which I call the basic syntheme. The remaining 90 may be obtained as follows in 10 batches of 9 each. Write down the 10 index distributions following:—

1 2 3; 4 5	1 4 5; 2 3
1 2 4; 3 5	2 3 4; 1 5
1 2 5; 3 4	2 3 5; 1 4
1 3 4; 2 5	2 4 5; 1 3
1 3 5; 2 4	3 4 5; 1 2.

Take any one of these distributions, as for instance 2 3 5; 1 4, and proceed

as follows:—In respect of 2, 3, 5, conjugate the three sets $\begin{matrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_5 & b_5 & c_5 \end{matrix}$; and in respect of 1, 4, conjugate the two remaining sets $\begin{matrix} a_1 & b_1 & c_1 \\ a_4 & b_4 & c_4 \end{matrix}$.

From the ternary conjugation form the nine arrangements,

$a_2\ a_3\ a_5$	$b_2\ b_3\ b_5$	$c_2\ c_3\ c_5$
$a_2\ a_3\ b_5$	$b_2\ b_3\ c_5$	$c_2\ c_3\ a_5$
$a_2\ a_3\ c_5$	$b_2\ b_3\ a_5$	$c_2\ c_3\ b_5$
$a_2\ b_3\ a_5$	$b_2\ c_3\ b_5$	$c_2\ a_3\ c_5$
$a_2\ b_3\ b_5$	$b_2\ c_3\ c_5$	$c_2\ a_3\ a_5$
$a_2\ b_3\ c_5$	$b_2\ c_3\ a_5$	$c_2\ a_3\ b_5$
$a_2\ c_3\ a_5$	$b_2\ a_3\ b_5$	$c_2\ b_3\ c_5$
$a_2\ c_3\ b_5$	$b_2\ a_3\ c_5$	$c_2\ b_3\ a_5$
$a_2\ c_3\ c_5$	$b_2\ a_3\ a_5$	$c_2\ b_3\ b_5$,

which call $L_1\ L_2\ L_3 \qquad L_4\ L_5\ L_6 \qquad L_7\ L_8\ L_9$.

Again, from the binary conjugation, form the nine arrangements,

$a_1\ b_1\ c_4$	$a_4\ b_4\ c_1$
$a_1\ b_1\ b_4$	$a_4\ c_1\ c_4$
$a_1\ b_1\ a_4$	$c_1\ b_4\ c_4$
$a_1\ c_1\ c_4$	$a_4\ b_4\ b_1$
$a_1\ c_1\ b_4$	$a_4\ b_1\ c_4$
$a_1\ c_1\ a_4$	$b_1\ b_4\ c_4$
$b_1\ c_1\ c_4$	$a_4\ b_4\ a_1$
$b_1\ c_1\ b_4$	$a_4\ a_1\ c_4$
$b_1\ c_1\ a_4$.	$a_1\ b_4\ c_4$,

which call $M_1\ M_2\ M_3 \qquad M_4\ M_5\ M_6 \qquad M_7\ M_8\ M_9$.

Now combine the L with the M system, each L with some M in any order whatever; the 9 combinations or appositions thus obtained will give a batch of 9 synthemes; and proceeding in like manner with each of the 10 distributions of the indices 1, 2, 3, 4, 5, we shall obtain 90 synthemes, which together with the basic syntheme complete the system required. The M system corresponding to any distribution of the indices is the system which contains the synthematic arrangement of the bipartite* triads which can be constituted out of six things, separated in two sets or parts, and is unique. The L system is *one* of those which represents the synthematic arrangement

* See note at end of paper.

of the tripartite* triads of nine things separated into three sets or parts. I have set out above one in particular of these for the sake of greater clearness; but any other system having the same property will serve the same purpose, and a careful study will serve to show that the total number of L's corresponding to a given distribution of indices will be ( )*. Consequently the total number of LM's that we can form for a given distribution will be ( ) $\times 1 . 2 . 3 . 4 . 5 . 6 . 7 . 8 . 9$; and the number of *distinct* synthematic arrangements satisfying the given conditions corresponding to any assumed basic syntheme will be this number raised to the tenth power; and as this vastly exceeds the total number of permutations of fifteen things, we see, without even taking into consideration the diversity that may be produced by a change of the base, that this method must give rise to many distinct types of solution (arrangements being defined to belong to the same or different types, according as they admit or not of being deduced from each other by a permutation effected among their monadic elements). The common character of all these allotypical aggregations, and which serves to constitute them into a natural order or family, consists in their being derived from a base formed out of five sets, such that the monopartite triads corresponding to the base form one syntheme, and the other 90 synthemes each contain a conjugation of the tripartite triads belonging to three out of the five sets of the base with the bipartite triads belonging to the other two sets thereof. There is, moreover, no reason to suppose, or at all events no safe ground for affirming, that this family exhausts the whole possible number of types to which the arrangements satisfying the proposed condition admit of being reduced. A further question which I have somewhere raised, and which brings the two problems of the school-girls into *rapport*, is the following:—"To divide the system of 91 synthemes satisfying the conditions above stated into thirteen minor systems, each of which satisfies the conditions of the old problem, that is, of containing all the duads that can be made out of the fifteen elements once and once only"; or to put the question in a more exact form, to exhibit thirteen systems, each satisfying this last condition, which shall together include between them all the triads that can be made out of the fifteen elements.

The reader would have reason to be dissatisfied with the author's reticence, were he to leave altogether unmentioned the synthematic aggregation of the *binomial* triads appertaining to the same three triliteral sets or *nomes*; but space forbids my doing more at present than giving one of these aggregates, and indicating the number and mode of generation of all from this one. It will readily be seen that any such aggregate will be made up of two sub-aggregates, which I shall call A and B respectively, of which one bears

* Some day or another a new combinatorial calculus must come into being to furnish general solutions to the infinite variety of questions of *multifariousness* to which the theory of combinatorial aggregation, *alias* compound permutations, gives rise.

the same relation to the disposition of the nomes in the order 123 456 789, as the other to their disposition in the order 123 789 456. Thus we may take for our A and B the following, which will each contain 9 synthemes, the total number of synthemes in the two together being 18*:—

(A)			(B)		
124	567	893	127	894	763
125	468	739	128	795	436
126	459	783	129	786	453
134	568	279	137	895	246
135	469	278	138	796	245
136	457	289	139	784	256
234	569	187	237	896	154
235	467	189	238	794	156
236	458	179	239	785	146

The system of triads contained in A may be arranged in twelve different aggregates similar to the one given, and the same will be true for the triads in the B; so that the total number of the combined systems will be 144. All the permutations which leave A or B (separately considered) unaltered will form a natural group,—the theory of groups in this, as in every other case, standing in the closest relation to the doctrine of combinatorial aggregation, or what for shortness may be termed syntax. I have elsewhere given the general name of Tactic to the third pure mathematical science, of which order is the proper sphere, as is number and space of the other two. Syntax and Groups are each of them only special branches of Tactic. I shall on another occasion give reasons to show that the doctrine of groups may be treated as the arithmetic of ordinal numbers. With respect to the twelve varieties of the A or B aggregates, they may be obtained from the one given by combining the substitutions corresponding to the six permutations of the three constituents of one nome, as 7, 8, 9, with the permutation of any two constituents of another, as 5, 6. But I have said enough for my present purpose, which is to point out the boundless untrodden regions of thought in the sphere of order, and especially in the department of *syntax*, which remain to be expressed, mapped out, and brought under cultivation. The difficulty indeed is not to find material, of which there is a superabundance, but to discover the proper and principal centres of speculation that may serve to reduce the theory into a manageable compass.

* Thus, since there is evidently one mononomial syntheme, the total number of synthemes of all three kinds will be $1+18+9=28=\frac{8\times 7}{2}$, as it should be, the total number of triads being $\frac{9.8.7}{3.2}$, and $\frac{9}{3}$ of them going to a syntheme.

I put on record (as a Christmas offering on the altar of science) for the benefit of those studying the theory of groups, or compound permutations (to which the prize shortly to be adjudicated by the Institute of France for the most important addition to the subject may tend to give a new impulse), and with an eye to the geometrical and algebraical verities with which, as a constant of reason, we may confidently anticipate it is pregnant, an exhaustive table of the monosynthematic aggregates of the trinomial triads that are contained in a system of three triliteral nomes. Let these latter be called respectively 123; 456; 789; then we have the annexed:—

Table of Synthemes of Trinomial Triads to base 3 . 3.

(1)	(2)	(3)	(4)
147 258 369	147 258 369	147 258 369	147 258 369
148 259 367	148 259 367	148 259 367	148 259 367
149 257 368	149 257 368	149 257 368	149 267 358
157 268 349	157 268 349	157 269 348	157 268 349
158 269 347	158 269 347	158 267 349	158 269 347
159 267 348	159 267 348	159 268 347	159 247 368
167 248 359	167 249 358	167 248 359	167 248 359
168 249 357	168 247 359	168 249 357	168 249 357
169 247 358	169 248 357	169 247 358	169 257 348
(5)	(6)	(7)	(8)
147 258 369	147 258 369	147 258 369	147 258 369
148 267 359	148 267 359	148 269 357	148 269 357
149 268 357	149 257 368	149 257 368	149 267 358
157 249 368	157 268 349	157 268 349	157 268 349
158 269 347	158 269 347	158 249 367	158 249 367
159 248 367	159 248 367	159 267 348	159 247 368
167 259 348	167 259 348	167 248 359	167 248 359
168 257 349	168 249 357	168 259 347	168 259 347
169 247 358	169 247 358	169 247 358	169 257 348

The discussion of the properties of this Table, and the classification of the eight aggregates into natural families, must be reserved for a future occasion.

Note.—A triad is called tripartite if its three elements are culled out of three different parts or sets between which the total number of elements is supposed to be divided; bipartite if the elements are taken out of two distinct sets; unipartite if they all lie in the same set. The more ordinary method for the reduction of synthematic arrangements from a given base to a linear one which I employ, consists in the separate synthematization *inter se* of all the combinations of the *same* kind as regards the number of parts

from which they are respectively drawn. Thus, for example, if the distribution of the $\frac{30 \times 29 \times 28}{6}$ triads to the base 30 into $\frac{29 \times 28}{2}$ synthemes be required, this may be effected by dividing the 30 elements in an arbitrary manner into 15 parts, each part containing 2 elements. These 15 parts being now themselves treated as elements, are first to be conjugated as in the old 15-school-girl problem, and each of these 7 conjugations can be made to furnish 6 synthemes containing exclusively bipartite triads. The same 15 parts are then to be conjugated as in the new school-girl problem, and the 91 conjugations thus obtained will each furnish 4 synthemes, containing exclusively the tripartite triads. These bipartite and tripartite synthemes will exhaust the entire number of triads of both kinds, and accordingly we shall find

$$\begin{aligned} 7 \times 6 + 91 \times 4 &= 406 \\ &= \frac{29 \times 28}{2}. \end{aligned}$$

A syntheme, I need scarcely add, is an aggregate of combinations containing between them all the monadic elements of a given system, each appearing once only. In the more general theory of aggregation, such an aggregate would be distinguished by the name of a monosyntheme. A disyntheme would then signify an aggregate of combinations containing between them the duadic elements, each appearing once only, and so forth. Thus the old 15-school-girl question in my nomenclature would be enunciated under the form of a problem "to construct a triadic disyntheme, separable into monosynthemes to the base 15"; the new school question, as a problem "to divide the whole of the triads to base 15 into monosynthemes"; the question which connects the two, as a problem "to exhibit the whole of the triads to base 15 under the form of 13 disynthemes, each separated into 7 monosynthemes."

A question of a more general kind, and embracing this last, would be the problem of dividing the whole of the same system of triads into 13 disynthemes, without annexing the further condition of monosynthematic divisibility. So there is the simpler question of constructing a single disyntheme to the base 15 without any condition annexed as to its decomposability into 7 synthemes.

47.

ON A PROBLEM IN TACTIC WHICH SERVES TO DISCLOSE THE EXISTENCE OF A FOUR-VALUED FUNCTION OF THREE SETS OF THREE LETTERS EACH.

[*Philosophical Magazine*, XXI. (1861), pp. 515—520.]

AT page 375 of the May Number of this *Magazine** (in that paragraph commencing at the middle of the page) I gave a Table of Synthemes, correct as far as it went, but left in a very imperfect state. It was intended to be supplemented with a material addition which escaped my recollection when, after a long delay, the proofs of the paper passed through my hands. The question to which this Table refers is the following :—

Three *nomes*, each containing three elements, are given; the number of *trinomial* triads (that is, ternary combinations, composed by taking one element out of each nome) will be 27, and these 27 may be *grouped* together into 9 synthemes (each syntheme consisting of 3 of the triads in question, which together include between them all the 9 elements). It is desirable to know :—1st. How many distinct *groupings* of this kind can be formed. 2nd. Whether there is more than one, and, if so, how many distinct types of groupings. The criterion of one grouping being cotypal or allotypal to another is its capability or incapability of being transformed into that other by means of an interchange of elements. Be it once for all stated that the question in hand is throughout one of combinations, and not of permutations; the order of the elements in a triad, of a triad in a syntheme, of a syntheme in a grouping is treated as immaterial. As we are only concerned with the elements as distributed into *nomes*, the number of interchanges of elements with which we are concerned is 6×6^3 or 1296; the factor 6^3 arises from the permutability of the elements of each nome *inter se*, the remaining factor 6 from the permutability of any nome with any other. I find, by a method which carries its own demonstration with it on its face, that the number of distinct *groupings* is 40, of which 4 belong to one *type* or *family*, and 36 to a second type or family.

[* p. 270 above.]

Let the nomes be 1.2.3, 4.5.6, 7.8.9, and let

c_1 denote 1.4, 2.5, 3.6		$\dot{c}_1$ denote 1.4, 2.6, 3.5
c_2 „ 1.5, 2.6, 3.4		$\dot{c}_2$ „ 1.5, 2.4, 3.6
c_3 „ 1.6, 2.4, 3.5		$\dot{c}_3$ „ 1.6, 2.5, 3.4

$$\gamma \text{ denote } \begin{matrix} 7, & 8, & 9 \\ 8, & 9, & 7 \\ 9, & 7, & 8 \end{matrix} \qquad \gamma' \text{ denote } \begin{matrix} 7, & 9, & 8 \\ 9, & 8, & 7 \\ 8, & 7, & 9 \end{matrix}$$

b_1 denote 1.7, 2.8, 3.9		$\dot{b}_1$ denote 1.7, 2.9, 3.8
b_2 „ 1.8, 2.9, 3.7		$\dot{b}_2$ „ 1.8, 2.7, 3.9
b_3 „ 1.9, 2.7, 3.8		$\dot{b}_3$ „ 1.9, 2.8, 3.7

$$\beta \text{ denote } \begin{matrix} 4, & 5, & 6 \\ 5, & 6, & 4 \\ 6, & 4, & 5 \end{matrix} \qquad \beta' \text{ denote } \begin{matrix} 4, & 6, & 5 \\ 6, & 5, & 4 \\ 5, & 4, & 6 \end{matrix}$$

a_1 denote 4.7, 5.8, 6.9		$\dot{a}_1$ denote 4.7, 5.9, 6.8
a_2 „ 4.8, 5.9, 6.7		$\dot{a}_2$ „ 4.8, 5.7, 6.9
a_3 „ 4.9, 5.7, 6.8		$\dot{a}_3$ „ 4.9, 5.8, 6.7

$$\alpha \text{ denote } \begin{matrix} 1, & 2, & 3 \\ 2, & 3, & 1 \\ 3, & 1, & 2 \end{matrix} \qquad \alpha' \text{ denote } \begin{matrix} 1, & 3, & 2 \\ 2, & 3, & 1 \\ 3, & 1, & 2. \end{matrix}$$

I take first the larger family of 36 groupings; these may be represented as follows:—

$a_1\alpha$	$a_1\alpha$	$a_1\alpha'$	$a_1\alpha$	$a_1\alpha'$	$a_1\alpha'$	$\dot{a}_1\alpha$	$\dot{a}_1\alpha$	$\dot{a}_1\alpha'$	$\dot{a}_1\alpha$	$\dot{a}_1\alpha'$	$\dot{a}_1\alpha'$
$a_2\alpha$	$a_2\alpha'$	$a_2\alpha$	$a_2\alpha'$	$a_2\alpha$	$a_2\alpha'$	$\dot{a}_2\alpha$	$\dot{a}_2\alpha'$	$\dot{a}_2\alpha$	$\dot{a}_2\alpha'$	$\dot{a}_2\alpha$	$\dot{a}_2\alpha'$
$a_3\alpha'$	$a_3\alpha$	$a_3\alpha$	$a_3\alpha'$	$a_3\alpha'$	$a_3\alpha$	$\dot{a}_3\alpha'$	$\dot{a}_3\alpha$	$\dot{a}_3\alpha$	$\dot{a}_3\alpha'$	$\dot{a}_3\alpha'$	$\dot{a}_3\alpha$
$b_1\beta$	$b_1\beta$	$b_1\beta'$	$b_1\beta$	$b_1\beta'$	$b_1\beta'$	$\dot{b}_1\beta$	$\dot{b}_1\beta$	$\dot{b}_1\beta'$	$\dot{b}_1\beta$	$\dot{b}_1\beta'$	$\dot{b}_1\beta'$
$b_2\beta$	$b_2\beta'$	$b_2\beta$	$b_2\beta'$	$b_2\beta$	$b_2\beta'$	$\dot{b}_2\beta$	$\dot{b}_2\beta'$	$\dot{b}_2\beta$	$\dot{b}_2\beta'$	$\dot{b}_2\beta$	$\dot{b}_2\beta'$
$b_3\beta'$	$b_3\beta$	$b_3\beta$	$b_3\beta'$	$b_3\beta'$	$b_3\beta$	$\dot{b}_3\beta'$	$\dot{b}_3\beta$	$\dot{b}_3\beta$	$\dot{b}_3\beta'$	$\dot{b}_3\beta'$	$\dot{b}_3\beta$
$c_1\gamma$	$c_1\gamma$	$c_1\gamma'$	$c_1\gamma$	$c_1\gamma'$	$c_1\gamma'$	$\dot{c}_1\gamma$	$\dot{c}_1\gamma$	$\dot{c}_1\gamma'$	$\dot{c}_1\gamma$	$\dot{c}_1\gamma'$	$\dot{c}_1\gamma'$
$c_2\gamma$	$c_2\gamma'$	$c_2\gamma$	$c_2\gamma'$	$c_2\gamma$	$c_2\gamma'$	$\dot{c}_2\gamma$	$\dot{c}_2\gamma'$	$\dot{c}_2\gamma$	$\dot{c}_2\gamma'$	$\dot{c}_2\gamma$	$\dot{c}_2\gamma'$
$c_3\gamma'$	$c_3\gamma$	$c_3\gamma$	$c_3\gamma'$	$c_3\gamma'$	$c_3\gamma$	$\dot{c}_3\gamma'$	$\dot{c}_3\gamma$	$\dot{c}_3\gamma$	$\dot{c}_3\gamma'$	$\dot{c}_3\gamma'$	$\dot{c}_3\gamma$.

An example of the development of any one of the above symbolisms into its correspondent grouping will serve to render perfectly intelligible the whole Table.

Let it be required to develope

$$\begin{matrix} \dot{b}_1\beta \\ \dot{b}_2\beta' \\ \dot{b}_3\beta'. \end{matrix}$$

Since

$$\begin{array}{llll} \dot{b}_1 = 1.7 \quad 2.9 \quad 3.8 & \qquad 4,\ 5,\ 6 & \qquad 4,\ 6,\ 5 \\ \dot{b}_2 = 1.8 \quad 2.7 \quad 3.9 & \qquad \beta = 5,\ 6,\ 4 & \qquad \beta' = 6,\ 5,\ 4 \\ \dot{b}_3 = 1.9 \quad 2.8 \quad 3.7 & \qquad 6,\ 4,\ 5 & \qquad 5,\ 4,\ 6, \end{array}$$

the development required is the following:—

$$\left|\begin{array}{ccc} 1.7.4 & 2.9.5 & 3.8.6 \\ 1.7.5 & 2.9.6 & 3.8.4 \\ 1.7.6 & 2.9.4 & 3.8.5 \\ 1.8.4 & 2.7.6 & 3.9.5 \\ 1.8.6 & 2.7.5 & 3.9.4 \\ 1.8.5 & 2.7.4 & 3.9.6 \\ 1.9.4 & 2.8.6 & 3.7.5 \\ 1.9.6 & 2.8.5 & 3.7.4 \\ 1.9.5 & 2.8.4 & 3.7.6 \end{array}\right|.$$

The whole of this family of 36 may be represented under the following condensed form, according to the notation usual in the theory of substitutions.

$$\left(\left|\begin{array}{c} a_1\alpha \\ a_2\alpha \\ a_3\alpha \end{array}\right| \times \begin{pmatrix} 123 & 123 & 123 \\ 123 & 231 & 312 \end{pmatrix} \times \begin{pmatrix} a & \dot{a} \\ a & a \end{pmatrix} \times \begin{pmatrix} \alpha\alpha' & \alpha\alpha' \\ \alpha\alpha' & \alpha'\alpha \end{pmatrix} \times \begin{pmatrix} a\alpha & b\beta & c\gamma \\ a\alpha & a\alpha & a\alpha \end{pmatrix}\right).$$

It remains to describe the principal and most symmetrical family. This contains only 4 groupings, and may be represented indifferently under any of the three following forms:—

$$\begin{array}{lllllllllllllll} a_1\alpha & a_1\alpha' & \dot{a}_1\alpha & \dot{a}_1\alpha' & & b_1\beta & b_1\beta' & \dot{b}_1\beta & \dot{b}_1\beta' & & c_1\gamma & c_1\gamma' & \dot{c}_1\gamma & \dot{c}_1\gamma' \\ a_2\alpha & a_2\alpha' & \dot{a}_2\alpha & \dot{a}_2\alpha' & \text{or} & b_2\beta & b_2\beta' & \dot{b}_2\beta & \dot{b}_2\beta' & \text{or} & c_2\gamma & c_2\gamma' & \dot{c}_2\gamma & \dot{c}_2\gamma' \\ a_3\alpha & a_3\alpha' & \dot{a}_3\alpha & \dot{a}_3\alpha' & & b_3\beta & b_3\beta' & \dot{b}_3\beta & \dot{b}_3\beta' & & c_3\gamma & c_3\gamma' & \dot{c}_3\gamma & \dot{c}_3\gamma'. \end{array}$$

In developing, it will be found that each of these three representations gives rise to the *same* family of groupings, which from its importance it is proper to set out in full as follows:—

1.4.7 2.5.8 3.6.9	1.4.7 2.5.9 3.6.8	1.4.7 2.6.8 3.5.9	1.4.7 2.6.9 3.5.8
1.4.8 2.5.9 3.6.7	1.4.9 2.5.8 3.6.7	1.4.8 2.6.9 3.5.7	1.4.9 2.6.8 3.5.7
1.4.9 2.5.7 3.6.8	1.4.8 2.5.7 3.6.9	1.4.9 2.6.7 3.5.8	1.4.8 2.6.7 3.5.9
1.5.7 2.6.8 3.4.9	1.5.7 2.6.9 3.4.8	1.5.7 2.4.8 3.6.9	1.5.7 2.4.9 3.6.8
1.5.8 2.6.9 3.4.7	1.5.9 2.6.8 3.4.7	1.5.8 2.4.9 3.6.7	1.5.9 2.4.8 3.6.7
1.5.9 2.6.7 3.4.8	1.5.8 2.6.7 3.4.9	1.5.9 2.4.7 3.6.8	1.5.8 2.4.7 3.6.9
1.6.7 2.4.8 3.5.9	1.6.7 2.4.9 3.5.8	1.6.7 2.5.8 3.4.9	1.6.7 2.5.9 3.4.8
1.6.8 2.4.9 3.5.7	1.6.9 2.4.8 3.5.7	1.6.8 2.5.9 3.4.7	1.6.9 2.5.8 3.4.7
1.6.9 2.4.7 3.5.8	1.6.8 2.4.7 3.5.9	1.6.9 2.5.7 3.4.8	1.6.8 2.5.7 3.4.9

It follows at once from the above Table, that if 3 cubic equations be given, we may form a function of the 9 roots, which, when any of the roots of any of the equations are interchanged *inter se*, or all the roots of one with all those of any other, will receive only *four distinct values.*

It also follows that we may form with 9 letters an intransitive group (of Cauchy) containing $\frac{216}{4}$, that is, 54, or a transitive group containing $\frac{1296}{4}$, or 324 substitutions. So the family of 36 groupings lead to the formation of an intransitive substitution group of $\frac{216}{12}$, that is, 18, and of a transitive group of $\frac{1296}{36}$, or 36 substitutions.

Since 9 letters may be thrown, in $\frac{8 \,.\, 7}{2} \times \frac{5 \,.\, 4}{2}$, that is, 280 different ways, into nomes of 3 letters each, it further follows that by repeating each of the above two families 280 times we shall obtain new families remaining unaltered by any substitution of any of the nine elements *inter se*, and consequently indicating the existence of substitution-groups containing

$$\frac{1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 5 \,.\, 6 \,.\, 7 \,.\, 8 \,.\, 9}{280 \times 36} \text{ and } \frac{1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 5 \,.\, 6 \,.\, 7 \,.\, 8 \,.\, 9}{280 \times 4},$$

that is, 36 and 324 substitutions respectively.

In the above solution a little consideration will show that the method is essentially based on the solution of a *previous* question, namely, of grouping together the synthemes of *binomial duads* of *two* nomes of three letters each, which can be done in two distinct modes, which (if, for example, we take $1 \,.\, 2 \,.\, 3$, $4 \,.\, 5 \,.\, 6$ as the two nomes in question) are represented in the notation used above by $\begin{matrix} c_1 \\ c_2 \\ c_3 \end{matrix}$ and $\begin{matrix} \dot{c}_1 \\ \dot{c}_2 \\ \dot{c}_3 \end{matrix}$ respectively. So, more generally, the groupings of the q-nomial q-ads of r nomes of s elements may be made to depend on the groupings of the $(q-1)$-nomial $(q-1)$-ads of $(r-1)$ nomes of s elements each. The more general question is to discover the groupings and their families of the synthemes composed of p-nomial q-ads of r nomes of s elements, of which the simplest example next that which has been considered and solved is to discover the groupings of the synthemes composed of 54 *binomial* triads of 3 nomes of 3 elements each*.

The chief difficulty of calculating *à priori* the number of such groupings is of a similar nature to that which lies at the bottom of the ordinary theory of the partition of numbers, namely, the liability of the same groupings to make their appearance under distinct symbolical representations. Of this we have seen an example in the threefold representation of the principal family

* I have ascertained, by a direct analytical method, since the above was written, that the number of different groupings of the synthemes composed of these binomial triads is 144. The number of distinct types or families is *three*, one containing 12, another 24, and the third 108 groupings.

of 4 groupings just treated of. But for the existence of this multiform representation of the same grouping we could have affirmed *à priori* the number of groupings to be $2 \times 3 \times 2^3$ or 48, whereas the true number is only 40. I believe that the above is the first instance of the doctrine of types making its appearance explicitly, and illustrated by example in the theory of tactic. It were much to be desired that some one would endeavour to collect and collate the various solutions that have been given of the noted 15-school-girl problem by Messrs Kirkman (in the *Ladies' Diary*), Moses Ansted (in the *Cambridge and Dublin Mathematical Journal*), by Messrs Cayley and Spottiswoode (in the *Philosophical Magazine* and elsewhere), and Professor Pierce, the latest and probably the best (in the *American Astronomical Journal*), besides various others originating and still floating about in the fashionable world (one, if not two, of which I remember having been communicated to me many years ago by Mr Archibald Smith, F.R.S.), with a view to ascertaining whether they belong to the same or to distinct types of aggregation.

48.

CONCLUDING PAPER ON TACTIC.

[*Philosophical Magazine,* XXII. (1861), pp. 45—54.]

IN my tactical paper in the May Number of this *Magazine* [p. 264 above], I considered the number of groupings and of types of groupings of *synthemes* formed out of triads of three *nomes* of three elements each. The first example of considering the *ensemble* of the groupings of a defined species of synthemes (each of such groupings being subjected to satisfy a certain exhaustive condition) was, as already stated, furnished by me in this *Magazine*, April, 1844. In that case the synthemes consisted of duads belonging to a single nome of 6 elements, and the total number of the groupings was observed to be 6, all contained in one type or family. The total number of synthemes in that instance being 15, and there being 6 groupings of 5 synthemes each, it followed that in the whole family every syntheme is met with twice over; once in one grouping, and once in another. In the case treated of in my last communication to this *Magazine*, the total number of the synthemes of the kind under consideration is 36 (for it may easily be shown that the number of synthemes of n-nomial n-ads of n nomes of q elements each is $(1.2.3...q)^{n-1}$); and as each grouping contains 9 synthemes, these 36 are distributed *without repetition* between the 4 groupings of the smaller of the two natural species,—a phenomenon of a kind here met with for the first time in the study of *syntax*. If now we go on (as a natural and irrepressible curiosity urges) to ascertain the groupings of the synthemes of *binomial* triads of the same 3 nomes of three elements each, we advance just one step further in the direction of type-complexity; that is to say, we meet with the existence of 3, and not more than 3, types or species in which all such groupings are comprised. The investigation by which this is made out appears to me well worthy to be given to the world, as affording an example of a new and beautiful kind of analysis proper to the study of *tactic*, and thus lighting the way to the further opening up of this fundamental doctrine of mathematic, the science of necessary relations, of which, combined with logic (if indeed the two be not identical), tactic appears to me to constitute the main stem from which all others, including even arithmetic itself, are derived and secondary branches. The key to success in dealing with the problems of

this incipient science (as I suppose of most others) must be sought for in the construction of an apt and expressive notation, and in the discovery of language by force of which the mind may be enabled to lay hold of complex operations and mould them into simple and easily transmissible forms of thought. I must then entreat the indulgence of the reader if, in this early grappling with the difficulties of a new language and a new notation, I may occasionally appear wanting in absolute clearness and fulness of expression.

Let us, as before, represent the nine elements by the numbers from 1 to 9, and suppose the *nomes* to be 1, 2, 3 : 4, 5, 6 : 7, 8, 9.

If we take any syntheme formed out of the *binomial* triads belonging to the above nomes, and if out of such syntheme we omit the elements 1, 2, 3 (belonging to the 1st nome) wherever they occur, the slightest consideration will serve to show that the synthemes thus denuded will assume the *form* $l.m.r, p.q, n$, where l, m, r may be regarded as belonging to one of the remaining nomes, and p, q, n to the other. The total number of synthemes in a grouping which contains all the binomial triads is 18, because the total number of these triads is 54; and consequently it will be seen that every grouping will in fact consist of the same *framework*, so to say, of combinations of elements belonging to the second and third nomes variously compounded with the elements of the first nome.

This framework may be advantageously divided into two parts, each containing nine terms, and which I shall call respectively U and $\dot{U}$. Thus by U I shall understand the nine arrangements following:—

4.5.7, 8.9, 6; 4.5.8, 7.9, 6; 4.5.9, 7.8, 6
5.6.7, 8.9, 4; 5.6.8, 7.9, 4; 5.6.9, 7.8, 4
6.4.7, 8.9, 5; 6.4.8, 7.9, 5; 6.4.9, 7.8, 5

each *imperfect* or defective syntheme being separated from the next by a semicolon, or else by a change of line. So by $\dot{U}$ I shall understand the *complementary* part of the framework, namely:—

8.9.6, 4.5, 7; 7.9.6, 4.5, 8; 7.8.6, 4.5, 9
8.9.4, 5.6, 7; 7.9.4, 5.6, 8; 7.8.4, 5.6, 9
8.9.5, 6.4, 7; 7.9.5, 6.4, 8; 7.8.5, 6.4, 9.

It is of cardinal importance to notice that the order in which the *imperfect synthemes* are arranged in U and $\dot{U}$ is one of absolute reciprocity. It is in this reciprocity, and in the fact of U or $\dot{U}$ being each in *strict regimen* (so to say) with the other, that the cause of the success of the method about to be applied essentially resides.

The slightest reflection will serve to show that every *complete* syntheme of the kind required will be of the form

$$\left| \begin{matrix} U \times P \\ \dot{U} \times \dot{P} \end{matrix} \right|,$$

where the symbolical multipliers P and $\dot{P}$ are each of them some one of the forms (by no means necessarily the *same*) represented generally by the framework of defective synthemes hereunder written (defective in the sense that all the elements of the second and third nomes are supposed to be omitted),

$$\begin{array}{lll} , a, bc; & , b, ca; & , c, ab \\ , b, ca; & , c, ab; & , a, bc \\ , c, ab; & , a, bc; & , b, ca, \end{array}$$

or else by the cognate framework

$$\begin{array}{lll} , a, bc; & , c, ab; & , b, ca \\ , b, ca; & , a, bc; & , c, ab \\ , c, ab; & , b, ca; & , a, bc, \end{array}$$

where a, b, c are identical in some order or another with the elements of the first nome, namely, 1, 2, 3; so that there are six different systems of a, b, c in each of these two frameworks.

No other combination of the elements in U or $\dot{U}$ (all of which belong to the second and third nomes) with the elements in the first nome is possible; for any such combination would involve the fact of a *repetition* of the same *triad* or triads in the same grouping, contrary to the nature of a grouping. Hence, then, the number of forms of P and of $\dot{P}$ being twice six, or 12, we at once perceive that the total number of groupings is 12×12, or 144.

But now comes the more difficult question of ascertaining between how many distinct species or types these groupings are distributed. If we study the form of P or $\dot{P}$, it is obvious that it will be completely and distinctively *denoted in brief* by the twelve forms arising from the development of

$$\begin{array}{ccc} a & b & c \\ b & c & a \\ c & a & b \end{array} \quad \text{and} \quad \begin{array}{ccc} a & c & b \\ b & a & c \\ c & b & a \end{array}; \quad \text{videlicet}$$

$$\begin{array}{cccccc} (1) & (2) & (3) & (4) & (5) & (6) \\ 1\,2\,3 & 2\,3\,1 & 3\,1\,2 & 2\,1\,3 & 1\,3\,2 & 3\,2\,1 \\ 2\,3\,1 & 3\,1\,2 & 1\,2\,3 & 1\,3\,2 & 3\,2\,1 & 2\,1\,3 \\ 3\,1\,2 & 1\,2\,3 & 2\,3\,1 & 3\,2\,1 & 2\,1\,3 & 1\,3\,2 \\ (7) & (8) & (9) & (10) & (11) & (12) \\ 1\,3\,2 & 2\,1\,3 & 3\,2\,1 & 2\,3\,1 & 1\,2\,3 & 3\,1\,2 \\ 2\,1\,3 & 3\,2\,1 & 1\,3\,2 & 1\,2\,3 & 3\,1\,2 & 2\,3\,1 \\ 3\,2\,1 & 1\,3\,2 & 2\,1\,3 & 3\,1\,2 & 2\,3\,1 & 1\,2\,3 \end{array}$$

which we may for facility of future reference denote by

$$\pi_1, \pi_2, \pi_3, \pi_4, \pi_5 \ldots \pi_{12}.$$

Now as regards the types: since the order of the elements in one nome is entirely independent of the order of the elements in any other, it is obvious that it is not the particular form of P or of $\dot{P}$ which can have any influence

on the form of the type, but solely the relation of P and $\dot{P}$ to one another. In order then to fix the ideas, I shall for the moment consider P equal to

$$\begin{matrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2. \end{matrix}$$

This at once enables us to fix a *limit* to the number of distinct types. In the first place, the essentially distinct FORMS of the first column in $\dot{P}$, with respect to that of P, may be sufficiently represented by taking the two columns identical, or differing by a single interchange, or else having no two elements in the same place. Hence $\dot{P}$, so far as the ascertainment of types is concerned, may be limited to the six forms following:—

$$\begin{matrix} (\alpha) & & (\gamma) & & (\epsilon) \\ 1\ 2\ 3 & & 2\ 1\ 3 & & 2\ 3\ 1 \\ 2\ 3\ 1 & & 1\ 3\ 2 & & 3\ 1\ 2 \\ 3\ 1\ 2 & & 3\ 2\ 1 & & 1\ 2\ 3 \\ (\beta) & & (\delta) & & (\eta) \\ 1\ 3\ 2 & & 2\ 3\ 1 & & 2\ 1\ 3 \\ 2\ 1\ 3 & & 1\ 2\ 3 & & 3\ 2\ 1 \\ 3\ 2\ 1 & & 3\ 1\ 2 & & 1\ 3\ 2. \end{matrix}$$

But again, since (β) and (η) are each derivable from (α) (the assumed form of P) by an interchange of two columns *inter se*, it is clear that, as regards distinction of type, $\eta = \beta$, and consequently there are only *at utmost* five types remaining, which may be respectively described by the symbols

$$\begin{vmatrix} U\alpha \\ \dot{U}\alpha \end{vmatrix} \quad \begin{vmatrix} U\alpha \\ \dot{U}\beta \end{vmatrix} \quad \begin{vmatrix} U\alpha \\ \dot{U}\gamma \end{vmatrix} \quad \begin{vmatrix} U\alpha \\ \dot{U}\delta \end{vmatrix} \quad \begin{vmatrix} U\alpha \\ \dot{U}\epsilon \end{vmatrix}.$$

It must be noticed that α comprehends or typifies the squares numbered 1; β those numbered 7, 8, 9; γ those numbered 4, 5, 6; δ those numbered 10, 11, 12; ϵ those numbered 2, 3.

I say designedly that the number of types is *at utmost* limited to these five. But it by no means follows that the number is so great as five; for it will not fail to be borne in mind that these differences have reference to the peculiar mode in which we have chosen to decompose in idea each syntheme, by viewing it as a symbolical product of an arrangement containing only the elements of the second and third nomes by an arrangement containing only those of the first nome. But the nomes are interchangeable, and therefore it may very well be the case that two types which appear to be distinct are in reality identical, their elements in the groupings appertaining to such types having absolutely analogous relations to different orderings of the

nomes, so that the groupings will be convertible into each other by permutations among the given elements. We must therefore ascertain how the above types, or any specific forms of them, come to be represented when we interchange the first nome with either of the other two, or, to fix the ideas, let us say with the second.

To effect this, let $U\alpha$, $\dot{U}\alpha$, $\dot{U}\beta$, $\dot{U}\gamma$, $\dot{U}\delta$, $\dot{U}\epsilon$ be actually expanded; by the performance of the symbolical multiplications we obtain—

$$U\alpha = \left| \begin{array}{l} 4.5.7\ \ 8.9.1\ \ 6.2.3;\ \ 4.5.8\ \ 7.9.2\ \ 6.1.3;\ \ 4.5.9\ \ 7.8.3\ \ 6.2.1 \\ 5.6.7\ \ 8.9.2\ \ 4.1.3;\ \ 5.6.8\ \ 7.9.3\ \ 4.1.2;\ \ 5.6.9\ \ 7.8.1\ \ 4.2.3 \\ 6.4.7\ \ 8.9.3\ \ 5.1.2;\ \ 6.4.8\ \ 7.9.1\ \ 5.2.3;\ \ 6.4.9\ \ 7.8.2\ \ 5.1.3 \end{array} \right|$$

$$\dot{U}\alpha = \left| \begin{array}{l|l|l} 8.9.6\ \ 4.5.1\ \ 7.2.3 & 7.9.6\ \ 4.5.2\ \ 8.1.3 & 7.8.6\ \ 4.5.3\ \ 9.2.1 \\ 8.9.4\ \ 5.6.2\ \ 7.1.3 & 7.9.4\ \ 5.6.3\ \ 8.1.2 & 7.8.4\ \ 5.6.1\ \ 9.2.3 \\ 8.9.5\ \ 6.4.3\ \ 7.2.1 & 7.9.5\ \ 6.4.1\ \ 8.2.3 & 7.8.5\ \ 6.4.2\ \ 9.1.3 \end{array} \right|$$

$$\dot{U}\beta = \left| \begin{array}{l|l|l} 8.9.6\ \ 4.5.1\ \ 7.2.3 & 7.9.6\ \ 4.5.3\ \ 8.1.2 & 7.8.6\ \ 4.5.2\ \ 9.1.3 \\ 8.9.4\ \ 5.6.2\ \ 7.1.3 & 7.9.4\ \ 5.6.1\ \ 8.2.3 & 7.8.4\ \ 5.6.3\ \ 9.2.1 \\ 8.9.5\ \ 6.4.3\ \ 7.2.1 & 7.9.5\ \ 6.4.2\ \ 8.1.3 & 7.8.5\ \ 6.4.1\ \ 9.2.3 \end{array} \right|$$

$$\dot{U}\gamma = \left| \begin{array}{l|l|l} 8.9.6\ \ 4.5.2\ \ 7.1.3 & 7.9.6\ \ 4.5.1\ \ 8.2.3 & 7.8.6\ \ 4.5.3\ \ 9.1.2 \\ 8.9.4\ \ 5.6.1\ \ 7.2.3 & 7.9.4\ \ 5.6.3\ \ 8.1.2 & 7.8.4\ \ 5.6.2\ \ 9.1.3 \\ 8.9.5\ \ 6.4.3\ \ 7.1.2 & 7.9.5\ \ 6.4.2\ \ 8.1.3 & 7.8.5\ \ 6.4.1\ \ 9.2.3 \end{array} \right|$$

$$\dot{U}\delta = \left| \begin{array}{l|l|l} 8.9.6\ \ 4.5.2\ \ 7.1.3 & 7.9.6\ \ 4.5.3\ \ 8.1.2 & 7.8.6\ \ 4.5.1\ \ 9.2.3 \\ 8.9.4\ \ 5.6.1\ \ 7.2.3 & 7.9.4\ \ 5.6.2\ \ 8.1.3 & 7.8.4\ \ 5.6.3\ \ 9.1.2 \\ 8.9.5\ \ 6.4.3\ \ 7.1.2 & 7.9.5\ \ 6.4.1\ \ 8.2.3 & 7.8.5\ \ 6.4.2\ \ 9.1.3 \end{array} \right|$$

$$\dot{U}\epsilon = \left| \begin{array}{l|l|l} 8.9.6\ \ 4.5.2\ \ 7.1.3 & 7.9.6\ \ 4.5.3\ \ 8.1.2 & 7.8.6\ \ 4.5.1\ \ 9.2.3 \\ 8.9.4\ \ 5.6.3\ \ 7.1.2 & 7.9.4\ \ 5.6.1\ \ 8.2.3 & 7.8.4\ \ 5.6.2\ \ 9.1.3 \\ 8.9.5\ \ 6.4.1\ \ 7.2.3 & 7.9.5\ \ 6.4.2\ \ 8.1.3 & 7.8.5\ \ 6.4.3\ \ 9.1.2 \end{array} \right|$$

Let us form a *framework* with the nomes $1.2.3$, $7.8.9$ exactly similar to that which we formed before with $4.5.6$, $7.8.9$, and let V, $\dot{V}$ be its two parts respectively analogous to U, $\dot{U}$, we thus obtain for $\dot{V}$,

$$\begin{array}{lll} 1.2.7,\ 8.9,\ 3; & 1.2.8,\ 7.9,\ 3; & 1.2.9,\ 7.8,\ 3 \\ 2.3.7,\ 8.9,\ 1; & 2.3.8,\ 7.9,\ 1; & 2.3.9,\ 7.8,\ 1 \\ 3.1.7,\ 8.9,\ 2; & 3.1.8,\ 7.9,\ 2; & 3.1.9,\ 7.8,\ 2, \end{array}$$

and for V,

$$\begin{array}{lll} 8.9.3,\ 1.2,\ 7; & 7.9.3,\ 1.2,\ 8; & 7.8.3,\ 1.2,\ 9 \\ 8.9.1,\ 2.3,\ 7; & 7.9.1,\ 2.3,\ 8; & 7.8.1,\ 2.3,\ 9 \\ 8.9.2,\ 3.1,\ 7; & 7.9.2,\ 3.1,\ 8; & 7.8.2,\ 3.1,\ 9. \end{array}$$

We must now perform the unwonted process of symbolical division, and obtain the quotients of $U\alpha$ by V, and of $\dot{U}\alpha$, $\dot{U}\beta$, $\dot{U}\gamma$, $\dot{U}\delta$, $\dot{U}\epsilon$ by $\dot{V}$ (it will of course be perceived that it is known *à priori* that the dividend forms of arrangement are actual multipliers of the divisors V and $\dot{V}$). In writing down the results of these divisions, which will consist exclusively of elements belonging to the nome $4.5.6$, and of which each term will be of the form d, $e.f$, we may, analogously to what we have done before for greater brevity, write down only the single element (d), and omit the residue (ef), which is

determined when (d) is determined. We shall thus obtain the quotients following:—

$$\frac{U\alpha}{V}=\begin{matrix}5&4&6\\6&5&4\\4&6&5\end{matrix}$$

$$\frac{\dot U\alpha}{\dot V}=\begin{matrix}5&4&6\\6&5&4\\4&6&5\end{matrix}\qquad \frac{\dot U\beta}{\dot V}=\begin{matrix}5&6&4\\6&4&5\\4&5&6\end{matrix}\qquad \frac{\dot U\gamma}{\dot V}=\begin{matrix}5&4&6\\4&6&5\\6&5&4\end{matrix}$$

$$\frac{\dot U\delta}{\dot V}=\begin{matrix}5&6&4\\4&5&6\\6&4&5\end{matrix}\qquad \frac{\dot U\epsilon}{\dot V}=\begin{matrix}4&6&5\\5&4&6\\6&5&4.\end{matrix}$$

It may be observed that these divisions may be effected with great rapidity; because when three out of the nine figures (in any quotient) not in the same line or column are known, all the rest are known. Thus, for example, to find $\frac{\dot U\epsilon}{\dot V}$ it is only necessary to seek in $\dot U\epsilon$ the syntheme which contains 1 . 2 . 7, and then to take out the figure in that syntheme associated with 8 . 9 in that line, namely, 4; then again to seek the syntheme which contains 1 . 2 . 8, and to take out the figure in that syntheme associated with 7 . 9, which is 6; and finally to seek the syntheme which contains 2 . 3 . 7, and then to take out the figure associated with 8 . 9, namely 5; we thus obtain the three corner figures of the square which represents $\frac{\dot U\epsilon}{\dot V}$ as thus:

$$\begin{matrix}4&6&.\\5&.&.\\.&.&.\end{matrix}$$

of which the six remaining figures are given by the condition that in no line and in no column must the same two figures be found. In order to compare these quotients, or rather the relations of the first of them to the remaining five with those of α to α, β, γ, δ, ϵ, it will be convenient to subtract the constant number 3 from each figure, and to transpose the first and second columns; we thus obtain

$$\frac{U\alpha}{V}\equiv\left|\begin{matrix}1&2&3\\2&3&1\\3&1&2\end{matrix}\right|\equiv\pi_1\equiv\alpha,$$

$$\frac{\dot U\alpha}{\dot V}\equiv\left|\begin{matrix}1&2&3\\2&3&1\\3&1&2\end{matrix}\right|\equiv\pi_1\equiv\alpha,\qquad \frac{\dot U\beta}{\dot V}\equiv\left|\begin{matrix}3&2&1\\1&3&2\\2&1&3\end{matrix}\right|\equiv\pi_9\equiv\beta,$$

$$\frac{\dot U\gamma}{\dot V}\equiv\left|\begin{matrix}1&2&3\\3&1&2\\2&3&1\end{matrix}\right|\equiv\pi_{11}\equiv\delta,\qquad \frac{\dot U\delta}{\dot V}\equiv\left|\begin{matrix}3&2&1\\2&1&3\\1&3&2\end{matrix}\right|\equiv\pi_6\equiv\gamma,$$

$$\frac{\dot U\epsilon}{\dot V}\equiv\left|\begin{matrix}3&1&2\\1&2&3\\2&3&1\end{matrix}\right|\equiv\pi_3\equiv\epsilon.$$

Thus, for greater brevity, considering the five types to be represented by

$$\begin{matrix} \alpha & \alpha & \alpha & \alpha & \alpha \\ \alpha & \beta & \gamma & \delta & \epsilon, \end{matrix}$$

or still more briefly by

$$\alpha \quad \beta \quad \gamma \quad \delta \quad \epsilon;$$

and calling the nomes N_1, N_2, N_3, we find that the effect of interchanging N_1 and N_2 with each other is to change

$$\alpha \quad \beta \quad \gamma \quad \delta \quad \epsilon$$

into

$$\alpha \quad \beta \quad \delta \quad \gamma \quad \epsilon.$$

In like manner it may be ascertained (and the student is advised to satisfy himself by actual trial of the fact) that the effect of interchanging N_1 and N_3 with each other is to convert

$$\alpha \quad \beta \quad \gamma \quad \delta \quad \epsilon$$

into

$$\alpha \quad \delta \quad \gamma \quad \beta \quad \epsilon.$$

From these two calculations it follows that the effect of any permutation between N_1, N_2, N_3 is to produce a permutation in β, γ, δ *inter se*, but will leave α and ϵ unaltered*. Hence then we have arrived at the goal of our inquiry, having demonstrated that

$$\begin{vmatrix} V\alpha \\ \dot{V}\alpha \end{vmatrix}$$

indicates one type,

$$\begin{vmatrix} V\alpha \\ \dot{V}\beta \end{vmatrix}, \quad \begin{vmatrix} V\alpha \\ \dot{V}\gamma \end{vmatrix}, \quad \begin{vmatrix} V\alpha \\ \dot{V}\delta \end{vmatrix}$$

each of them another *the same* type, and

$$\begin{vmatrix} V\alpha \\ \dot{V}\epsilon \end{vmatrix}$$

a third type,—and bearing in mind that

(α)	belongs to	π_1	exclusively,
(ϵ)	,,	π_2, π_3	,,
(β)	,,	π_7, π_8, π_9	,,
(γ)	,,	π_4, π_5, π_6	,,
(δ)	,,	π_{10}, π_{11}, π_{12}	,,

* This result, by the aid of a fine observation, may be more rapidly established *uno ictu* (I mean by *one* calculation instead of two) as follows. Let $N_1 N_2 N_3$ be made to undergo a *cyclical* interchange, then it will be found that β, γ, δ also undergo a cyclical interchange, whilst α and ϵ remain unchanged. This proves that β, γ, δ are only different phases of the same type, which is *sufficient*; for as regards α and ϵ, the fact of the number of individuals which they represent being unequal *inter se*, and also unequal to the number contained in β, γ, δ, renders it *à priori* impossible to allow that they can either pass into each other or into the forms β, γ, δ, by virtue of any interchange among the elements.

and that each form of π comprehends 12 groupings due to the 12 forms of $V\alpha$, we are enabled to affirm that the total number of groupings of the binomial triads of 3 nomes of 3 elements each is 144, and that the number of types or species between which these 144 are distributed is 3, comprising 12, 24, and 108 respectively,—a conclusion which it would almost have exceeded the practical limits of human labour and perspicuity to have established by the direct comparison of the 144 groupings of 18 synthemes each with each other, with a view to ascertain which admit of being permutable into each other, and which not.

The largest species of 108 groupings, it may be observed, is subdivisible into 3 *varieties*, not really allotypical, of 36 each,—the characteristic of those groupings which belong to the same variety being that they permute *exclusively* into each other when the permutations of the elements are confined to perturbations of the order of the elements in the same nome or nomes, and the different nomes are subject to no interchange of elements between themselves.

Just so the species of 36 groupings of trinomial triads, treated of in my preceding paper, subdivides into 3 varieties of sub-families characterized by a similar property.

The total number of modes of subdivision of 9 elements between 3 nomes being 280, it follows, from considerations of the same kind as stated in the May Number of this *Magazine* [p. 264 above], that there exist transitive substitution-groups belonging to 9 elements of

$$\frac{\pi(9)}{280\times 12},\quad \frac{\pi(9)}{280\times 24},\quad \frac{\pi(9)}{280\times 108},$$

that is, 108, 24 and 12 substitutions respectively.

Again, let us consider the question of forming the synthemes of the triads of a *single nome* of 9 elements into groupings where *every* triad shall be found without repetition. We may obtain such groupings by choosing arbitrarily any one of the 280 sets of 3 nomes into which the 9 elements may be segregated*, and then forming one syntheme with the three monomial triads (corresponding to such set so chosen), 18 synthemes (in any one of the 144 possible ways) of exclusively binomial triads, and 9 synthemes (in any one of the 40 possible ways) of exclusively trinomial triads; we shall thus obtain in all $280\times 144\times 40$, or 1,612,800 solutions of the question proposed; I mean

* 280 is also evidently the number of synthemes of triads belonging to one nome of 9 elements. In general the number of r-ads belonging to one nome of mn elements is

$$\frac{\pi(mn-1)\,\pi\{(m-1)n-1\}\,\pi\{(m-2)n-1\}\ldots\pi(n-1)}{\{\pi(n-1)\}^m\,\pi\{(m-1)n\}\,\pi\{(m-2)n\}\ldots\pi(n)}.$$

1,612,800 groupings, all satisfying the imposed condition, and reducible to 6 *genera**, comprising respectively

$$4 \times 12 \times 280, \quad 4 \times 24 \times 280, \quad 4 \times 108 \times 280, \quad 36 \times 12 \times 280,$$
$$36 \times 24 \times 280, \quad 36 \times 108 \times 280,$$

that is, 13,440, 26,880, 120,960, 120,960, 241,920, 1,088,640 individual groupings. I conclude with putting a grand question, more easy to propose than to answer, namely, are these one million six hundred thousand (and upwards) groupings (classifiable under six distinct genera) all the possible modes and types of grouping which will satisfy the conditions of the question? and if not, what other mode or type of grouping can be found? Were I compelled to give an answer to this question, I would say that the balance of my mind leans to the opinion that the six types in question are the sole possible types of solution; but I do not pretend to rest this judgment upon any solid grounds of demonstration, nor to entertain it with any strong degree of assurance. It is a question which the effort to resolve cannot but react powerfully on our knowledge of the principles of tactic in general, and of the theory of substitution-groups in particular; and as such I submit it to the consideration of the rising chivalry of analysis, seeking myself meanwhile fresh fields and pastures new of meditation.

* The above *genera* must not be confounded with types or species. (In my preceding communications I may inadvertently have used the word *family* as coincident with type: *species* is the proper term.) The type of a total grouping in the problem referred to in the text will depend not only on the particular combination of the types of the binomial and trinomial partial groupings which give rise to these 6 ($=2 \times 3$) genera, but also on the relative *phases* of the types so combined. The number of groupings in one type or species is always a submultiple of the number of permutations of the elements; whereas it will be seen that the number of groupings in one of the above genera greatly exceeds that number, which in the present case is only

$$1 . 2 . 3 . 4 . 5 . 6 . 7 . 8 . 9, \text{ or } 362,880.$$

Whatever may be the case in natural history, the nature of a type or species, as distinguished from a genus, family, or any other higher kind of aggregation of individuals, in *pure syntax* is perfectly clear and unambiguous; those groupings form a species which are commutable into one another by an interchange of elements: thus the different *phases* of the same type or species are in analogy with the different values of the same function arising out of a change in a constant parameter. If it should turn out that the above sixteen hundred thousand and odd groupings are not the sole solutions of which the question admits, then it will follow that even in this early instance we shall have an example not only of species and genera, but of distinct families of genera, for it is certain that the above six genera constitute within themselves a complete natural family. It will form an interesting subject of inquiry to ascertain how many types are included within each of the six genera belonging to this family; and be it never forgotten that to each species corresponds, and from it is, so to say, capable of being extracted or sublimated, a Cauchian substitution-group.

49.

REMARK ON THE TACTIC OF NINE ELEMENTS.

[*Philosophical Magazine*, XXII. (1861), pp. 144—147.]

AT the end of my preceding paper in this *Magazine* for July [p. 284 above], I hazarded an opinion that any grouping of 28 synthemes comprising the 84 triads belonging to a system of 9 elements, might be regarded as made up of 1 syntheme of monomial triads, 18 synthemes of binomial triads, and 9 of trinomial triads, the denominations (monomial, binomial, and trinomial) having reference to a duly chosen distribution of the 9 elements into 3 nomes of 3 elements each. This conjecture is capable of being brought to a very significant, although not decisive test, by examining a peculiar and important distribution of the 28 synthemes into 7 sets of 4 synthemes each, the property of each set being that its 12 triads contain amongst them all the 36 *duads* appertaining to the 9 elements. I discovered this mode of distribution very many years ago; but it was first published independently by a mathematician whose name I forget, either in the *Philosophical Magazine* or in the *Cambridge and Dublin Mathematical Journal*, I think at some time between the years 1847–53. A similar mode of distribution exists for any system of elements of which the number is a power of 3. Without pausing to give the law of formation, I shall simply observe that for 9 elements we may take as a basic arrangement the square

$$\begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{matrix}$$

and form from this, by a symmetrical method, the annexed six derived arrangements :—

$$\begin{matrix} 7\,1\,2 & 7\,2\,3 & 9\,2\,3 \\ 3\,5\,6 & 1\,4\,5 & 1\,5\,6 \\ 4\,8\,9 & 6\,8\,9 & 4\,7\,8 \\ & & \\ 4\,3\,1 & 5\,2\,3 & 4\,2\,3 \\ 7\,5\,6 & 7\,6\,4 & 8\,5\,6 \\ 2\,8\,9 & 1\,8\,9 & 1\,9\,7 \end{matrix}$$

and reading off each of these squares in lines, in columns, and in right and left diagonal fashion, we obtain the 7 sets of 4 synthemes each referred to, namely

	123 456 789	
	147 258 369	
	159 267 348	
	168 249 357	
712 356 489	723 145 689	923 156 478
734 158 269	716 248 359	914 257 368
759 164 238	749 256 318	958 264 317
768 139 254	758 219 346	967 218 354
431 756 289	523 764 189	423 856 197
472 358 169	571 268 349	481 259 367
459 362 178	569 241 378	457 261 389
468 379 152	548 279 361	469 287 351

If, now, we take any distribution of the 9 elements into nomes other than 123, 456, 789, we shall find that some of the synthemes will contain trinomials, some binomials only, but others (in number either 9 or 18, according to the distribution chosen) will contain binomials and trinomials mixed; but if we adopt 123, 456, 789 as the nomes, then it will be found that the remaining 27 synthemes (after excluding the monomial syntheme 123, 456, 789) will consist of 18 purely binomial triads, and 9 purely trinomial triads. The former will consist of the first, second, and fourth synthemes of the 6 derived groups; the latter of the second, third and fourth of the basic group, and of the second synthemes of each of the 6 derived groups.

It may be remembered that there are two types or species of groupings of trinomial triad synthemes appertaining to 3 nomes of 3 elements; one of these species contains 4, the other 36 individual groupings. It may easily be ascertained that the grouping above indicated belongs to the first (the less numerous) of these species. Again, there are 3 types or species of groupings of binomial triad synthemes appertaining to the same system of nomes; one containing 12, one 24, and the third 108 groupings. The grouping with which we are here concerned will be found to belong to the *second* of these species,—that denoted by the symbols $\begin{vmatrix} \alpha \\ \epsilon \end{vmatrix}$ in my paper of last month. Hence, then, we derive a very considerable presumption in favour of the opinion which I advanced at the close of my preceding paper on Tactic, and derived, too, from a case apparently unfavourable to the verisimilitude of the conjecture; for a natural subdivision of 28 things into 7 sets of 4 each seems

at first sight hardly compatible with another natural division into 3 sets of 1, 18, and 9 respectively. Notwithstanding this seeming incompatibility, we have found that the two methods of decomposition do coexist, owing essentially to the fact that the 7 sets (of 4 synthemes each) stand not in a relation of indifference set to set, but are to be considered as composed of a base and 6 derivatives indifferently related to the base and to each other. The theory of these 7 sets is extremely curious, and well worthy of being fully investigated by the student of tactic, but cannot be gone into within the limits suitable to the pages of a philosophical miscellany.

Before taking final leave of the subject (at all events for the present, and in the pages of this *Magazine*), as I have been questioned as to the meaning of the important word "syntheme," derived from $\sigma\upsilon\nu$ $\theta\eta\mu\alpha$, I repeat that a "syntheme" is the general name for any consociation of the single or combined elements of a given system of elements in which each element is once and once only contained. A *nome*, from $\nu\epsilon\mu\omega$ (*to divide*), means a consociation of a certain number out of a given system of elements; and a binomial, trinomial, or r-nomial combination of any specified sort, means a combination whose elements are dispersed between 2, 3, or r of the nomes between which the entire system of elements is supposed to have been divided.

P.S. I have found the date and place of the resolution into 7 sets referred to in the text; it is given in a paper by Mr Kirkman, Vol. v. p. 261 of the *Cambridge and Dublin Mathematical Journal* for 1850. His 7 squares, whose horizontal, vertical, and two diagonal readings (like mine) constitute the 7 sets in question, are substantially as follows :—

	1 2 3 4 5 6 7 8 9	
1 2 4 5 6 7 8 9 3	4 5 7 8 9 1 2 3 6	7 8 1 2 3 4 5 6 9
7 1 2 3 4 5 6 8 9	1 4 5 6 7 8 9 2 3	4 7 8 9 1 2 3 5 6*.

* By changing the positions of the lines and columns of the six derivative squares, which may be done without affecting the value of their *readings*, they may be represented under the form following, which will be seen to render much clearer their relation to the primitive square :—

412	623	423	239	129	127
756	745	956	451	563	453
389	189	178	786	784	896.

On assuming 123, 456, 789 as the three nomes, the 28 synthemes contained in the sets will be found to consist of purely monomial, binomial, and trinomial synthemes.

Thus there would be an additional presumption in favour of the supposed law of *homonomial resolubility*, provided that Mr Kirkman's solution were essentially distinct in type from my own; his binomial and trinomial systems, taken separately, coincide in type with those afforded by my solution, notwithstanding which it would not be lawful to assume (indeed I had at first some reasons for doubting) the identity of type of the total groupings of which these systems form part; all we could have positively inferred from that fact would have been, that these two groupings both belong to the same class or genus containing 26,880 individuals, the second of the six referred to at the close of my last paper; a comparison of the two solutions has, however, satisfied me that they are absolutely identical in form.

50.

ON A GENERALIZATION OF A THEOREM OF CAUCHY ON ARRANGEMENTS*.

[*Philosophical Magazine*, XXII. (1861), pp. 378—382.]

IN a paper "On the Theory of Determinants" in the *Philosophical Magazine* for March in this year, Mr Cayley has referred and added to a theorem of Cauchy deduced from the latter's method of *arrangements*, namely, that if we resolve an integer n in every possible way into parts, to wit α parts of a, β parts of b, ... λ of l, ($a, b, c \ldots l$ being all distinct integers), then

$$\Sigma \frac{1}{\Pi\alpha \, . \, a^\alpha \Pi\beta \, . \, b^\beta \ldots \Pi\lambda \, . \, l^\lambda} = 1.$$

Now both Cauchy's theorem and Mr Cayley's addition to it (which essentially consists in the observation, that if before the numerator 1 in the above quantity under the sign of summation we write $(-)^{\alpha+\beta+\ldots+\lambda}$, the sum becomes zero) are no more than particular cases of the following theorem: namely, that if instead of 1 we write $\rho^{\alpha+\beta+\ldots+\lambda}$ in the numerator of the quantity under the sign of summation (ρ being any quantity whatever), the sum becomes expressible as a known function of ρ. Nothing can be easier than the proof.

Let the $\alpha, \beta, \gamma, \ldots \lambda$ in the preceding statement be supposed subject to the further condition that their sum is r; then for any assigned value of r (a positive integer) it is easy to see that the sum of the terms within the sign of summation in Cauchy's theorem is

$$S\left(\frac{1}{x_1 x_2 \ldots x_r} \cdot \frac{\rho^r}{\Pi(r)}\right),$$

where $x_1, x_2, \ldots x_r$ mean *every* system of values of $x_1, x_2, \ldots x_r$ (permutations *admitted*) which satisfy the equation

$$x_1 + x_2 + \ldots + x_r = n.$$

(It should here be observed that $\alpha, \beta, \gamma, \ldots \lambda$; $a, b, c, \ldots l$ are the systems

[* See above p. 245.]

which satisfy $\alpha a + \beta b + \gamma c + \ldots + \lambda l = n$, permutations being *excluded*; that is to say, if, for example, α, β, γ should happen to be equal for any partition of n, the values α, a; α, b; α, c would figure only *once*, and not *six* times, among the systems included under the sign of Σ.) Hence then we see that

$$\Sigma \frac{\rho^{\alpha+\beta+\gamma+\ldots+\lambda}}{\Pi\alpha . a^\alpha \Pi\beta . b^\beta \ldots \Pi\lambda . l^\lambda} = \sum_{r=\infty}^{r=1} S_r \frac{\rho^r}{\Pi(r)} \text{*},$$

where S_r is the coefficient of t^n in $\left(\frac{t}{1} + \frac{t^2}{2} + \frac{t^3}{3} + \text{\&c. } ad\ infin.\right)^r$, that is, in $\left\{\log\left(\frac{1}{1-t}\right)\right\}^r$; and the total sum designated by Σ will be consequently the coefficient of t^n in

$$\log\left(\frac{1}{1-t}\right)\rho + \left(\log\frac{1}{1-t}\right)^2 \frac{\rho^2}{1 . 2} + \text{\&c.},$$

that is, in $e^{\rho \log\left(\frac{1}{1-t}\right)}$, that is, in $\left(\frac{1}{1-t}\right)^\rho$.

Thus if $\rho = 1$, we have Cauchy's theorem, namely $\Sigma = 1$;

Thus if $\rho = -1$, we have Cayley's theorem, namely $\Sigma = 0$†;

and in general for any value of ρ,

$$\Sigma = \frac{\rho(\rho+1)\ldots(\rho+n-1)}{1 . 2 \ldots n} \ddagger.$$

* For if we take a system of values satisfying the above equation, consisting of α equal values a, β equal values b, ... λ equal values l, such a system will give rise in $\Sigma\frac{1}{x_1 x_2 \ldots x_r}$ to $\frac{\pi(r)}{(\pi\alpha)(\pi\beta)\ldots(\pi\lambda)}$ repetitions of the term $\frac{1}{a^\alpha . b^\beta \ldots l^\lambda}$, and consequently in $\Sigma\frac{1}{x_1 x_2 \ldots x_r} . \frac{1}{\pi r}$ to a total value $\frac{1}{(\pi\alpha) a^\alpha (\pi\beta) b^\beta \ldots (\pi\lambda) l^\lambda}$, condensed into a single term in Cauchy's theorem.

† Provided, however, that n exceeds 1, a limitation accidentally omitted in Mr Cayley's paper; and so in general

$$\Sigma \frac{(-\rho)^{\alpha+\beta+\ldots+\lambda}}{\Pi\alpha . a^\alpha \ldots \Pi\lambda . l^\lambda} = 0,$$

ρ being *any* positive integer provided n is greater than ρ.

‡ If $\rho = \frac{1}{2}$, we obtain

$$\Sigma = \frac{1 . 3 . 5 \ldots (2n-1)}{2 . 4 . 6 \ldots 2n};$$

from which it is easy to infer that the number of substitutions of $2n$ things representable by the product of cyclical substitutions, all of an even order, is $\{1 . 3 . 5 \ldots (2n-1)\}^2$. If $\rho = -\frac{1}{2}$, we obtain

$$\Sigma = \frac{1 . 1 . 3 \ldots (2n-1)}{2 . 4 . 6 \ldots (2n)},$$

combining which with the preceding result, it is easy to infer that the number of substitutions of $2n$ things representable by the product of an odd number of cyclical substitutions, all of an even order, is to the number of such representable by the product of an even number of cyclical substitutions, all of an even order, in the ratio of n to $(n-1)$. The former of these two theorems

In this theorem is in fact included another, namely, that if

$$\alpha a + \beta b + \ldots + \lambda l = n \text{ and } \alpha + \beta + \ldots + \lambda = r$$

(permutations *not* admissible), then

$$\Sigma \frac{\Pi n}{\Pi\alpha \,.\, a^\alpha \,.\, \Pi\beta \,.\, b^\beta \ldots \Pi\lambda \,.\, l^\lambda}$$

is equal to the coefficient of ρ^{r-1} in

$$(\rho + 1)(\rho + 2) \ldots (\rho + n - 1).$$

This coefficient is accordingly (to return to Cauchy's theory of arrangements) the number of substitutions of n elements capable of being expressed by the product of r cyclical substitutions. As, for instance, the number of substitutions of four elements a, b, c, d capable of expression by the product of two cyclical substitutions ought to be the coefficient of λ in $(\lambda + 1)(\lambda + 2)(\lambda + 3)$, that is, 11, which is right; for the number of substitutions of the form $(a, b)(c, d)$ will be 3, and of the form $(a, b, c)(d)$ will be 8. In conclusion, I may notice that by an obvious deduction from this last theorem, we are led to the well-known one in the theory of numbers, that every coefficient in the development of

$$\Sigma (\rho + 1)(\rho + 2) \ldots (\rho + n - 1),$$

except the first and last, and the sum of these two, is divisible by n when n is a prime number; and indeed we can actually express by aid of it the quotient of every intermediate coefficient divided by n as the sum of separate integer terms free from the sign of addition.

Postscript. By an extension of the method of generating functions contained in the text above, it may easily be seen that the number* of substitutions of n letters represented by the products of r cyclical substitutions, where the number of letters of each cycle leaves a given residue e in respect

is intimately allied with Mr Cayley's celebrated theorem on "skew," or what, for good reasons hereafter to be alleged, I should prefer to call *polar* determinants, namely, that every such of the $2n$th order is the square of a *Pfaffian*. A Pfaffian is in fact a sum of quantities typifiable completely, both as to sign and magnitude, by a duadic *syntheme* of $2n$ elements, the number of which is readily seen to be $1 \,.\, 3 \,.\, 5 \ldots (2n - 1)$. I believe I shall soon be in a condition to announce a remarkable extension of this theory to embrace the case of Polar *Commutants* and *Hyperpfaffians*.

* For this number, divided by $\Pi(n)$, is the coefficient of x^n in

$$\frac{1}{\Pi r}\left(\int_0^x \frac{dx\,x^{e-1}}{1-x}\right)^r, \text{ say } \frac{1}{\Pi r}(\phi x)^r,$$

and therefore of $x^n \rho^r$ in $e^{\rho\phi x}$, say $\psi(x, \rho)$, and therefore $\left(\text{since } \frac{d\psi}{dx} = \frac{x^{e-1}}{1-x^m} \text{ and } \psi \text{ may be put under the form } \Sigma \frac{u_n}{n} x^n\right)$ of ρ^r in $\frac{u_n}{n}$, where u_n is defined as in the text.

to a given *modulus* μ, may be made to depend on the solution of the equation in differences

$$u_n - u_{n+\mu} = \frac{\rho}{n-e} u_{n-e}.$$

The case where $e=1$ is deserving of particular notice.

It may be shown by means of the above equation in differences, that the number of substitutions of n letters formed by r cycles each of the form $\mu K+1$ (μ being constant), say $\phi(n, r, \mu, 1)$, where $\frac{n-r}{\mu}$ is necessarily an integer, may be found by taking in every possible way $\frac{n-r}{\mu}$ *distinct* groups of μ consecutive terms of the series $1, 2, 3, \ldots (n-1)$; the sum of the products of every such combination of groups is the value required. For example, if

$$n=8, \quad r=3, \quad \mu=2,$$

$$\begin{aligned}\phi(8, 3, 2, 1) = &1.2.3.4.5.6+1.2.3.5.6.7+1.2.3.5.6.8\\ &+1.2.3.6.7.8+2.3.4.5.6.7+2.3.4.6.7.8\\ &+3.4.5.6.7.8.\end{aligned}$$

And as a corollary, since it may easily be seen that $\phi(n, r, \mu, e)$ is always divisible by n when n is a prime and $\mu r+e<n$, it follows that the sum of all the possible products of (any given number) i distinct groups of a given number r of consecutive terms of the series $1, 2, 3, \ldots (n-1)$ will be divisible by n when n is a prime and $ir<n-1$*. When $r=1$, this theorem becomes identical with Wilson's, already referred to.

Finally, it may be noticed that the number of substitutions of n letters formed by *any* number of cycles, all of an *odd* order, will be the coefficient of x^n in $\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}$, that is, $\{1.3.5 \ldots (n-1)\}^2$ (the same as the number that can be formed with cycles all of an *even* order) when n is even, and

$$\{1.3.5 \ldots (n-2)\}^2 n$$

when n is odd†.

* For instance, making $n=7$, $r=2$, $i=2$,

$$1.2.3.4+1.2.4.5+1.2.5.6+2.3.4.5+2.3.5.6+3.4.5.6=784$$

and is *divisible by* 7.

† By taking $\mu=2$ in the general theorem, it is an easy inference that if we write

$$(\tan^{-1} x)^r = x^r - \frac{A_2 x^{r+2}}{(r+1)(r+2)} + \frac{A_4 x^{r+4}}{(r+1)(r+2)(r+3)(r+4)} \mp \&c.,$$

A_{2i} will be the sum of all the products of $2i$ integers comprised between 1 and $r+2i-1$ that can be formed with combinations of i distinct pairs of consecutive integers; thus, for example, the coefficient of x^{2m} in $(\tan^{-1} x)^2$ ought to be

$$\frac{1}{m}\left(1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2m-1}\right),$$

which may be easily verified.

51.

NOTE ON A DIRECT METHOD OF OBTAINING THE EXPANSION OF THE SINE OR COSINE OF MULTIPLE ARCS IN TERMS OF POWERS OF THE SINES OR COSINES OF THE SIMPLE ARC BY MEANS OF DE MOIVRE'S THEOREM.

[*Quarterly Journal of Mathematics*, IV. (1861), pp. 159—163.]

THE annexed appears to be the most direct and natural method for obtaining the known formulæ for the expansion of the sines and cosines of multiple arcs.

We know by De Moivre's theorem, that

$$\cos 2nx = (\cos x)^{2n} - 2n \, . \, \frac{2n-1}{2} (\sin x)^2 (\cos x)^{2n-2} + \&c.$$

Let $(\sin x)^2 = \gamma$, then

$$\cos 2nx = (1-\gamma)^n - 2n \frac{2n-1}{2} \gamma (1-\gamma)^{n-1}$$
$$+ 2n \frac{2n-1}{2} \, . \, \frac{2n-2}{3} \, . \, \frac{2n-3}{4} \gamma^2 (1-\gamma)^{n-2}, \&c.$$
$$= A_0 - A_1\gamma + A_2\gamma^2 - A_3\gamma^3, \&c.$$

I use $\omega_r \phi x$ to indicate the coefficient of x^r in ϕx expanded in a series of powers of x. We have then

$$A_r = P_0Q_0 + P_1Q_1 + P_2Q_2 + \&c.,$$

where
$$P_0 = \omega_r (1+t)^n = \omega_r (1-t)^{-(n-r+1)} = \omega_{2r} (1-t^2)^{-(n-r+1)},$$
$$P_1 = \omega_{r-1} (1+t)^{n-1} = \omega_{r-1} (1-t)^{-(n-r+1)} = \omega_{2r-2} (1-t^2)^{-(n-r+1)},$$
$$P_2 = \omega_{r-2} (1+t)^{n-2} = \omega_{r-2} (1-t)^{-(n-r+1)} = \omega_{2r-4} (1-t^2)^{-(n-r+1)},$$
$$\&c. \qquad\qquad \&c.$$

$$Q_0 = 1 = \omega_0 (1+t)^{2n},$$
$$Q_1 = 2n \frac{2n-1}{2} = \omega_2 (1+t)^{2n},$$
$$Q_2 = 2n \frac{2n-1}{2} \, . \, \frac{2n-2}{3} \, . \, \frac{2n-3}{4} = \omega_4 (1+t)^{2n},$$
$$\&c. \qquad\qquad \&c.$$

Hence evidently,

$$A_r = \omega_{2r}\{(1-t^2)^{-(n-r+1)} \times (1+t)^{2n}\} = \omega_{2r}\{(1-t)^{-(n-r+1)} \times (1+t)^{n+r-1}\}\,*.$$

To fix the ideas, suppose $r=2$, then

$$A_2 = \omega_4\left\{\begin{array}{l}\left\{1+(n-1)t+\dfrac{(n-1)n}{2}t^2\right.\\ \qquad\left.+\dfrac{(n-1)n(n+1)}{2.3}t^3+\dfrac{(n-1)n(n+1)(n+2)}{2.3.4}t^4\right\}\\ \qquad\times\left\{1+(n+1)t+\dfrac{(n+1)n}{2}t^2\right.\\ \qquad\left.+\dfrac{(n+1)n(n-1)}{2.3}t^3+\dfrac{(n+1)n(n-1)(n-2)}{2.3.4}t^4\right\}\end{array}\right\}$$

$$=\frac{(n+1)n(n-1)(n-2)+(n+2)(n+1)n(n-1)}{1.2.3.4}$$

$$+\frac{(n+1)n(n-1)(n-1)+(n+1)(n+1)n(n-1)}{1.2.3\times 1}$$

$$+\frac{n(n-1)n(n+1)}{1.2\times 1.2}$$

$$=\frac{2n(n-1)n(n+1)}{1.2.3.4}+\frac{2n(n-1)n(n+1)}{1.2.3\times 1}+\frac{n(n-1)n(n+1)}{1.2\times 1.2}$$

$$=\omega_4\left(1+t+\frac{t^2}{1.2}+\frac{t^3}{1.2.3}+\frac{t^4}{1.2.3.4}\right)^2(n^2-1)n^2;$$

and so in general, we shall have

$$A_r = n(n-r+1)(n-r+2)\dots(n+r-1)$$

$$\times\,\omega_{2r}\left(1+t+\frac{t^2}{1.2}+\dots+\frac{t^{2r}}{1.2\dots 2r}\right)^2$$

$$=\omega_{2r}\,e^{2t}\times n\{(n-r+1)\dots(n+r-1)\}$$

$$=\frac{2^{2r}}{1.2.3.4\dots 2r}n^2(n^2-1)(n^2-4)\dots\{n^2-(r-1)^2\},$$

and thus

$$\cos 2nx = 1-\frac{n^2}{1.2}(2\sin x)^2+\frac{n^2(n^2-1)}{1.2.3.4}(2\sin x)^4 \mp \&c.$$

In like manner we have

$$\cos(2n+1)x = \cos x\{(1-\gamma)^n - \tfrac{1}{2}(2n+1)\,2n\gamma(1-\gamma)^{n-1}+\&c.\}$$

$$=\cos x\{B_0 - B_1\gamma + B_2\gamma^2 + \text{etc.}\},$$

where

$$B_r = \omega_{2r}\{(1-t^2)^{-(n-r+1)}(1+t)^{2n+1}\}$$

$$=\omega_{2r}\{(1-t)^{-(n-r+1)}\times(1+t)^{n+r}\};$$

* Note well this simple change in the form of the generating function; in it the point and pith of the method resides.

and making, as before, $r = 2$, we see that

$$B_2 = \omega_4 \left\{ \begin{aligned} &\left\{1 + (n-1)t + \frac{(n-1)n}{2}t^2 \right. \\ &\qquad \left. + \frac{(n-1)n(n+1)}{1.2.3}t^3 + \frac{(n-1)n(n+1)(n+2)}{1.2.3.4}t^4\right\} \\ &\times \left\{1 + (n+2)t + \frac{(n+2)(n+1)}{2}t^2 \right. \\ &\qquad \left. + \frac{(n+2)(n+1)n}{1.2.3}t^3 + \frac{(n+2)(n+1)n(n-1)}{1.2.3.4}t^4\right\} \end{aligned} \right\}$$

$$= \omega_4(e^{2t}) \times (n-1)n(n+1)(n+2),$$

and so in general,

$$B_r = \omega_{2r} e^{2t} \{(n-r+1)(n-r+2) \ldots n \ldots (n+r-1)(n+r)\}$$

$$= \frac{(n-r+1)(n-r+2)\ldots(n+r)}{1.2.3\ldots 2r} 2^{2r},$$

and thus

$$\cos(2n+1)x = \cos x \left\{1 - \frac{n(n+1)}{2}(2\sin x)^2 \right. \\ \left. + \frac{(n-1)n(n+1)(n+2)}{1.2.3.4}(2\sin x)^4 \ldots \&c.\right\}.$$

We might in like manner, and by precisely the same process, obtain the expressions for $\cos 2mx$, $\cos(2m+1)x$ in terms of $\cos x$, and of $\sin 2mx$, $\sin(2m+1)x$ in terms of $\sin x$ or of $\cos x$, but these results may, of course, be most readily found by means of obvious processes of differentiation in respect to the arc and by substitution of the complement for the arc itself in the results already obtained.

It may be worth while to show here how the same elementary theorem as we have employed above, furnishes, *uno ictu*, another important formula connected with multiple arcs:

$$\left(\frac{d}{dx}\right)^{n-1}(1-x^2)^{\frac{2n-1}{2}} = \left(\frac{d}{dx}\right)^{n-1}\left\{(1+x)^{\frac{2n-1}{2}}(1-x)^{\frac{2n-1}{2}}\right\},$$

by Leibnitz's Theorem,

$$= \frac{2n-1}{2} \cdot \frac{2n-3}{2} \cdots \frac{3}{2}\sqrt{(1-x^2)}(1-x)^{n-1}$$

$$-(n-1) \times \frac{2n-1}{2} \cdot \frac{2n-3}{2} \cdots \frac{5}{2} \times \frac{2n-1}{2}\sqrt{(1-x^2)}(1-x)^{n-2}(1+x)$$

$$+(n-1)\frac{n-2}{2} \times \frac{2n-1}{2} \cdot \frac{2n-3}{2} \cdots \frac{7}{2}$$

$$\times \frac{2n-1}{2} \cdot \frac{2n-3}{2}\sqrt{(1-x^2)}(1-x)^{n-3}(1+x)^2 \mp \&c.$$

$$=\frac{1.3.5\ldots(2n-1)}{2^{n-1}}\sqrt{(1-x^2)}$$

$$\times\{(1-x)^{n-1}-A_1(1-x)^{n-2}(1+x)+A_2(1-x)^{n-3}(1+x)^2\mp\&c.\},$$

where
$$A_r=\frac{2n-1}{2}\cdot\frac{2n-3}{2}\ldots\frac{2n-(2r-1)}{2}$$

$$\times\frac{(n-1)(n-2)\ldots(n-r)}{1.2\ldots r}$$

$$\times\frac{2}{3}\cdot\frac{2}{5}\ldots\frac{2}{2r+1}$$

$$=\frac{(2n-1)(2n-2)(2n-3)(2n-4)\ldots\{2n-(2r-1)\}}{2.3.4.5\ldots(2r+1)}$$

$$=\frac{1}{2n}\left[\frac{2n(2n-1)\ldots(2n-2r)}{1.2\ldots(2r+1)}\right].$$

Hence, making $x=\cos 2\phi$,

$$\left(\frac{d}{dx}\right)^{n-1}(1-x^2)^{\frac{2n-1}{2}}=\frac{1.3.5\ldots(2n-1)}{n}$$

$$\times\left\{2n(\sin\phi)^{2n-1}\cos\phi-\frac{2n(2n-1)(2n-2)}{1.2.3}(\sin\phi)^{2n-3}(\cos\phi)^3\pm\&c.\right\}$$

$$=\frac{1.3.5\ldots(2n-1)}{n}\cdot\frac{\{\sin\phi+\sqrt{(-1)}\cos\phi\}^{2n}-\{\sin\phi-\sqrt{(-1)}\cos\phi\}^{2n}}{2\sqrt{(-1)}}$$

$$=(-)^{n-1}\frac{1.3.5\ldots(2n-1)}{n}\sin 2n\phi$$

$$=(-)^{n-1}\frac{1.3.5\ldots(2n-1)}{n}\sin\{n\sin^{-1}\sqrt{(1-x^2)}\},$$

or if we please to pass to the more general form by a linear transformation,

$$\left(\frac{d}{dx}\right)^{n-1}(A+2Bx-Cx^2)^{\frac{2n-1}{2}}$$

$$=(-)^{n-1}\frac{1.3.5\ldots(2n-1)}{n}C^{\frac{2n-3}{2}}\sqrt{(AC+B^2)}\sin n\sin^{-1}\sqrt{\left(\frac{A+2Bx-Cx^2}{A+\frac{B^2}{C}}\right)}.$$

52.

NOTE ON CERTAIN DEFINITE INTEGRALS.

[*Quarterly Journal of Mathematics*, IV. (1861), pp. 319—324.]

IN the *Institutiones Calculi Integralis*, Euler has investigated the value of the definite integral $\int_1^0 \frac{\log x dx}{\sqrt{(1-x^2)}}$, and his mode of statement seems to imply that the result, as well as the demonstration, was his own. How this may be, in fact, I do not pretend to know: in the *Philosophical Magazine** of December, 1860, I have (under another notation) investigated the values of

$$\int_1^0 \frac{\log x dx}{\sqrt{(1-x^2)(1-c^2x^2)}} \text{ and } \int_1^0 \frac{\log\{1+\sqrt{(1-c^2x^2)}\}}{\sqrt{\{(1-x^2)(1-c^2x^2)\}}} dx,$$

and shown them to be equal, but of course with contrary signs, and the former to be expressed† by $\frac{1}{2}\log cF(c) + \frac{1}{4}\pi F(b)$. The relation of which (in regard to the form of the functions of which it is composed) to the integral of its differential without $\log x$ in the numerator is so strikingly analogous to the relation of Euler's more simple integral (namely, $\frac{1}{2}\pi \log 2$) to $\int_1^0 dx \frac{1}{\sqrt{(1-x^2)}}$ as to suggest the existence of some general theorem in which both these results are comprised.

In proving the equality of the two definite integrals in question, a third integral of different form from either came to light. In fact, it is shown in the paper referred to, that

$$\frac{\log\{1+\sqrt{(1-t)}\}}{\sqrt{(1-t)}}$$
$$= \frac{2}{\pi}\int_{\frac{1}{2}\pi}^0 d\phi \,\{\log(\cos\phi) + (\cos\phi)^2 \log(\cos\phi)\, t + (\cos\phi)^4 \log(\cos\phi)\, t^2 + \text{etc.}\},$$

with a tacit supposition that t is contained within the limits (both *inclusive*) $+1$ and -1, within which limits the series which expresses $\frac{\log\{1+\sqrt{(1-t)}\}}{\sqrt{(1-t)}}$ in powers of t remains convergent.

[* p. 213 above.] [† where $b^2 = 1 - c^2$.]

If now we make $1-t=c^2$, and write θ in place of ϕ, we have

$$\int_{\frac{1}{2}\pi}^{0}\frac{\log\cos\theta d\theta}{1-t(\cos\theta)^2}=\int_{\frac{1}{2}\pi}^{0}\frac{\log\cos\theta d\theta}{(\sin\theta)^2+c^2(\cos\theta)^2}=\frac{\pi}{2}\frac{\log(1+c)}{c},$$

with the restriction that c must be positive.

The limits of convergency imply furthermore (as far as the demonstration given is concerned) that c^2 should not be greater than 2, but since neither side of the equation passes through *a critical* phase in any sense (that is, as regards either themselves or their successive differentials) for this, or indeed for any positive value of c, it seems to follow that the equation must continue good from $c=0$ to $c=\infty$, and that the limitation which the cessation of the convergency of the intermediary series might have required to be placed upon the subsistence of the equation may in effect be disregarded. Perhaps also it would be desirable to inquire whether the equality may not continue to subsist for imaginary values of c with a positive real part.

Knowing the value of $\frac{2}{\pi}\int_{\frac{1}{2}\pi}^{0}\frac{d\theta\log\cos\theta}{\{(\sin\theta)^2+c^2(\cos\theta)^2\}^i}$, namely, $\frac{\log(1+c)}{c}$ when $i=1$, we may obviously obtain an expression involving only logarithms and algebraical quantities for all integer values of i; indeed, calling the above integral u_i, we easily obtain the formula of reduction,

$$u_{i+1}-u_i=-\frac{1-c^2}{2ic}\frac{d}{dc}u_i,$$

from which it may readily be shown that u_i will be of the form

$$\left(\frac{A_1}{c}+\frac{A_2}{c^3}+\ldots+\frac{A_i}{c^{2i-1}}\right)\log(1+c)+\frac{B_1}{c}+\frac{B_2}{c^2}+\frac{B_3}{c^3}+\ldots+\frac{B_{2i-2}}{c^{2i-2}},$$

where the two sets of numerators are *constant*, the law of which it may be desirable at some future time to investigate. It should be noticed that although $(1+c)$ appears in the denominator of $\frac{du_i}{dc}$, it does not make its appearance in u_i by reason of the numerator $1-c^2$ in the expression for Δu_i.

The series expressing $\frac{\log\{1+\sqrt{(1-t)}\}}{\sqrt{(1-t)}}$ from which the value of u_1 has been derived, is the following:

$$\frac{\log\{1+\sqrt{(1-t)}\}}{\sqrt{(1-t)}}=\log 2\left(1+\frac{1}{2}t+\frac{1.3}{2.4}t^2+\frac{1.3.5}{2.4.6}t^3+\&c.\right)$$
$$-\left\{\frac{1}{1.2}\frac{1}{2}t+\left(\frac{1}{1.2}+\frac{1}{3.4}\right)\frac{1.3}{2.4}t^2\right.$$
$$\left.+\left(\frac{1}{1.2}+\frac{1}{3.4}+\frac{1}{5.6}\right)\frac{1.3.5}{2.4.6}t^3+\&c.\right\},$$

(see *Philosophical Magazine**, December, 1860, p. 530, where t^2 is used in place of t) but since

$$\frac{d}{dt}\log\{1+\sqrt{(1-t)}\} = -\frac{1}{2\sqrt{(1-t)}}\,\frac{1-\sqrt{(1-t)}}{t} = \frac{1}{2t} - \frac{1}{2t\sqrt{(1-t)}},$$

and consequently

$$\log\{1+\sqrt{(1-t)}\} = \log 2 - \frac{1}{2^2}t - \frac{1.3}{2.4^2}t^2 - \frac{1.3.5}{2.4.6^2}t^3 - \&c.,$$

the more natural (as the more obvious) mode of deducing the coefficients of the powers of t in the expansion of $\dfrac{\log\{1+\sqrt{(1-t)}\}}{\sqrt{(1-t)}}$, would *seem* to be to multiply the series written above by the series

$$1 + \frac{1}{2}t + \frac{1.3}{2.4}t^2 + \frac{1.3.5}{2.4.6}t^3 + \&c.$$

This method of proceeding would however in fact have left obscure the true nature of those coefficients. Let us perform the multiplication indicated; we shall then obtain, by comparison with the expansion already obtained, the following very far from obvious, indeed very unlikely to be suspected identity, which it is desirable to put on record: namely,

$$\frac{1}{2}\cdot\frac{1}{2}\,\frac{1.3.5\ldots(2i-3)}{2.4.6\ldots(2i-2)} + \frac{1}{4}\cdot\frac{1.3}{2.4}\,\frac{1.3.5\ldots(2i-5)}{2.4.6\ldots(2i-4)} + \ldots$$
$$+ \frac{1}{2i-2}\,\frac{1.3.5\ldots(2i-3)}{2.4.6\ldots(2i-2)} + \frac{1}{2i}\,\frac{1.3.5\ldots(2i-1)}{2.4.6\ldots 2i}$$
$$= \frac{1.3.5\ldots(2i-1)}{2.4.6\ldots 2i}\left\{\frac{1}{1.2} + \frac{1}{3.4} + \ldots + \frac{1}{(2i-1)\,2i}\right\}.$$

Thus, for example, if $i=4$,

$$\frac{1}{2^2}\,\frac{1.3.5}{2.4.6} + \frac{1.3}{2.4^2}\,\frac{1.3}{2.4} + \frac{1.3.5}{2.4.6^2}\,\frac{1}{2} + \frac{1.3.5.7}{2.4.6.8^2}$$
$$= \frac{1.3.5.7}{2.4.6.8}\left(\frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + \frac{1}{7.8}\right).$$

The expansion above referred to leads to the value of u_1 through the intervention of the equality [see p. 212 above]

$$\frac{2}{\pi}\int_{\frac{1}{2}\pi}^{0}\log(\cos\theta)(\cos\theta)^{2n}\,d\theta = \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}\sum_{k=\infty}^{k=1}\frac{1}{(2n+2k-1)(2n+2k)},$$

established in the paper referred to: and it is not uninteresting to notice that the above equality enables us to determine the value of the integral on the left-hand side of the equation in a series of descending powers of n, from

[* p. 213 above.]

which doubtless many new conclusions may be deduced. The first term in this descending series being $\frac{1}{4n}$, we are enabled to fix the degree of the integral, when n becomes infinite, for in that case

$$\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n} = \sqrt{\left(\frac{2}{\pi}\cdot\frac{1}{2n}\right)} = \frac{1}{\sqrt{(\pi n)}},$$

so that for $n=\infty$,

$$\int_{\frac{1}{2}\pi}^{0} (\cos\theta)^{2n} \log\cos\theta d\theta = \frac{\pi}{2}\frac{1}{\sqrt{(\pi n)}}\cdot\frac{1}{4n} = \frac{\sqrt{(\pi)}}{(2n^{\frac{1}{2}})^3}.$$

The consideration of the expansion above referred to, namely, of

$$\frac{1}{m+1} - \frac{1}{m+2} + \frac{1}{m+3} + \&c.$$

in a series of descending powers of $\frac{1}{m}$, or which is the same thing, of

$$\frac{\mu}{\mu+1} - \frac{\mu}{2\mu+1} + \frac{\mu}{3\mu+1},$$

in a series of ascending powers of μ, suggests an observation which may appear to amount to a mere futile distinction, but which, closely examined, will be found to have a real signification and importance.

The above series being obtained by means of the equivalence $\Sigma = e^{\frac{d}{dx}} - 1$ will readily be seen to import Bernoulli's numbers in such a manner into the development that the latter would commonly be said (like all the series of the same class) to be absolutely divergent, incapable, that is to say, of constituting an arithmetical equivalent to its generatrix for any value whatever of the variable μ. The distinction I would draw would be to say not that the circle of convergence of μ ceases to exist, but that it becomes indefinitely small, or which is the same thing, the *corona* of convergence for the series treated as a function of m, has its inner radius indefinitely large: so that for $\mu=0$ or $m=\infty$, we may reason, and reason with perfect safety, upon the equality between the generating function and the series as subsisting in an arithmetical sense as regards not only μ or m, but all successive powers of the same. [I mean that supposing

$$\phi\mu = a_0 + a_1\mu + a_2\mu^2 + \ldots,$$

we may affirm not only the equality

$$\phi\mu - a_0 = 0,$$

but also

$$\phi\mu - a_0 = 0, \quad (\phi\mu - a_0)\div\mu = a_1, \quad (\phi\mu - a_0 - a_1\mu)\div\mu^2 = a_2,$$

and so on when $\mu = 0$.]

This fact of the character of arithmetical equivalence within certain limits not absolutely departing even in the case of series considered irreclaimably divergent, may, I think, serve to account, *à priori,* for the phenomenon of many conclusions being capable of being truthfully drawn from reasonings upon them in which they are treated as though they were in an ordinary sense convergent, because, in fact, part of the attributes of ordinary convergency (all such indeed as are not nullified by the radius of convergency becoming infinitely small) must continue to adhere to such series.

The expansion for

$$\frac{1}{2n+1} - \frac{1}{2n+2} + \frac{1}{2n+3} \text{ \&c.},$$

which occurs in the expression for $\int_{\frac{1}{2}\pi}^{0} \log\cos\theta (\cos\theta)^{2n} d\theta$ in a series proceeding according to powers of $\frac{1}{n}$, may be most readily obtained by means of the differences of zero, as follows: calling $2n = x$, we have

$$\frac{1}{x+1} - \frac{1}{x+2} + \frac{1}{x+3} - \frac{1}{x+4} + \text{\&c.}$$

$$= \{(1+\Delta) - (1+\Delta)^2 + (1+\Delta)^3 \ldots\} \frac{1}{x+0}$$

$$= \frac{1+\Delta}{2+\Delta}\left(\frac{1}{x} - \frac{1}{x^2} 0 + \frac{1}{x^3} 0^2 \pm \text{\&c.}\right)$$

$$= \frac{1}{x} - \frac{1}{2+\Delta}\left(\frac{1}{x} - \frac{1}{x^2} 0 + \frac{1}{x^3} 0^2 \pm \text{\&c.}\right),$$

so that the first term will be $\frac{1}{2x}$ or $\frac{1}{4n}$. This might also be shown in a strictly arithmetical method as follows: let

$$s = \frac{1}{(x+1)(x+2)} + \frac{1}{(x+3)(x+4)} + \ldots \textit{ ad inf.} = s_1 + s_2 + s_3 + \ldots,$$

where

$$s_1 = \frac{1}{(x+1)(x+2)} + \frac{1}{(x+3)(x+4)} + \ldots + \frac{1}{(\epsilon x - 1)\,\epsilon x},$$

$$s_2 = \frac{1}{(\epsilon x+1)(\epsilon x+2)} + \frac{1}{(\epsilon x+3)(\epsilon x+4)} + \ldots + \frac{1}{(\epsilon^2 x - 1)\,\epsilon^2 x},$$

$$s_3 = \frac{1}{(\epsilon^2 x+1)(\epsilon^2 x+2)} + \frac{1}{(\epsilon^2 x+3)(\epsilon^2 x+4)} + \ldots + \frac{1}{(\epsilon^3 x - 1)\,\epsilon^3 x},$$

&c. = &c.,

ϵ being any real positive value superior to unity, and x being infinite. Then, observing that each partial series is a mean between the products of the number of terms in it, by the first and last respectively, we have obviously s always intermediate to

$$\frac{\epsilon-1}{2x}+\frac{(\epsilon-1)}{2\epsilon x}+\frac{(\epsilon-1)}{2\epsilon^2 x}+\text{\&c.},$$

and

$$\frac{(\epsilon-1)}{2\epsilon x}+\frac{(\epsilon-1)}{2\epsilon^2 x}+\frac{(\epsilon-1)}{2\epsilon^3 x}+\text{\&c.};$$

that is, between $\frac{\epsilon}{2x}$ and $\frac{1}{2x}$, and consequently, is equal to $\frac{1}{2x}$, as before.

53.

ON THE INVOLUTION OF AXES OF ROTATION.

[*Manchester British Association Report* (1861), p. 12.]

AFTER a brief statement as to the most general mode of representing the displacement of a rigid body in space by means of angular rotations about six distinct axes fixed in position, it was shown that under peculiar conditions the six axes would become insufficient, being, in fact, equivalent to a smaller number, in which case they would be said to form a system in involution. Various constructions for representing such and similar systems were stated, and the remarkable conclusion presented, that the necessary and sufficient condition for three, four, five, or six lines being thus mutually, as it were, implicated and involved, consists in their lying in ruled surfaces of the first, second, third, and fourth orders respectively. The theory of involution originated with Prof. Möbius, by whom, however, it had been left in an imperfect condition. The author referred for further information on the subject to some recent notes by himself in the *Comptes Rendus** of the Academy of Sciences of Paris, and to certain masterly geometrical investigations of M. Chasles and Mr Cayley, to which these had given rise.

[* p. 236 above.]

54.

ADDITION À LA DÉMONSTRATION DU THÉORÈME DE LAGRANGE SUR LES MINIMA D'UNE FONCTION LINÉAIRE À COEFFICIENTS ENTIERS D'UNE QUANTITÉ IRRATIONNELLE, DONNÉE DANS LA SÉANCE PRÉCÉDENTE*.

[*Comptes Rendus de l'Académie des Sciences,* LIV. (1862), pp. 53—55.]

On peut à juste titre élever quelque objection contre la forme donnée au théorème cité en tant que j'ai posé comme *criterium* des réduites $\frac{p}{q}$ de l'irrationnelle ν, la condition que la valeur de $p - q\nu$ restera plus petite que toute valeur qui résulte de la diminution ou de p, ou de q, ou de p et q simultanément dans cette fonction, tandis que le *criterium* de Lagrange ne considère que l'effet de la substitution simultanée des nombres inférieurs à p et à q. On remédie à cet inconvénient et en même temps on simplifie la démonstration du théorème dont il est question en donnant un peu plus d'extension à la conclusion nommée A dans la Note précédente.

Dans l'équation (3), c'est-à-dire,

$$D\Delta' = (-1)^i (\theta - s\theta + r - ks),$$

si l'on pose $$s = l + 1, \quad r = ks + 1 = kl + k + 1$$

(de sorte que $p - \lambda$, $q - \mu$ deviennent simultanément $-p'$, $-q'$), on aura

$$s\theta = (1 + l)\,\theta \begin{matrix} > 1 \\ < 1 + \theta \end{matrix} \quad \text{et} \quad \theta - s\theta + (r - ks) \begin{matrix} < \theta \\ > \theta' \end{matrix},$$

donc $\Delta'^2 < \Delta^2$, c'est-à-dire que les minima $p - q\nu$, $p' - q'\nu$, etc., vont toujours en diminuant; mais si, s restant égale à $l + 1$, r n'est pas prise égale à $ks + 1$, λ et μ tous les deux excéderont $p + p'$, $q + q'$ respectivement. Tel est donc l'effet des conditions caractéristiques du système p, q; pour qu'il soit possible que Δ'^2 soit moindre que Δ^2, $(p - \lambda)^2$ ne peut pas devenir p'^2 sans qu'en même temps $(q - p)^2$ devienne q'^2 et réciproquement.

[* p. 250 above.]

Conséquemment à la place de ladite conclusion A, on peut substituer l'énoncé suivant, c'est-à-dire $\frac{p}{q}$ *étant une réduite quelconque de* ν, $p-q\nu$ *s'augmentera en substituant pour* p *un nombre quelconque moindre que* p' *ou pour* q *un nombre moindre que* q', *pourvu qu'on ne substitue pas en même temps* p' *pour* p *et* q' *pour* q.

Avec cet énoncé, on peut se passer tout à fait de la conclusion B. La preuve que la condition de Lagrange est nécessaire découle et avec surabondance de cet énoncé: cela saute aux yeux; et quant à la suffisance ou *criterium*, on n'a qu'à remarquer que si $\frac{a}{b}$ n'est pas une réduite de ν, on peut prendre

$$a > p_e,\ a \mathrel{\overline{\gtrless}} p_{e+1};\quad b > q_i,\ b \mathrel{\overline{\gtrless}} q_{i+1},$$

et alors
$$p_e - q_e\nu \text{ et } p_i - q_i\nu,$$
seront tous les deux $< a - b\nu$. De plus, on aura

$$p_e < a \text{ et } q_e < b,$$

ou bien
$$p_i < a \text{ et } q_i < b\ *;$$

donc, dans tous les cas, $a - b\nu$ diminuera quand on diminuera dans une manière convenable a et b simultanément: ce qui démontre la différence du *criterium* dont il a été question.

* On n'a pas besoin de dire que rien n'empêche que e ne soit égal à i; mais dans ce cas, comme on ne peut pas avoir simultanément $a = p_{e+1}$, $b = q_{e+1}$, la conclusion du texte reste bonne.

55.

ON THE SOLUTION OF THE LINEAR EQUATION OF FINITE DIFFERENCES IN ITS MOST GENERAL FORM.

[*Cambridge British Association Report* (1862), p. 188.]

THE author exhibited (and illustrated with examples) a simple and readily applied method of obtaining the general term of, and consequently the complete, solution of an equation of finite differences with any number of independent variables, a question which, although touched upon by Libri and laboriously investigated by Binet, had hitherto, to the best of his knowledge, remained unsolved even in the case of an equation with but one independent variable with non-constant coefficients; when the coefficients are supposed constant, the well-known solution flows as an immediate corollary from the author's general form. Essentially the method depends upon the adoption of a natural principle of notation for the given coefficients, according to which each coefficient is to be denoted by a *twofold* group of indices, the number of the double indices in a group being equal to the number of independent variables in the given equation. Thus, supposing $u_{m,n,p\ldots}$ to be expressible, by means of the given general equation, as a sum of u's with inferior indices, the coefficient of $u_{\mu,\nu,\pi\ldots}$ in that sum must be denoted by the double index group

$$\begin{bmatrix} m, & n, & p\ldots \\ \mu, & \nu, & \pi\ldots \end{bmatrix}.$$

The process for obtaining the general term in $u_{x,y,z\ldots}$ is then shown to be reducible virtually to the problem of effecting the simultaneous *decomposition* of the integer variables $x, y, z\ldots$ into *parts* in every possible manner and order of relative arrangement, the magnitudes of such parts being limited by the degree or degrees of the given equation in respect of these variables. The collective value of the terms thus obtained constituting the complete solution may be termed, in the author's nomenclature, a hyper-cumulant, whose properties and their applications remain to be studied, as those of the elementary kinds of common cumulants have been, to a considerable extent, in the ordinary theory of continued fractions. The first stage in the process of constructing the terms of a general cumulant or general hyper-cumulant is almost identical with that of finding the coefficients in the expansion of a power of a polynomial function of one or several variables, differing from it indeed only in the circumstance that permutations which lead to repetitions in the latter case, represent distinct values in the former.

56.

SUR UNE CLASSE NOUVELLE D'ÉQUATIONS DIFFÉRENTIELLES ET D'ÉQUATIONS AUX DIFFÉRENCES FINIES D'UNE FORME INTÉGRABLE.

[*Comptes Rendus de l'Académie des Sciences*, LIV. (1862), pp. 129—132.]

COMMENÇONS par le cas des différences finies. Représentons par Δ_x le déterminant

$$\begin{vmatrix} u_x, & u_{x+1}, & u_{x+2} & \dots u_{x+i-1} \\ u_{x+1}, & u_{x+2}, & u_{x+3} & \dots u_{x+i} \\ u_{x+2}, & u_{x+3}, & u_{x+4} & \dots u_{x+i+1} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ u_{x+i-1}, & u_{x+i}, & u_{x+i+1} & \dots u_{x+2i-2} \end{vmatrix},$$

et considérons l'équation

$$\Delta_x = C, \dots, \tag{1}$$

ce qui au fond est aussi général que si nous écrivions $\Delta_x = C\gamma^x$.

Je dis que l'équation (1) pourra être satisfaite par la même intégrale que celle qui satisfait à l'équation

$$u_x - p_1 u_{x+1} + p_2 u_{x+2} \dots (-1)^{i-1} p_{i-1} u_{x+i-1} + (-1)^i u_{x+i} = 0, \tag{2}$$

$p_1, p_2, \dots, p_{i-1}$, étant des constantes. Car si cette dernière équation a lieu, on peut dans la première ligne du déterminant substituer à

$$u_x,\ u_{x+1} \dots u_{x+i-1}$$

les quantités

$$(-1)^{i-1} u_{x+i},\ (-1)^{i-1} u_{x+i+1} \dots (-1)^{i-1} u_{x+2i-1},$$

sans changer la *valeur* de ce déterminant.

Donc on voit immédiatement que Δ_x devient égal à Δ_{x+1}, c'est-à-dire Δ_x sera constant ; donc l'intégrale de $\Delta_x = C$ sera

$$u_x = a_1 \alpha_1^x + a_2 \alpha_2^x + \dots + a_i \alpha_i^x, \tag{3}$$

avec la condition $\alpha_1\alpha_2 \dots \alpha_i = 1$. Cette condition est une conséquence de la forme du dernier coefficient, $(-1)^i$, dans l'équation (2); de plus une autre condition se présente à cause de la valeur spéciale qu'il faut attribuer à la constante C dans l'équation donnée.

Pour obtenir cette dernière condition nous pouvons considérer les a et les α comme étant données et C comme une fonction de ces quantités. Or en faisant un quelconque des a égal à zéro, le degré de l'équation (2) s'abaisse d'une unité, c'est-à-dire les i fonctions $u_x, u_{x+1}, \dots, u_{x+i-1}$ seront liées entre elles par une équation linéaire et conséquemment le déterminant Δ_x s'évanouira. Donc C contient le produit $a_1a_2 \dots a_i$ comme facteur. Mais on trouve aussi, en prenant $x=0$, C égal au déterminant à i lignes

$$\begin{vmatrix} \Sigma a, & \Sigma a\alpha & \dots \Sigma a\alpha^{i-1} \\ \Sigma a\alpha, & \Sigma a\alpha^2 & \dots \Sigma a\alpha^{i} \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ \Sigma a\alpha^{i-1}, & \Sigma a\alpha^{i} & \dots \Sigma a\alpha^{2i-2} \end{vmatrix}$$

qui est du degré i par rapport aux quantités a.

Donc $$C = a_1a_2 \dots a_i\, F(\alpha_1, \alpha_2 \dots \alpha_i).$$

Pour déterminer F, on n'a qu'à supposer

$$a_1 = a_2 = \dots = a_i = 1,$$

et on obtient immédiatement par un théorème bien connu

$$F = (\alpha_1-\alpha_2)^2(\alpha_1-\alpha_3)^2(\alpha_2-\alpha_3)^2 \dots (\alpha_{i-1}-\alpha_i)^2.$$

Donc finalement on aura pour l'intégrale complète de l'équation (1), qui est de l'ordre $(2i-2)$, le système d'équations

avec les conditions
$$\left.\begin{aligned} u_x = a_1\alpha_1^x + a_2\alpha_2^x + \dots + a_i\alpha_i^x, \\ \alpha_1\alpha_2 \dots \alpha_i = 1, \\ a_1a_2 \dots a_i\,[(\alpha_1-\alpha_2)^2 \dots (\alpha_{i-1}-\alpha_i)^2] = C, \end{aligned}\right\} \qquad (4)$$

système qui contient $(2i-2)$ constantes, le nombre qu'on doit avoir.

On peut appliquer cette même méthode à un système d'équations beaucoup plus général. Car si on désigne par $P_1, P_2, \dots, P_{i-1}$ les fonctions algébriques de $u_x, u_{x+1}, \dots, u_{x+2i-2}$ qui satisfont au système simultané des $(i-1)$ équations

$$\begin{aligned} &u_x - P_1u_{x+1} + P_2u_{x+2} \dots - (-1)^i P_{i-1}u_{x+i-1} + (-1)^i u_{x+i} = 0, \\ &u_{x+1} - P_1u_{x+2} + P_2u_{x+3} \dots - (-1)^i P_{i-1}u_{x+i} + (-1)^i u_{x+i+1} = 0, \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &u_{x+i-2} - P_1u_{x+i-1} + P_2u_{x+i} \dots - (-1)^i P_{i-1}u_{x+2i-3} + (-1)^i u_{x+2i-2} = 0, \end{aligned}$$

et si, en conservant à Δ_x la même valeur que dans l'équation (1), on écrit

$$\Delta_x + \phi(P_1, P_2, \ldots, P_{i-1}) = 0, \tag{5}$$

il est évident qu'en faisant

$$u_x - p_1 u_{x+1} + p_2 u_{x+2} \ldots - (-1)^i p_{i-1} u_{x+i-1} + (-1)^i u_{x+i} = 0,$$

Δ_x sera égal à Δ_{x+1} et ϕ sera toujours constant, car on aura

$$P_1 = p_1,\ P_2 = p_2,\ \ldots,\ P_{i-1} = p_{i-1}.$$

Donc l'équation (5) sera satisfaite par l'intégrale

$$\left.\begin{array}{c} u_x = a_1\alpha_1^x + a_2\alpha_2^x + \ldots + a_i\alpha_i^x, \\ \text{avec les conditions} \qquad \alpha_1\alpha_2\ldots\alpha_i = 1, \\ (a_1a_2\ldots a_i)\{(\alpha_1-\alpha_2)^2(\alpha_1-\alpha_3)^2(\alpha_2-\alpha_3)^2\ldots(\alpha_{i-1}-\alpha_i)^2\} \\ + \phi(\Sigma\alpha_1, \Sigma\alpha_1\alpha_2, \ldots, \Sigma\alpha_i\ldots\alpha_{i-1}) = 0. \end{array}\right\} \tag{6}$$

Passons au cas de la forme analogue des équations différentielles. En supposant y une fonction de x, j'écrirai $\dfrac{d^iy}{dx^i} = y_i$, et je nommerai $D_x{}^iy$ le déterminant

$$\left|\begin{array}{l} y,\quad y_1,\ y_2\ \cdots y_{i-1} \\ y_1,\ \ y_2,\ y_3\ \cdots y_i \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ y_{i-1},\ y_i,\ y_{i+1}\cdots y_{2i-2} \end{array}\right|.$$

Considérons d'abord l'équation

$$D_x{}^iy = C. \tag{7}$$

Sans prendre la peine de passer par les moyens connus du cas des différences finies à des différences infiniment petites, il suffit de faire le rapprochement de la valeur de $\dfrac{u_{x+1}}{u_x}$ quand $u_x = \alpha^x$ avec celle de $\dfrac{d_xy}{y}$ quand $y = e^{\alpha x}$ pour conclure immédiatement de la forme de l'intégrale (1) celle de l'équation (7) qui sera évidemment

$$\left.\begin{array}{c} y = a_1e^{\alpha_1x} + a_2e^{\alpha_2x} + \ldots + a_ie^{\alpha_ix} \\ \text{avec les conditions} \qquad \alpha_1 + \alpha_2 + \ldots + \alpha_i = 0, \\ a_1a_2\ldots a_i(\alpha_1-\alpha_2)^2(\alpha_1-\alpha_3)^2(\alpha_2-\alpha_3)^2\ldots(\alpha_{i-1}-\alpha_i)^2 = C. \end{array}\right\} \tag{8}$$

Avant de considérer quelques modifications très-intéressantes de cette équation, il sera utile d'établir un théorème élémentaire sur les rapports des formes consécutives $D_x{}^iy$ entre elles.

Pour fixer les idées, bornons-nous pour le moment à la considération du déterminant

$$\begin{vmatrix} y, & y_1, & y_2, & y_3 \\ y_1, & y_2, & y_3, & y_4 \\ y_2, & y_3, & y_4, & y_5 \\ y_3, & y_4, & y_5, & y_6 \end{vmatrix},$$

c'est-à-dire $D_x^4 y$, et des déterminants *mineurs* qu'il renferme.

Posons

$$D_x^3 y = \begin{vmatrix} y, & y_1, & y_2 \\ y_1, & y_2, & y_3 \\ y_2, & y_3, & y_4 \end{vmatrix}.$$

En différentiant les quantités qui entrent dans ce déterminant *ligne* sur *ligne*, on formera trois déterminants nouveaux dont tous s'évanouiront identiquement à cause de l'égalité de deux lignes (terme à terme) qui en résultera, sauf toutefois le dernier qui sera

$$\begin{vmatrix} y, & y_1, & y_2 \\ y_1, & y_2, & y_3 \\ y_3, & y_4, & y_5 \end{vmatrix}$$

et qui exprimera conséquemment la valeur de $\frac{d}{dx}(D_x^3 y)$.

De même en différentiant ce dernier déterminant (*colonne à colonne*), on obtiendra

$$\begin{vmatrix} y, & y_1, & y_3 \\ y_1, & y_2, & y_4 \\ y_3, & y_4, & y_6 \end{vmatrix}$$

comme la valeur de $\frac{d^2}{dx^2}(D_x^3 y)$.

On remarquera que tous les termes du nouveau déterminant

$$\begin{vmatrix} D_x^3 y, & \frac{d}{dx}(D_x^3 y) \\ \frac{d}{dx}(D_x^3 y), & \frac{d^2}{dx^2}(D_x^3 y) \end{vmatrix}$$

seront des déterminants mineurs de $D_x^4 y$, et par un théorème tres-connu on conclut que ce déterminant composé sera égal au produit $D_x^2 y \times D_x^4 y$, c'est-à-dire

$$D_x^2 y \times D_x^4 y = D_x^2 (D_x^3 y),$$

et dans la même manière on peut établir l'équation générale qui lie ensemble trois termes consécutifs quelconques de la série

$$D^1,\ D^2,\ D^3,\ D^4,\ D^5 \ldots,$$

c'est-à-dire

$$D_x{}^{i-1}y \times D_x{}^{i+1}y = D_x{}^2\,(D_x{}^i y). \tag{9}$$

Avec l'aide de cette équation on parvient facilement à l'intégration d'une classe très-intéressante d'équations différentielles du quatrième ordre, parmi lesquelles on peut distinguer les équations

$$D_x{}^3 y = Cy^3, \quad (D_x{}^3 y)^2 = C\,(D_x{}^2 y)^3,$$

lesquelles ne sont que deux cas particuliers d'équations qu'on peut intégrer par le moyen des fonctions elliptiques inverses.

57.

ÉQUATIONS DIFFÉRENTIELLES. ADDITION À UNE NOTE INSÉRÉE DANS LE COMPTE RENDU DE LA SÉANCE PRÉCÉDENTE SUR UNE FORME NOUVELLE D'ÉQUATIONS DIFFÉRENTIELLES ET INTÉGRABLES.

[*Comptes Rendus de l'Académie des Sciences,* LIV. (1862), pp. 170—174.]

En se rappelant la forme du coefficient différentiel de $D_x{}^i y$ par rapport à x que j'ai donnée dans la Note indiquée ci-dessus*, il est aisé de voir qu'on peut arriver, par une méthode directe, à la solution de l'équation

$$D_x{}^i y = Le^{\lambda x} \tag{1}$$

en se servant de l'équation auxiliaire

$$y_i - \lambda y_{i-1} + \mu y_{i-2} + \nu y_{i-3} \ldots + \omega y = 0, \tag{2}$$

μ, ν, ..., ω étant des constantes arbitraires.

En prenant, par exemple, $i = 3$, on voit qu'en supposant l'équation $y_3 - \lambda y_2 + \mu y_1 + \nu y = 0$ satisfaite, le déterminant

$$\begin{vmatrix} y, & y_1, & y_2 \\ y_1, & y_2, & y_3 \\ y_2, & y_3, & y_4 \end{vmatrix} \text{ deviendra égal à } \frac{1}{\lambda}\begin{vmatrix} y, & y_1, & y_2 \\ y_1, & y_2, & y_3 \\ y_3, & y_4, & y_5 \end{vmatrix};$$

car, au lieu de y_2, y_3, y_4, dans la dernière ligne du premier de ces déterminants, on peut alors substituer respectivement $\frac{1}{\lambda}y_3$, $\frac{1}{\lambda}y_4$, $\frac{1}{\lambda}y_5$, tout simplement.

Donc $\frac{d}{dx}(D_x{}^3 y)$ devient égal à $\lambda D_x{}^3 y$, et l'équation $D_x{}^3 y = Le^{\lambda x}$ peut être satisfaite par l'intégrale de l'équation (2) en déterminant convenablement

[* p. 310 above.]

les rapports entre les constantes arbitraires qui y figurent. De même on voit, en général, que l'intégrale de $D_x{}^i y = Le^{\lambda x}$ sera

$$y = a_1 e^{\alpha_1 x} + a_2 e^{\alpha_2 x} + \ldots + a_i e^{\alpha_i x} \tag{3}$$

avec les conditions
$$\alpha_1 + \alpha_2 + \ldots + \alpha_i = \lambda, \tag{4}$$

$$a_1 a_2 \ldots a_i (\alpha_1 - \alpha_2)^2 (\alpha_1 - \alpha_3)^2 \ldots (\alpha_{i-1} - \alpha_i)^2 = L. \tag{5}$$

Rien n'empêche de prendre un nombre quelconque des α et d'en faire différer les valeurs infiniment peu les unes des autres; on peut aussi, en général, former plusieurs groupes distincts de cette espèce. En agissant de cette façon, on arrive, par une analyse facile à retrouver et par le moyen d'un lemme que j'exposerai tout à l'heure, à des formes spéciales (pour ne pas dire singulières) de l'équation (3), dont voici le type le plus général:

$$y = Xe^{\alpha x} + X_1 e^{\alpha_1 x} + \ldots \tag{6}$$

où
$$X = ax^{n-1} + bx^{n-2} \ldots + l,$$

$$X_1 = a_1 x^{n_1 - 1} + b_1 x^{n_1 - 2} \ldots + l_1,$$

....

avec les conditions suivantes:

$$\Sigma n = i, \tag{7}$$

$$\Sigma \alpha n = \lambda, \tag{8}$$

$$\Gamma n : a^n . \Gamma n_1 . a_1^{n_1} . \Gamma n_2 . a_2^{n_2} \ldots (\alpha - \alpha_1)^{2nn_1} (\alpha - \alpha_2)^{2nn_2} (\alpha_1 - \alpha_2)^{2n_1 n_2} \ldots = L. \tag{9}$$

Le nombre total de ces formes (l'intégrale générale y comprise) sera le nombre des partitions indéfinies du nombre i; le nombre de ces formes qui ne doivent contenir qu'un nombre donné $2i - 2 - \omega$ de constantes arbitraires sera le nombre de partitions de i en $i - \omega$ parties, lequel, quand ω n'excède pas $\frac{i}{2}$, est identique avec le nombre des partitions indéfinies de ω.

Le lemme dont il a été question est qui sert pour obtenir l'équation (9) est le suivant:

Si on a un système de n équations de la forme

$$\lambda_1{}^\omega x_1 + \lambda_2{}^\omega x_2 + \ldots + \lambda_n{}^\omega x_n = p_\omega \quad \text{où} \quad \omega = 0, 1, 2, \ldots, n-1, \tag{10}$$

alors, quand les λ deviennent tous infiniment petits, la fonction

$$(\lambda_1 - \lambda_2)^2 (\lambda_1 - \lambda_3)^2 \ldots (\lambda_{n-1} - \lambda_n)^2 x_1 x_2 \ldots x_n,$$

reste finie et aura pour limite une valeur indépendante de $p_0, p_1, \ldots, p_{n-2}$, savoir:

$$(-)^{n\frac{n-2}{2}} p^n_{n-1}.$$

J'ajoute que la même méthode suffit également pour trouver et l'intégrale générale et les intégrales spéciales d'une équation d'une forme plus générale, à savoir l'équation

$$D_x{}^i y = \phi(P_1, P_2, \ldots, P_{i-1})\, e^{\lambda x},$$

ϕ exprimant une forme de fonction quelconque donnée, et les P étant les fonctions algébriques de $y, y_1, y_2, \ldots, y_{2i-2}$, qui satisfont identiquement aux $(i-1)$ équations

$$y_{i+\omega-2}P_1 + y_{i+\omega-3}P_2 + \ldots + y_\omega P_{i-1} = \lambda y_{i+\omega-1} - y_{i+\omega}, \tag{11}$$

où $\qquad \omega = 0,\ 1,\ 2,\ \ldots,\ i-2.$

Sans insister là-dessus, je passe à la considération plus intéressante de certaines équations différentielles qu'on peut immédiatement réduire à une forme intégrable par le moyen de la formule établie à la fin de la Note précédente, c'est-à-dire

$$D_x{}^{i-1}y \,.\, D_x{}^{i+1}y = D_x{}^i y \,.\left(\frac{d}{dx}\right)^2 D_x{}^i y - \left(\frac{d}{dx} D_x{}^i y\right)^2. \tag{12}$$

Pour plus de brièveté, je me servirai du symbole λ pour exprimer $\left(\frac{d}{dx}\right)^2 \log$, de sorte que la loi d'opération de λ sur des produits sera identique avec celle de log, c'est-à-dire qu'on aura

$$\lambda(uvw \ldots) = \lambda u + \lambda v + \lambda w + \ldots.$$

Je me servirai aussi du symbole D_i pour exprimer ce que j'ai précédemment désigné par $D_x{}^i y$; on aura ainsi par la formule (12)

$$yD_3 = (D_2)^2 \lambda D_2, \quad D_2 = y^2 \lambda y;$$

donc $$D_3 = y^3(\lambda y)^2 \lambda(y^2\lambda y) = y^3(\lambda y)^2(2\lambda y + \lambda^2 y).$$

Ainsi on voit que la solution de l'équation $D_3 = y^3\phi\left(\frac{D_2}{y^2}\right)$ dépendra de celle de l'équation

$$(\lambda y)^2(\lambda^2 y + 2\lambda y) = \phi(\lambda y); \tag{13}$$

ou bien (en mettant $\lambda y = u$, ce qui donne $y = e^{\int\int u dx^2}$) pourra être ramenée à celle de l'équation

$$u^2\left[\frac{d}{dx}\left(\frac{u'}{u}\right) + 2u\right] = \phi u. \tag{14}$$

Cette dernière équation, on le voit immédiatement, aura pour intégrale

$$x = \int \frac{du}{\sqrt{\left\{2u^2 \int du \left(\frac{\phi u}{u^3}\right) - 4u^3\right\}}}. \tag{15}$$

Ainsi on voit que si

$$\phi u = A + Bu + Du^3 + Eu^4, \tag{16}$$

x deviendra une fonction elliptique de u, et de même que si

$$\phi u = \alpha u + \beta u^{\frac{3}{2}} + \delta u^{\frac{5}{2}} + \epsilon u^3, \tag{17}$$

x deviendra une fonction elliptique de $u^{\frac{1}{2}}$.

Ainsi l'équation

$$D_3 = y^3 \phi\left(\frac{D_2}{y^2}\right) \tag{18}$$

sera intégrable par le moyen de fonctions elliptiques, toutes les fois que l'une ou l'autre des suppositions (16), (17) aura lieu. En prenant successivement $\phi u = A$, $\phi u = A^{\frac{1}{2}} u^{\frac{3}{2}}$, on obtient deux équations que je signalerai (quoiqu'elles ne soient que des cas particuliers) à cause de leur grande simplicité; ce sont les équations

$$D_3 = Ay^3, \tag{19}$$

$$D_3^2 = AD_2^3, \tag{20}$$

où $\quad D_2 = yy'' - y'^2$ et $D_3 = yy''y^{\text{IV}} - yy'''^2 - y'^2 y^{\text{IV}} - y''^3 + 2y'y''y'''$.

Considérons d'abord l'équation (19); en faisant $\xi = A^{\frac{1}{6}}x$, elle prend la forme

$$'D_3 = y^3 \tag{21}$$

($'D_3$ ne différant de D_3 qu'en ce que ξ y remplace x). Alors par la formule (15) nous aurons $y = e^{\iint u dx^2}$;

$$\xi + \gamma = \int \frac{du}{\sqrt{(-1 + \lambda_1 u^2 - 4u^3)}} = -\frac{1}{2^{\frac{2}{3}}} \int \frac{dv}{\sqrt{(-1 + \lambda v^2 + v^3)}}$$

$$= -\frac{1}{2^{\frac{2}{3}}} \int \frac{dv}{\sqrt{\left\{(2c + 4c^2 v + v^2)\left(-\frac{1}{2c} + v\right)\right\}}},$$

c et γ étant des constantes arbitraires. En écrivant

$$w = 2c^2 + v = 2c^2 - 2^{\frac{2}{3}}u, \quad C = -\sqrt{(4c^4 - 2c)}, \quad C_1 = 2c^2 + \frac{1}{2c},$$

on obtient immédiatement (en se servant de la substitution $w = C\cos 2\theta$)

$$\frac{C - 2c^2 + 2^{\frac{2}{3}}u}{2C} = \sin^2 \operatorname{am}\left(2^{-\frac{1}{3}}\sqrt{(C + C_1)}\,(\xi + \gamma),\ \sqrt{\frac{2C}{C + C_1}}\right); \tag{22}$$

ce qui donne

$$\log y = 2^{\frac{1}{3}} C \iint dx^2 \left[\sin^2 \operatorname{am}\left\{2^{-\frac{1}{3}}\sqrt{\left(\frac{C + C_1}{A^{\frac{1}{3}}}\right)}(x + \gamma'),\ \sqrt{\frac{2C}{C + C_1}}\right\}\right]$$

$$+ (2c^2 - C)\frac{x^2}{2^{\frac{8}{3}}}.$$

Cette équation est l'intégrale complète de l'équation donnée $D_3 = Ay^3$.

Maintenant considérons l'équation (20), c'est-à-dire $D_3{}^2 = A D_2{}^3$. La formule (15) donne

$$y = e^{\int\int dx^2 u}\,; \quad x + \gamma = \int \frac{du}{\sqrt{(-4A^{\frac{1}{2}} u^{\frac{3}{2}} + \lambda_1 u^2 - 4u^3)}},$$

ce qui, en mettant $u = \dfrac{1}{v^2}$, devient

$$x + \gamma = \int \frac{-dv}{\sqrt{(-A^{\frac{1}{2}} v^3 + \lambda v^2 - 1)}}. \tag{23}$$

En supposant $A = 1$ la somme en (23) devient identique avec celle qui a été trouvée plus haut en déterminant la valeur de $\xi + \gamma$, d'où il sera facile de conclure la valeur de log y qui contiendra la reciproque d'une double somme du carré d'une fonction linéaire du $\sin^2 \mathrm{am}$ d'une fonction linéaire de x, et quant au cas général où A a une valeur quelconque, il se réduit au cas précédent en écrivant $\xi = A^{\frac{1}{6}} x$.

Il existe encore une infinité d'équations d'une forme symétrique et analogue à celle des équations (19) et (20) auxquelles on peut appliquer une pareille méthode, non pas, il est vrai en général, pour les intégrer complétement, mais au moins pour en abaisser le degré de 4 unités. C'est toujours l'équation fondamentale (12) qui sert à effectuer cette réduction.

58.

ON THE INTEGRAL OF THE GENERAL EQUATION IN DIFFERENCES.

[*Philosophical Magazine*, XXIV. (1862), pp. 436—441.]

THE most general form which can be given to a linear equation in differences may easily be seen to be reducible to the following,

$$a_x u_x + b_x u_{x-1} + c_x u_{x-2} + \&\text{c. } ad\ lib. = 0,$$

with the initial conditions

$$u_0 = 1, \quad u_{-e} = 0.$$

Consequently to find u_n, or let us rather say to find

$$(-)^n a_1 a_2 \ldots a_n u_n,$$

is really the problem of finding the value of a determinant belonging to a matrix of n^2 terms, whereof all the places below the diagonal line, with the exception of those in the oblique line immediately under the diagonal, are occupied by zeros, but of which all the other places are or may be occupied by finite quantities. For instance, supposing n to be 4, such a determinant would be

$$\begin{vmatrix} b_4, & c_4, & d_4, & e_4 \\ a_3, & b_3, & c_3, & d_3 \\ 0, & a_2, & b_2, & c_2 \\ 0, & 0, & a_1, & b_1 \end{vmatrix}.$$

Let us for a moment consider more particularly this determinant. If, using double indices to denote each coefficient, we were to write the above according to the usual method of notation as below,

$$\begin{vmatrix} 4.4, & 4.3, & 4.2, & 4.1 \\ 3.4, & 3.3, & 3.2, & 3.1 \\ 0, & 2.3, & 2.2, & 2.1 \\ 0, & 0, & 1.2, & 1.1 \end{vmatrix}$$

the law of formation of the general term would be very far from becoming evident on a cursory inspection; but a slight change, suggested by the very

system of equations in which the determinant originates, makes the law at once obvious. Nothing is more natural than that we should use $r.s$ or $s.r$, where $r > s$, to denote the coefficient of u_s in the equation of which r is the highest subindex of u; with this modification, the above determinant changes into the following:—

$$\begin{vmatrix} 4.3, & 4.2, & 4.1, & 4.0 \\ 3.3, & 3.2, & 3.1, & 3.0 \\ . & 2.2, & 2.1, & 2.0 \\ . & . & 1.1, & 1.0 \end{vmatrix}$$

(the terms with equal indices appearing not now in the diagonal, but in the oblique line below it). With this notation it becomes apparent (and the *reason* of the rule may be deduced by the most simple reasoning from following the course of the successive substitutions in the system of equations giving rise to the determinant) that to find the general term we must write all the descending series of integers which can be formed, beginning with 4 and ending with zero, namely,

43210
4310
4210
4320
430
420
410
40

and read them off respectively into products as below:—

$$\begin{gathered} 4.3 \times 3.2 \times 2.1 \times 1.0 \\ (4.3 \times 3.1 \times 1.0) \times (-2.2) \\ (4.2 \times 2.1 \times 1.0) \times (-3.3) \\ (4.3 \times 3.2 \times 2.0) \times (-1.1) \\ (4.3 \times 3.0) \times (2.2 \times 1.1) \\ (4.2 \times 2.0) \times (3.3 \times 1.1) \\ (4.1 \times 1.0) \times (2.2 \times 3.3) \\ (4.0) \times (1.1 \times 2.2 \times 3.3). \end{gathered}$$

The sum of the above terms is the value of the determinant in question. And so in general, if we define u_n by means of the equation

$$(n.n)\, u_n + (n.n-1)\, u_{n-1} + (n.n-2)\, u_{n-2} + \ldots = 0\,;$$

with the initial conditions as above stated, the value of u_n to a factor *près* will be represented by

$$\Sigma\, (n,\, n_1,\, n_2,\, \ldots\, n_\omega,\, 0),$$

where $n > n_1 > n_2 \ldots > n_\omega$ $[\omega = 0, 1, 2, \ldots (n-1)]$ and $(n, n_1, n_2, \ldots n_\omega, 0)$ is to be interpreted as meaning

$$M \times n \,.\, n_1 \times n_1 \,.\, n_2 \times \ldots \times n_\omega \,.\, 0,$$

where to find M we write the complementary integers

$$m_1,\ m_2,\ m_3,\ \ldots\ m_{n-\omega+1},$$

which together with $n_1, n_2, \ldots n_\omega$ make up the complete tally of all the integers from 1 to $(n-1)$, and then write

$$M = (-)^{n-\omega+1} (m_1 \,.\, m_1) \,.\, (m_2 \,.\, m_2) \ldots (m_{n-\omega+1} \,.\, m_{n-\omega+1}).$$

In order to form by an exhaustive process all the descending series above described, we may if we please consider the differences of the terms of any such series, and write

$$\delta = n - n_1,\ \ \delta_1 = n_1 - n_2 \ldots \delta_\omega = n_\omega,$$

we have then

$$\delta + \delta_1 + \delta_2 + \ldots + \delta_\omega = n.$$

So that the question is reducible to that of finding all the partitions of n, and of permuting in every possible manner the terms in each such system of partitions; for it is obvious that in general the value of $(n, n_1, n_2, \ldots n_\omega, 0)$ depends not only on the magnitudes, but on the order of sequence of $\delta, \delta_1, \delta_2, \ldots \delta_\omega$.

If we suppose that the order of the differences is limited, as, for example, that the equation is of the ith order, then any such coefficient as $r \,.\, s$ is to be considered as zero when $r \sim s > i$, and consequently the partitions of n are to be limited to parts none greater than i. Moreover, if in such case the coefficients become constant, so that $r \,.\, s = \phi(r - s)$, it is apparent that the order of the arrangement of $\delta_1, \delta_2, \ldots \delta_\omega$ becomes indifferent, and consequently the value of u_n, defined by the equation

$$u_n = (1)\, u_{n-1} + (2)\, u_{n-2} + \ldots + (i)\, u_{n-i},$$

becomes the coefficient of t^n in $\dfrac{1}{1 - (1)\, t - (2)\, t^2 \ldots - (i)\, t^i}$, as is well known.

The above rule may easily be extended to a linear equation in differences with any number of variables. Thus suppose, for greater simplicity, that we write

$$u_{x,\,y} = \Sigma \begin{pmatrix} x, & x' \\ y, & y' \end{pmatrix} u_{x',\,y'} \qquad \begin{bmatrix} x' = x-1,\ x-2,\ \ldots\ 0 \\ y' = y-1,\ y-2,\ \ldots\ 0 \end{bmatrix},$$

with the initial conditions $u_{0,0} = 1$, $u_{e,f} = 0$ wherever one or both of e, f are negative units; then to find the value of $u_{m,n}$ we must form all the possible descending series $\begin{bmatrix} m, & m_1, & m_2, & \ldots & m_\omega, & 0 \\ n, & n_1, & n_2, & \ldots & n_\omega, & 0 \end{bmatrix}$, subject only to the law that there

is a descent either from m_i to m_{i+1}, or from n_i to n_{i+1}, or at one and the same time from m_i to m_{i+1} and from n_i to n_{i+1}. The value of $u_{m,n}$ then becomes

$$\Sigma \begin{pmatrix} m, & m_1, & m_2, & \dots & m_\omega, & 0 \\ n, & n_1, & n_2, & \dots & n_\omega, & 0 \end{pmatrix},$$

with the understanding that the term within the parenthesis is to be read as meaning

$$\begin{pmatrix} m, & m_1 \\ n, & n_1 \end{pmatrix} \times \begin{pmatrix} m_1, & m_2 \\ n_1, & n_2 \end{pmatrix} \times \begin{pmatrix} m_2, & m_3 \\ n_2, & n_3 \end{pmatrix} \dots \times \begin{pmatrix} m_\omega, & 0 \\ n_\omega, & 0 \end{pmatrix}.$$

And in like manner and under a similar form we obtain the value of $u_{n_1, n_2 \dots n_\epsilon}$ defined by the general equation

$$u_{n_1, n_2, \dots n_\epsilon} = \Sigma \begin{pmatrix} n_1, & \nu_1 \\ n_2, & \nu_2 \\ \vdots & \vdots \\ n_\epsilon, & \nu_\epsilon \end{pmatrix} u_{\nu_1, \nu_2, \dots \nu_\epsilon}.$$

In defining the relations which connect one u with another, we may suppose that (r, s) means the coefficient of u_s in the equation

$$u_r = \Sigma (r, s)\, u_s \qquad [r > s,\ u_0 = 1,\ u_{-e} = 0];$$

but we may also suppose that (r, s) means the coefficient of v_r in the equation

$$v_s = \Sigma (r, s)\, v_r \qquad [r > s,\ v_n = 1,\ v_{n+\epsilon} = 0];$$

the value of u_0, on the latter supposition, it is obvious, becomes equal to that of u_n on the former—a fact that is well known, and deducible from the circumstance that u_n and v_0 will be represented by the same determinant turned round into a new position. But by means of our general representation for the case of any number ϵ of variables, we see that there is an analogous theorem which connects together 2^ϵ different results, and which is not so immediate a consequence of the theory of determinants.

To make my meaning more clear, if we suppose the four following systems of equations, in each of which $m > \mu$, $n > \nu$,

$$u_{m,n} = \Sigma \begin{pmatrix} m, & \mu \\ n, & \nu \end{pmatrix} u_{\mu,\nu}\ [u_{0,0} = 1,\ u_{-e,f} = 0,\ u_{e,-f} = 0,\ u_{-e,-f} = 0]^*,$$

$$v_{\mu,n} = \Sigma \begin{pmatrix} m, & \mu \\ n, & \nu \end{pmatrix} v_{m,\nu}\ [v_{m,0} = 1,\ v_{m+e,0} = 0,\ v_{m-e,-f} = 0,\ v_{m+e,-f} = 0],$$

$$w_{m,\nu} = \Sigma \begin{pmatrix} m, & \mu \\ n, & \nu \end{pmatrix} w_{\mu,n}\ [w_{0,n} = 1,\ w_{0,n+f} = 0,\ w_{-e,n-f} = 0,\ w_{-e,n+f} = 0],$$

$$\omega_{\mu,\nu} = \Sigma \begin{pmatrix} m, & \mu \\ n, & \nu \end{pmatrix} \omega_{m,n}\ [\omega_{m,n} = 1,\ \omega_{m+e,n-f} = 0,\ \omega_{m-e,n+f} = 0,\ \omega_{m+e,n+f} = 0],$$

we shall have $u_{m,n} = v_{0,n} = w_{m,0} = \omega_{0,0}$.

* Or, more simply and rather more accurately, in place of the three equations within the bracket it is better to write $u_{p,q} = 0$ when p or q or each of them is negative, and so analogously for the cases following:—

$v_{p,q} = 0$ when $m - p$ or q or each of them is negative,
$w_{p,q} = 0$ when m or $n - q$ or each of them is negative,
$\omega_{p,q} = 0$ when $m - p$ or $n - q$ or each of them is negative.

The theorem $u_n = v_0$ above given, when the equation of differences is of the second order, expresses the well-known theorem that the cumulant $[a, b, c, \dots h, k, l]$

$$\left(\text{the denominator of the continued fraction } \frac{1}{a+}, \ \frac{1}{b+}, \ \frac{1}{c+}, \dots \frac{1}{k+}, \ \frac{1}{l}\right)$$

is the same as the cumulant $[l, k, h, \dots c, b, a]$.

There is no known property either of cumulants of this kind or those of the higher orders, nor can there be any found, but what does and must flow as an immediate consequence from the representation of the linear-difference integral above given. For instance, the law of formation of the above cumulant by rejecting consecutive pairs of terms becomes intuitive; for to meet this case we must write descending series of integers $n, n_1, n_2, \dots n_\omega, 0$, such that each difference between consecutive terms n_i, n_{i+1} is always 1 or 2, and when the latter, $(n_i, n_{i+1}) = 1$.

So more generally if we write $u_n = a_n u_{n-1} + u_{n-r}$, we obtain an analogous law for throwing out in every possible way groups of r consecutive terms in order to express u_n in terms of $a_n, a_{n-1}, a_{n-2}, \dots a_0$. So, too, if we write $u_n = u_{n-1} + b_n u_{n-\mu}$, we obtain Binet's law of "*discontiguous*" products given in his long memoir on the subject published in the *Mémoires* of the Institute,—the law of descent upon this supposition being that the difference between n_i and n_{i+1} is 1 or r; and if the former, $(n_i, n_{i+1}) = 1$.

We have seen above the convenience of shifting the system of subindices so as, for instance, to be able to treat the question of finding u_0 when we suppose $u_n = 1$ and $u_{n+e} = 0$, as well as that of finding u_n when we suppose $u_0 = 1$, $u_{-e} = 0$. More generally there is an advantage in writing $u_m = 1$ and $u_{m-e} = 0$ when it is a question of expressing u_n, which may then be conveniently denoted indifferently by $m:n$ or $n:m$,—the law being that regularly descending or ascending series are to be formed beginning with n and ending with m in every possible manner, each of which expresses a known product consisting of two parts—one made up of factors denoted by the conjunction of the consecutive terms in every such series, the other by the duplication of the integers between n and m not appearing in the series.

It is, moreover, convenient in some cases to express the limit which the descents are not to exceed (corresponding to the order of the equation). Thus $\dfrac{n:m}{i}$ may be used to denote the limitation of the differences in $n:m$ not to exceed i. The well-known theorem in continued fractions ordinarily denoted by the equation $pq' - p'q = \pm 1$ may then be expressed in a somewhat more general form in the manner following.

(*To be continued.*)

59.

ON THE QUANTITY AND CENTRE OF GRAVITY OF FIGURES GIVEN IN PERSPECTIVE, OR HOMOGRAPHY.

[*Newcastle-on-Tyne British Association Report* (1863), p. 2.]

IN the first instance, the author showed how to find the point in the perspective representation of a plane figure into which the centre of gravity of such figure is projected. For this purpose it is only necessary to be furnished with the direction of the vanishing-line corresponding to the plane of the object put into perspective. The rule for finding the point in question is the following: every element of the picture is to be charged with a density equal to the inverse fourth power of its distance from the vanishing-line; the centre of gravity of the figure so charged will be the point required, and may of course be found by the rules of the integral calculus.

Next, as to the area of the unknown object. To determine this another datum (but only one other) is required besides the direction of the vanishing-line, which may be termed the constant of perspective, being determined when the position of the eye and that of the object-plane in reference to the picture are given. This constant is the product of the eye's distance from the vanishing-line into the square of the distance of the intersections of the object- and picture-planes from the same line. If now every element of the picture be charged with a density equal to the constant of perspective divided by the *cube* of the element's distance from the vanishing-line, the mass of the figure so charged will be the area of the unknown object-figure.

The author then proceeded to show how the area and the perspective centre, by aid of the preceding principles, admit of being reduced to depend on one single integral, closely analogous to the *potential* used in the theory of attractions to which he gives the name of *polar potential.* The polar potential of a plane figure in respect to a given line is defined to be the sum

of the quotients of the elements by their respective distances from the line, and consequently the polar potential of the picture in respect to a vanishing-line in its plane becomes a function of the two parameters by which its position may be determined. The parameters which the author finds most convenient to employ are the distance of the vanishing-line from an arbitrary fixed point in the picture and the angle which it makes with a fixed line therein.

The author then supplied the formulæ (which are of a very simple character) for calculating the area of the object and the coordinates of its perspective centre of gravity, by means of differentiation processes performed upon the polar potential of the picture treated as a function of these parameters. He afterwards proceeded to extend the same method to figures, plane or solid, connected by the more general relation known under the name of homography, of which the relation between figures generated through the medium of perspective is only a particular kind. In the case of a solid figure, its polar potential in respect to a variable plane becomes a function of three parameters; and by means of differentiations performed upon it in respect to these parameters, the content and the coordinates of the point corresponding homographically to the centre of gravity of a solid figure may be expressed when its homograph and the position of a plane corresponding to the points at infinity in the otherwise unknown figure are given in addition (as regards the content) to a certain constant termed the homographic determinant.

Professor Rankine threw out a suggestion as to the possibility of a practical application of the preceding theory to the stability of structures standing to each other in a certain simple relation of homography.

60.

ON A QUESTION OF COMPOUND ARRANGEMENT.

[*Proceedings of the Royal Society of London*, XII. (1862–3), pp. 561—563.]

MY successful but as yet unpublished researches into the Theory of Double Determinants have involved the consideration of the following curious case of arrangements.

There are given $m+n-1$ counters of n distinct *colours* just capable of being packed into m *urns*. The question refers to the distribution of the counters among the urns, subject to the condition that it shall *not* be possible to form a closed circuit of double colours between any number of the urns chosen arbitrarily; for example, we must allow no distribution of counters in which one urn contains blue and yellow, a second yellow and red, a third red and green, and a fourth green and blue, because here *blue, yellow, red,* and *green* would form a closed circuit. This condition, it is evident, excludes the same combination of colours from existing in any two of the urns, and also the repetition of any one colour in the same urn. Any distribution of counters obeying this condition may be called an excyclic distribution.

I annex two propositions, one qualitative, the other quantitative, referring to such distributions.

Qualitative Theorem.

In any excyclic distribution between m urns of $m+n-1$ counters of n different colours, any set of counters selected at will must be fewer in number than the number of distinct colours which they contain added to the number of urns from which they are drawn.

Before going on to enunciate the second proposition I must premise one or two simple definitions.

The *capacity* of an urn means the number of counters it will contain, the *frequency* of a colour the number of counters of that colour, so that the sum of all the capacities and the sum of all the frequencies must be each equal to the number of the counters.

Again, by the *diminished* capacity of any urn or *diminished* frequency of any colour, I mean such capacity or frequency respectively diminished by *unity.*

Finally, by the *polynomial function* of any set of numbers $a, b, \ldots l$, I mean the coefficient of $x^a . y^b \ldots z^l$ in the expansion of

$$(x + y + \ldots + z)^{a+b+\ldots+l}.$$

I can now enunciate the following

Quantitative Theorem.

The number of modes of excyclic distribution between m urns of $m + n - 1$ counters of n different colours is equal to the product of the polynomial function of the diminished frequencies *of all the several colours multiplied by the polynomial function of the* diminished capacities *of all the several urns.*

Observation.

A double determinant means the resultant of a system of $(m + n - 1)$ homogeneous equations, each containing mn terms and linear in respect to each of two systems of m and n variables taken separately, but of the second order in respect to the variables of these two systems taken collectively. Any such resultant is of the degree $\frac{(m + n - 1)!}{(m - 1)!\,(n - 1)!}$ in respect of the given coefficients, and may be represented by an ordinary determinant of the $(m + n - 1)$th order, every one of whose terms corresponds to a particular system of capacities of the m urns and of repetitions of the n colours in the question above treated.

The total number of such systems or terms will be

$$\left\{\frac{(m + n - 2)!}{(m - 1)!\,(n - 1)!}\right\}^2.$$

Every term in this determinant will itself be a sum of simple determinants of the $(m + n - 1)$th order, corresponding (each to each) with the totality of the excyclic distributions of $(m + n - 1)$ counters in respect of the particular systems of m capacities and n frequencies appertaining to that term; so that the number of simple determinants whose sum constitutes a term in the grand total determinant is always the product of two polynomial coefficients. In the particular case, where one of the systems contains only *two* variables, one of these polynomial coefficients becomes unity, and the other sinks down to a binomial coefficient. The only instance of a double determinant which is believed to have been considered up to the present moment is that given by Mr Cayley in the *Cambridge and Dublin Mathematical Journal,* vol. IX. 1854, for the case of $m = 2$, $n = 2$.

61.

ON A THEOREM RELATING TO POLAR UMBRÆ.

[*Proceedings of the Royal Society of London*, XII. (1862–3), pp. 563—565.]

BY polar umbræ I mean such as obey in the strictest manner the polar law of sign, so that not only any two appositions or products of such umbræ derivable from one another by an interchange of two of their elements are to be considered each as the negative of the other, but also any such apposition or product becomes zero if the same element is found in it more than once.

Thus Sir W. Hamilton's i, j, k are not polar umbræ, because although $ijk = -jik = kij$, &c., ii, jj, kk, instead of being *nulls*, are in the Calculus of Quaternions taken as *unities**.

Let us now define any set arranged either in line or column of such *umbral* quantities to be multiplied by a corresponding set of *actual* quantities when each term of the one set is multiplied by the corresponding one of the other, and the sum taken of the products so obtained as in the ordinary case of the multiplication of the lines or columns of two determinants *inter se*. Thus, for example, $(a, b, c \between x, y, z)$, as also $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \between \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is to mean the same product, namely,

$$ax + by + cz.$$

Again, imagine a rectangular (square or oblong) matrix of polar umbræ, and that each line thereof is multiplied by the same line of *actual* quantities, the product of the products so obtained I call a Factorial of the Matrix. I also call the product similarly obtained when the columns of the matrix are substituted for the lines, a Factorial of the same, but distinguish between the two by giving to one the name of a Transverse, the second of a Longitudinal Factorial of the matrix. We are now in a position to enunciate the following remarkable theorem:—

* If we use Vandermonde's condensed notation for a determinant $\begin{bmatrix} 1, 2, \ldots n \\ 1, 2, \ldots n \end{bmatrix}$ to represent a "déterminant gauche," then, since on this supposition $rs = -sr$ and $rr = 0$, the elements $1, 2, 3, \ldots n$ will be polar umbræ by definition.

The product of any longitudinal by any transverse factorial of the same polar umbral matrix is identically zero.

For example, let $\begin{vmatrix} a, & b, & c \\ d, & e, & f \end{vmatrix}$ be a matrix of polar umbræ, but x, y, z and also ξ, η actual quantities. Then

$$(ax + by + cz)(dx + ey + fz)$$

is a transverse factorial,

$$(a\xi + d\eta)(b\xi + e\eta)(c\xi + f\eta)$$

a longitudinal factorial of the above matrix, and by the theorem their product should be zero. This is easily verified.

The two *factorials* expanded are respectively

$$adx^2 + bey^2 + cfz^2 + (ae + bd)\,xy + (bf + ce)\,yz + (af + cd)\,zx,$$

$$abc\xi^3 + (abf + aec + dbc)\,\xi^2\eta + (dec + dbf + aef)\,\xi\eta^2 + def\eta^3;$$

in their product the coefficient of

$$\begin{aligned} x^2\xi^3 &= abcad = 0, \\ xy\xi^3 &= abcae + abcbd = 0, \\ x^2\xi^2\eta &= abfad + aecad + dbcad = 0, \\ xy\xi^2\eta &= abfae + abfbd + aecae + aecbd + dbcae + dbcdb \\ &= aecbd + dbcae = aecbd - aecbd = 0, \end{aligned}$$

and so for all the other terms.

This is the fundamental theorem by aid of which I obtain the resultant of a lineo-linear system of equations in its most perfect form. It is easy to obtain two different solutions, each of them unsymmetrical in respect of the data of the question; the conversion and fusion of each of these into one and the same determinant, symmetrical in all its relations to the data, is effected instantaneously by a process derived from the above theorem. In that particular application of it, the umbræ involved each represent columns of actual quantities in number equal to the number of places in the width and length of the umbral matrix to which they belong, so that each coefficient in the product of a lateral by a longitudinal factorial represents an ordinary determinant made up of these columns, from which it is evident that the polar law of sign and nullity necessary for the truth of the theorem is satisfied in the case supposed.

62.

ON THE DEGREE AND WEIGHT OF THE RESULTANT OF A MULTIPARTITE SYSTEM OF EQUATIONS.

[*Proceedings of the Royal Society of London*, XII. (1862–3), pp. 674—676.]

LET there be $(1+n)$ equations each homogeneous in any number of sets of variables, and suppose that the degrees of the several equations in respect to these sets are respectively

$$\begin{array}{l} a,\ \ b,\ \ c,\ \ldots,\ l, \\ a_1,\ b_1,\ c_1,\ \ldots,\ l_1, \\ a_2,\ b_2,\ c_2,\ \ldots,\ l_2, \\ \cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots \\ a_n,\ b_n,\ c_n,\ \ldots,\ l_n, \end{array}$$

where the a, b, c, &c. are any positive integers, zero not excluded.

Let the number of variables in the several sets be respectively $1+\alpha$, $1+\beta$, $1+\gamma$, ... $1+\lambda$, then in order that the system may have a resultant, since the number of ratios to be eliminated is $\alpha+\beta+\gamma+\ldots+\lambda$, this sum must be equal to n.

Let $$a_i\rho + b_i\sigma + c_i\tau + \ldots + l_i\omega = L_i,$$

and let $$LL_1L_2 \ldots L_n = P,$$

then 1st, the degree of the resultant in question in regard to the coefficients of the rth equation will be the coefficient of $\rho^\alpha . \sigma^\beta . \tau^\gamma \ldots \omega^\lambda$ in $\dfrac{P}{L_r}$.

2nd. As regards weight. By the weight of any letter in respect to any given variable is to be understood the exponent of that variable in the term affected with the coefficient; and by the weight of any term of the resultant in respect to such variable, the sum of the weights of its several simple factors;

each term in the resultant in respect to any given variable has the same weight; and this weight may also be proved to be alike for each variable in the same set, and may be taken as the weight of the resultant in respect to such set. This being premised, we have the following theorem :—

The value of the weight of the resultant in respect to any particular set of the variables, for example, the $(1+\alpha)$ set, will be the coefficient of

$$\rho^{1+\alpha} . \sigma^{\beta} . \tau^{\gamma} \dots \omega^{\lambda} \text{ in } P.$$

In the particular case where $\alpha = \beta = \gamma \dots = \lambda$, the above expressions for the degree and weight evidently become polynomial coefficients. Thus, for example, if we suppose each equation *linear* in respect to the variables of each set, the degree of the resultant in respect to the coefficients of any equation will be

$$\frac{(\alpha+\beta+\gamma+\dots+\lambda)\,!}{\alpha\,!\,\beta\,!\,\gamma\,!\dots\lambda\,!},$$

and its weight in respect to the $(1+\alpha)$ set will be

$$\frac{(1+\alpha+\beta+\dots+\lambda)\,!}{(1+\alpha)\,!\,\beta\,!\,\gamma\,!\dots\lambda\,!}.$$

In particular if each set is binary, so that $\alpha = \beta = \gamma \dots = \lambda = 1$, the degree becomes $n\,!$, and the weight $\frac{1}{2}(n+1)\,!$.

The above theorems are, I believe, altogether new.

It may just be noticed (as a passing remark) that the total degree in the general case is the coefficient of

$$\rho^{\alpha} . \sigma^{\beta} . \tau^{\gamma} \dots \omega^{\lambda} \text{ in } P\left\{\frac{1}{L}+\frac{1}{L_1}+\dots+\frac{1}{L_n}\right\},$$

and the *total* weight the coefficient of the same argument in

$$P\left\{\frac{1}{\rho}+\frac{1}{\sigma}+\dots+\frac{1}{\omega}\right\}.$$

63.

SEQUEL TO THE THEOREMS RELATING TO "CANONIC ROOTS" GIVEN* IN THE MARCH NUMBER OF THIS MAGAZINE.

[*Philosophical Magazine*, XXV. (1863), pp. 453—460.]

THE theorems kindly communicated from me by Mr Cayley in the March Number of this *Magazine* were originally designed to appear as a note or *excursus* to a memoir in preparation on the extension of Gauss's method of approximation from single to multiple integrals by a method which invariably leads to the construction of a *canonizant* whose roots are all real. To establish this reality, recourse may advantageously be had to a theorem of Jacobi, given at the end of his well known memoir "De Eliminatione Variabilis e duabus æquationibus," *Crelle*, vol. XV. p. 101, a very slight inspection of which at once leads to the further and interesting inference that the resultant of the canonizant of an odd-degreed function of x and *unity*, and of the canonizant of the second differential coefficient of that function in respect to x, is an exact power of the *catalecticant* of the first differential coefficient of x in respect to the same. This is the essence of the matter communicated by Mr Cayley; but subsequent successive generalizations of the theorem have led me on, step by step, to the discovery of a vast general theory of double determinants, that is, resultants of bipartite lineo-linear equations, constituting, I venture to predict, the dawn of a new epoch in the history of modern algebra and the science of pure tactic.

I will begin this note upon a note, by reproducing in brief the first of my two demonstrations of the simple theorem in question†. Let us write

$$X_0 = 1, \quad X_1 = \begin{vmatrix} 1, & x \\ a, & b \end{vmatrix}, \quad X_2 = \begin{vmatrix} 1, & x, & x^2 \\ a, & b, & c \\ b, & c, & d \end{vmatrix}, \quad X_3 = \begin{vmatrix} 1, & x, & x^2, & x^3 \\ a, & b, & c, & d \\ b, & c, & d, & e \\ c, & d, & e, & f \end{vmatrix},$$

and so on. And again, let

$$\lambda_1 = a, \quad \lambda_2 = \begin{vmatrix} a, & b \\ b, & c \end{vmatrix}, \quad \lambda_3 = \begin{vmatrix} a, & b, & c \\ b, & c, & d \\ c, & d, & e \end{vmatrix},$$

* [Cayley, *Coll. Math. Papers*, vol. V. p. 104.]

† The second has been communicated by Mr Cayley in the March number of this *Magazine*.

and so on. The theorem in effect to be proved is simply this, that the resultant of X_i and X_{i-1} is an exact power of λ_i, which (as will at once be seen) is the coefficient of x^i in X_i. In what follows, I shall use $R(P, Q)$ or $R(Q, P)$ to denote indifferently the positive or negative resultant of any two functions P and Q, ignoring for greater simplicity all considerations as to the proper algebraical sign to be affixed to a resultant of two functions taken in assigned order.

Jacobi's theorem above referred to, stated so far as necessary for the purpose in hand, is as follows:

$$X_n = (Ax + B) X_{n-1} - \frac{\lambda_n^2}{\lambda_{n-1}^2} X_{n-2}.$$

Hence, by virtue of a general theorem of elimination*,

$$R(X_n, X_{n-1}) = \lambda_{n-1}^2 R\left(-\frac{\lambda_n^2}{\lambda_{n-1}^2} X_{n-2}, X_{n-1}\right);$$

or, neglecting as premised all considerations of algebraical sign,

$$= (\lambda_{n-1})^2 \left(\frac{\lambda_n}{\lambda_{n-1}}\right)^{2(n-1)} . R(X_{n-1}, X_{n-2}),$$

that is

$$\frac{R(X_n, X_{n-1})}{\lambda_n^{2(n-1)}} = \frac{R(X_{n-1}, X_{n-2})}{\lambda_{n-1}^{2(n-2)}} = \frac{R(X_{n-2}, X_{n-3})}{\lambda_{n-2}^{2(n-3)}}$$

$$= \&\text{c.} = \frac{R(X_1, X_0)}{\lambda_1} = 1;$$

or if any one of my readers finds a difficulty in admitting that $R(ax-b, 1)=a$, he can stop short at $\frac{R(X_2, X_1)}{\lambda_2^2}$, which may easily be verified to be equal to unity. Hence

$$R(X_n, X_{n-1}) = \lambda_n^{2n-2}. \quad \text{Q.E.D.}$$

Thus we see that if X_n, X_{n-1} have one root in common, λ_n must vanish; but then, by the cited theorem of Jacobi, it follows that X_n completely contains X_{n-1}; from this it was easy to infer the necessity of the function† of which X_n is the canonizant, having infinity for one of its "canonic roots"—or, in other words, of its being reducible to the form

$$k_1(x + h_1)^{2n-1} + k_2(x + h_2)^{2n-1} + \ldots + k_{n-1}(x + h_{n-1})^{2n-1} + k_n.$$

* This theorem is best seen by dealing in the first instance with U, V, any two homogeneous functions of x, y of degrees n, $n-1$ respectively satisfying the identity $U=(Ax+By)V+y^2W$; we have then

$$R(U, V) = R(V, y^2W) = R(V, W) \times \{R(V, y)\}^2,$$

where evidently $R(V, y)$ is the coefficient of x^{n-1} in V; let y become unity, then on calling U, V, $R(V, y)$ respectively X_n, X_{n-1}, λ_{n-1}, and giving to W its corresponding value, we have the theorem as it is used in the text.

† For in general if X_n be the canonizant to F, X_{n-1} will be the canonizant to $\frac{d^2F}{dx^2}$.

And so it became natural to establish *à priori* the existence of this condition, and thus to obtain the proof virtually reproduced by Mr Cayley in the article referred to.

In what precedes, X_{n-1} was *a first principal minor* of X_n; and it occurred to me to institute an inquiry into the form of the resultant of two functions related to each other as X_{n-1} is to X_n, with the sole but important difference that the constants in X_n are not to be contained in a concatenated order from one line to another, but to be taken perfectly independent as in the example

$$X_3 = \begin{vmatrix} 1, & x, & x^2, & x^3 \\ a, & b, & c, & d \\ a', & b', & c', & d' \\ a'', & b'', & c'', & d'' \end{vmatrix}$$

and

$$X_4 = \begin{vmatrix} 1, & x, & x^2, & x^3, & x^4 \\ a, & b, & c, & d, & e \\ a', & b', & c', & d', & e' \\ a'', & b'', & c'', & d'', & e'' \\ a''', & b''', & c''', & d''', & e''' \end{vmatrix}$$

Or, according to a suggestion of Mr Cayley, putting the question into a more general and simple form, we may inquire into the resultant of any two complete determinants, functions of x of the $(n-1)$th degree, which belong to the rectangular matrix

$$\begin{array}{lllll} 1, & x, & x^2 & \ldots & x^{n-1}, \\ a_{1,1}, & a_{1,2}, & a_{1,3} & \ldots & a_{1,n}, \\ a_{2,1}, & a_{2,2}, & a_{2,3} & \ldots & a_{2,n}, \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n-1,1}, & a_{n-1,2}, & a_{n-1,3} & \ldots & a_{n-1,n}, \\ a_{n,1}, & a_{n,2}, & a_{n,3} & \ldots & a_{n,n}; \end{array}$$

as, for instance, the resultant of the two determinants which may be obtained by suppressing successively the last and last but one line in the matrix above written: and by aid of the most elementary principles of the calculus of determinants the instructed reader will find no difficulty in proving that this resultant will resolve itself into two distinct parts—one a power of the determinant obtained by suppressing the uppermost (or x) line in the above matrix, the other the Resultant of the matrix obtained by suppressing

simultaneously the two lowermost lines*. This last suppression leaves a rectangular matrix which, written in a homogeneous form, becomes

$$\begin{array}{lllll} y^{n-1}, & y^{n-2}x, & y^{n-3}x^2 & \ldots x^{n-1}, \\ a_{1,1}, & a_{1,2}, & a_{1,3} & \ldots a_{1,n}, \\ a_{2,1}, & a_{2,2}, & a_{2,3} & \ldots a_{2,n}, \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ a_{n-2,1}, & a_{n-2,2}, & a_{n-2,3} & \ldots a_{n-2,n}, \end{array}$$

consisting of n columns and $(n-1)$ lines†.

The Resultant of this matrix means the quantity R which, equated to zero, will indicate the possibility of the simultaneous nullity of *all* its first minors, so that R will be the factor common to the resultants of every couple of these minors. If we name the columns of the matrix taken in any arbitrary order $C_1, C_2 \ldots C_n$, and call R' the resultant of

$$C_1 C_3 \ldots C_{n-1} C_n,\ C_2 C_3 \ldots C_{n-1} C_n,$$

it may readily be made out that $\frac{R'}{R}$ is equal to a power of the determinant obtained by suppressing the uppermost (or x) line of the rectangular matrix $C_3 \ldots C_{n-1} C_n$.

To find R, we may proceed in the general case in the manner indicated in the example following, where $n-1$ is made 4. Taking the two extreme first minors and dividing them respectively by y and x, we have two equations of the following form for determining R, namely,

$$\begin{vmatrix} y^3, & y^2x, & yx^2, & x^3 \\ a, & b, & c, & d \\ a', & b', & c', & d' \\ a'', & b'', & c'', & d'' \end{vmatrix} = 0, \quad \begin{vmatrix} y^3, & y^2x, & yx^2, & x^3 \\ b, & c, & d, & e \\ b', & c', & d', & e' \\ b'', & c'', & d'', & e'' \end{vmatrix} = 0.$$

By rejecting, as we have done, the factors x and y from the above equations, certain factors, it is true, are lost to their resultant (R'); but it will easily be seen that these factors are each of them powers of one and the same determinant, namely, the determinant

$$\begin{vmatrix} b, & c, & d \\ b', & c', & d' \\ b'', & c'', & d'' \end{vmatrix}$$

* For, on making the last-named determinant referred to in the text zero, it may easily be shown, by aid of a familiar theorem in compound determinants, that the two determinants whose resultant is under investigation have all the coefficients of the one in the same ratio to each other as the corresponding coefficients of the other.

† The reader may notice that the real interest of the subject under consideration commences with the independent inquiry into the form of the Resultant of the above matrix—the original question, as to the quasi-canonizant, being important only as leading up to the appearance of this Resultant.

and that their product is contained in the irrelevant factor $\frac{R'}{R}$, itself a power of that determinant, as above explained. To find R, we may write down the oblong matrix

$$\begin{matrix} y^2, & yx, & x^2, \\ b, & c, & d, \\ b', & c', & d', \\ b'', & c'', & d'', \end{matrix}$$

and make its three first minors respectively equal to u, v, w, that is,

$$\begin{vmatrix} y^2, & yx, & x^2 \\ b', & c', & d' \\ b'', & c'', & d'' \end{vmatrix} = u, \qquad \begin{vmatrix} y^2, & yx, & x^2 \\ b'', & c'', & d'' \\ b, & c, & d \end{vmatrix} = v, \qquad \begin{vmatrix} y^2, & yx, & x^2 \\ b, & c, & d \\ b', & c', & d' \end{vmatrix} = w;$$

then we shall obtain the equations following, of which the intermediate ones result solely from the equations last assumed, but the first and last from those combined with the original two given ones, namely,

$$\begin{aligned} (bu + b'v + b''w)\, y - (au + a'v + a''w)\, x &= 0, \\ (cu + c'v + c''w)\, y - (bu + b'v + b''w)\, x &= 0, \\ (du + d'v + d''w)\, y - (cu + c'v + c''w)\, x &= 0, \\ (eu + e'v + e''w)\, y - (du + d'v + d''w)\, x &= 0. \end{aligned}$$

These equations may be satisfied by making simultaneously

$$u = 0, \quad v = 0, \quad w = 0,$$

all of which (since u, v, w are minors of the same rectangular matrix) may exist simultaneously, provided

$$\begin{vmatrix} b, & c, & d \\ b', & c', & d' \\ b'', & c'', & d'' \end{vmatrix} = 0.$$

Rejecting (as before) this irrelevant factor, it remains to find the resultant of the system of equations in x, y; u, v, w, above written, defined as the characteristic of the possibility of their coexistence for some particular system of values of x, y; u, v, w, but with joint and *several exclusion* of the system $x = 0$, $y = 0$, and of the system $u = 0$, $v = 0$, $w = 0$.

So, in like manner, in the general case we shall obtain a similar system of $(m+1)$ homogeneous equations linear in x, y, and also in $u_1, u_2, \ldots u_m$; and R will be the resultant of this system, subject to the same condition as to the exclusion of zero systems of x, y, and $u_1, u_2, \ldots u_m$ as in the particular instance above treated. Such a resultant, as hinted at the outset, is entitled to the name of a double determinant. In general a double determinant will refer to two systems of variables, one p, the other q in number, and to $(p+q-1)$ equations between them.

In the particular instance before us, one of these quantities, say q, is the number 2. There is, moreover, a further particularity (but which as it happens does not at all influence the form of the solution), consisting in the fact that the equations are of the *recurring* form

$$\begin{aligned} &L_1 y \quad - L_0 x = 0, \\ &L_2 y \quad - L_1 x = 0, \\ &L_3 y \quad - L_2 x = 0, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &L_{p+1} y - L_p x = 0, \end{aligned}$$

where $L_0, L_1, \ldots L_{p+1}$ are each of them linear homogeneous functions of $u_1, u_2, \ldots u_p$. This gives rise to an identification of the resultants of two matrices of very different appearance—one matrix, for example, being

$$\begin{array}{ccccc} y^4, & y^3x, & y^2x^2, & yx^3, & x^4, \\ a, & b, & c, & d, & e, \\ a', & b', & c', & d', & e', \\ a'', & b'', & c'', & d'', & e'', \end{array}$$

and the other being

$$\begin{array}{llll} au + a'v + a''w, & bu + b'v + b''w, & cu + c'v + c''w, & du + d'v + d''w, \\ bu + b'v + b''w, & cu + c'v + c''w, & du + d'v + d''w, & eu + e'v + e''w. \end{array}$$

I have ascertained, and hope shortly to publish, the method of obtaining the explicit value of double determinants in the most general case and under their most symmetrical form: for the particular case before our eyes, this resultant will be as follows:—

$$\left\| \begin{array}{ccc} \begin{vmatrix} a, & b, & a', & a'' \\ b, & c, & b', & b'' \\ c, & d, & c', & c'' \\ d, & e, & d', & d'' \end{vmatrix}, \begin{vmatrix} a, & b, & b', & a'' \\ b, & c, & c', & b'' \\ c, & d, & d', & c'' \\ d, & e, & e', & d'' \end{vmatrix} & + & \begin{vmatrix} a, & b, & a', & b'' \\ b, & c, & b', & c'' \\ c, & d, & c', & d'' \\ d, & e, & d', & e'' \end{vmatrix}, \begin{vmatrix} a, & b, & b', & b'' \\ b, & c, & c', & c'' \\ c, & d, & d', & d'' \\ d, & e, & e', & e'' \end{vmatrix} \\ \begin{vmatrix} a', & b', & a'', & a \\ b', & c', & b'', & b \\ c', & d', & c'', & c \\ d', & e', & d'', & d \end{vmatrix}, \begin{vmatrix} a', & b', & b'', & a \\ b', & c', & c'', & b \\ c', & d', & d'', & c \\ d', & e', & e'', & d \end{vmatrix} & + & \begin{vmatrix} a', & b', & a'', & b \\ b', & c', & b'', & c \\ c', & d', & c'', & d \\ d', & e', & d'', & e \end{vmatrix}, \begin{vmatrix} a', & b', & b'', & b \\ b', & c', & c'', & c \\ c', & d', & d'', & d \\ d', & e', & e'', & e \end{vmatrix} \\ \begin{vmatrix} a'', & b'', & a, & a' \\ b'', & c'', & b, & b' \\ c'', & d'', & c, & c' \\ d'', & e'', & d, & d' \end{vmatrix}, \begin{vmatrix} a'', & b'', & b, & a' \\ b'', & c'', & c, & b' \\ c'', & d'', & d, & c' \\ d'', & e'', & e, & d' \end{vmatrix} & + & \begin{vmatrix} a'', & b'', & a, & b' \\ b'', & c'', & b, & c' \\ c'', & d'', & c, & d' \\ d'', & e'', & d, & e' \end{vmatrix}, \begin{vmatrix} a'', & b'', & b, & b' \\ b'', & c'', & c, & c' \\ c'', & d'', & d, & d' \\ d'', & e'', & e, & e' \end{vmatrix} \end{array} \right\|$$

And it may be noticed that if we return to the original question, in which the coefficients are no longer independent, but where the column $a'b'c'd'e'$ is

identical, term for term, with *bcdef*, and $a''b''c''d''e''$ with *cdefg*, the above determinant becomes

$$\left|\begin{array}{ccc} * & * & \begin{vmatrix} a, & b, & c, & d \\ b, & c, & d, & e \\ c, & d, & e, & f \\ d, & e, & f, & g \end{vmatrix} \\ * & \begin{vmatrix} b, & c, & d, & a \\ c, & d, & e, & b \\ d, & e, & f, & c \\ e, & f, & g, & d \end{vmatrix} & * \\ \begin{vmatrix} c, & d, & a, & b \\ d, & e, & b, & c \\ e, & f, & c, & d \\ f, & g, & d, & e \end{vmatrix} & * & * \end{array}\right|$$

that is to say, it becomes a power of the determinant

$$\begin{vmatrix} a, & b, & c, & d \\ b, & c, & d, & e \\ c, & d, & e, & f \\ d, & e, & f, & g \end{vmatrix}$$

as we know *à priori* it ought to do, by virtue of the theorem originating out of Jacobi's theorem stated at the beginning of this paper: in fact the two factors of the resultant of X_3, X_4 each of them becomes equal to $\lambda_4{}^3$; and so in general we shall find, if we use n instead of 4, each factor of the corresponding resultant becomes λ_n^{n-1}, giving λ_n^{2n-2} as the complete resultant for that singular case, as previously determined.

The author is conscious that some apology may appear due for the cursory mode of elucidation pursued in the preceding extended note, and for the absence as regards certain points of the appropriate proofs; but to have gone into all the details of demonstration would have swollen the paper to a length out of proportion to its importance. Let him be permitted also in all humility to add (as can be vouched by more than one contributor to this *Magazine*), that in consequence of the large arrears of algebraical and arithmetical speculations waiting in his mind their turn to be called into outward existence, he is driven to the alternative of leaving the fruits of his meditations to perish (as has been the fate of too many foregone theories, the still-born progeny of his brain, now for ever resolved back again into the primordial matter of thought), or venturing to produce from time to time such imperfect sketches as the present, calculated to evoke the mental cooperation of his readers, in whom the algebraical instinct has been to some extent developed, rather than to satisfy the strict demands of rigorously systematic exposition.

64.

OBSERVATIONS ON THE METHOD FOR FINDING THE CENTRE OF GRAVITY OF A QUADRILATERAL GIVEN IN THE PRESENT NUMBER OF THE JOURNAL.

[*Quarterly Journal of Mathematics*, VI. (1863), pp. 130—133.]

THE method given in *Mechanical Solutions of Geometrical Problems*, p. 127, for finding the centre of gravity of a quadrilateral, leaves nothing to be wished for in point of elegance and conciseness; it is new* to the Editors and stands in advantageous contrast with all other methods of effecting the same end. It involves only four lines of construction and two bisections; in some elementary works on Mechanics, in use at our Universities, a method is given involving no less than 9 or 11 auxiliary lines. It must henceforth take rank as the best method of effecting the end in view; the second best is that which has been treated analytically, by Mr Stephen Fenwick, of the Royal Military Academy, in the *Mathematician*, 1847, Vol. II., p. 292, but admits of a simple and pleasing geometrical proof.

Let us call the intersection of the two diagonals the *cross-centre*, and the intersection of the two bisectors of opposite pairs of sides the *mid-centre* of a quadrilateral.

If we take the centres of gravity of the four triangles into which a given quadrilateral is divided by its two diagonals, it is clear that the cross-centre of the new quadrilateral, of which these four points are the summits, will be the centre of gravity of the original quadrilateral. But it may easily be seen that this new quadrilateral is only a miniature image of the original one, and that each of the two quadrilaterals has the same mid-centre; in a word, the new quadrilateral may be obtained by reducing the linear dimensions of the original one in the ratio of 1 to 3, and then swinging it through half a revolution round the mid-centre. Hence the new cross-centre will be in opposition with the original one, in respect to the mid-centre, and at a distance from it equal to one-third of the distance of the former one from the same.

* I should say new in *form*; in substance it is identical with that given, Vol. II., p. 292, of the *Mathematician*.

This method involves only four auxiliary lines, but requires four bisections and one trisection, instead of merely two bisections, according to the method of the text above.

The substitution of heavy points for areas or volumes admits of an extension which the author of this note believes to be new, and which occurred to him incidentally in treating of the extension of Gauss' method of approximation from simple to multiple quadratures.

It will be convenient to call the sum of the masses of any system of bodies into the nth powers of their distances from a fixed plane their nth *moments* in respect to the plane. (Thus the second moments will mean the sum of the masses into their squared distances.) It may then be affirmed as a universal proposition that such nth moments of a line, triangle, and tetrahedron (and so on for the higher dimensions of space) may always be replaced by suitable weights at fixed points symmetrically situated about the centre of gravity of such figures. For example, the *second* moments of lines, triangles, and tetrahedra (say each of mass unity) in respect to any plane may be replaced by masses of $\frac{1}{6}$ at the two angular points for the line, of $\frac{1}{12}$ at the three angular points for the triangle, and of $\frac{1}{20}$ at the four angular points for the tetrahedron, the balance $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$ being of course placed at the centre of gravity in these cases respectively. Hence it follows obviously that the same law will be true for the *moments of inertia* of a line, triangle, or tetrahedron about any *axis*, and consequently the centres and times of oscillation of these figures about any axis will be the same as for equal weights placed at the angles and weights respectively 4, 9, and 16 times as great placed at their centres of gravity. The ingenious author of the matter which has called forth these observations may probably be able to draw interesting inferences from this equivalence, and also from combining his own unrivalled method for finding the centre of gravity of a quadrilateral with the miniature-image method hereinbefore explained.

One word more before I conclude; the rule given in the text may be expressed in general terms by aid of a simple verbal definition. Let two points situated in a limited line be said to be *opposite* when their respective distances from opposite ends of the line are equal.

The centre of gravity of a quadrilateral may then be stated to be identical with the centre of gravity of a triangle whose apices are the point of intersection of the two diagonals and the opposite points thereto on these two diagonals*.

* It is difficult to resist the impression that some similar construction must apply to the determination of the centre of gravity of the frustum of a pyramid. Two points in a line with the centre of gravity of a triangle and at equal distances on opposite sides may be defined as opposite points in respect to the triangle. As a mere conjecture to be subjected to ulterior verification I suggest the possibility of the following construction being applicable; if not true it may at least serve to set the mind athinking in the right direction for the discovery of the truth.

The Barycentric principle employed in the text leads me to make an observation which will be found somewhat prolific in consequences and may be made instrumental (as I have satisfied myself by actual trial) in the edification of a complete theory of the parabola by processes greatly exceeding in simplicity those depending on Cartesian coordinates.

The secret of the utility of the Barycentric principle consists essentially in the plasticity of the axes to which the moments may be referred. Equally advantageous will be found the introduction of the laws of motion into the theory of the parabola, aided by the *plastic* condition that the motion of a projectile acted on by a constant force, *reckoned in any direction,* depends only on the actual velocity and force respectively estimated in such direction.

Let Aa, Bb, Cc be the edges of the frustum.

Let the three diagonal triangles Abc, Bca, Cab intersect in the point P and let the opposites to P in these three triangles be respectively P', P'', P'''. Similarly, by means of the other system of diagonal planes aBC, bCA, cAB, let a second system of four points p, p', p'', p''' be obtained. I conjecture that the centre of gravity of equal weights at these eight points, or, which is the same thing, the centre of gravity of the pyramid, whose apices bisect respectively the lines Pp, $P'p'$, $P''p''$, $P'''p'''$, *may be* the centre of gravity of the frustum. This intermediate pyramid appears to be the natural measure of the distortion of the frustum from the prismatic form, as the triangle formed by the cross-centre and its two opposites is that of the distortion of the quadrilateral, which may be regarded as the frustum of a triangle.

I have verified the conjectural construction for the case where the frustum becomes a prism and also for the case where it becomes a tetrahedron by the vanishing of one of its triangular faces.

À propos of the relation between the trapezium and the pyramidal frustum, I am not aware whether it has been observed that as a trapezium may be divided in two ways into a pair of triangles, so may the frustum of a pyramid be divided in six ways into a triplet of tetrahedrons. Using the same letters as before, one such division will be represented by the table following:

a	b	c	A
b	c	A	B
c	A	B	C

and permuting simultaneously and conformably the two systems of letters a, b, c; A, B, C, we obtain all the six systems in question. This stereotomic division leads to a direct and almost instantaneous geometrical proof of the known expression

$$\delta = \frac{h}{4}\frac{a^2+2ab+3b^2}{a^2+ab+b^2},$$

for finding the position, of the centre of gravity of a frustum bounded by parallel faces, in the line joining the centres of the parallel faces.

Obviously for space of any number of dimensions, say n, an analogous dissection may be effected in $n!$ different ways; the scheme above given serving fully to disclose the tactical law of the symbols.

For example, for space of four dimensions one such dissection out of the twenty-four will be denoted by the scheme

a	b	c	d	A
b	c	d	A	B
c	d	A	B	C
d	A	B	C	D

By aid of this principle I have reconstructed all the essential properties of the curve in respect to its directrix, focus, and tangents, and obtained, as it were instantaneously, various theorems, some of which, if not new, could only be obtained by long processes through the ordinary methods, whether of Geometry or of Cartesian coordinates.

Since penning the above observations, the author has found without difficulty the two geometrical constructions for the centre of gravity of a pyramidal frustum, precisely analogous to those alluded to for the centre of gravity of a quadrilateral: which will probably appear* in the August Number (or, if not, in the September Number) of the *Philosophical Magazine*. The true *mid-centre* is the centre of gravity of six equal weights placed at the six angles of the frustum; the true *cross-centre* is the point of intersection of either of two ternary systems of planes, which have the property of intersecting in the same point; one of these planes, for example, will be the plane passing through the middle point of ab, the middle point of aC, and the middle point of BC.

This brings to mind an analogous generalization long ago made known by the writer of this note, namely, that as a quadratic surface is cut by any tangent plane in two straight lines, so is a cubic hyper-surface by a tangent hyper-plane in six, a quartic transhyper-surface by a tangent transhyper-plane in twenty-four right lines, and so on indefinitely. Passing by an abrupt flight from a transcendental analogy to what many may regard as a mere platitude, let me notice that it is not a truism but a proposition and no insignificant one to affirm that a convex figure of five [plane] faces capable of being formed by joining conformably the angles of one triangle with those of another can only be the *frustum of a pyramid;* it is in fact equivalent to the assertion that three right lines of which every two intersect must either lie in one plane or pass through one point.

I ought not to conclude without alluding to a second conjectural method for finding the centre of gravity of such frustum which will in all probability stand or fall with that already given, bearing to it the same relation as the second best bears to the best determination of the analogous problem for the quadrilateral. Taking as Q (the *cross-centre*) the point mid-way between P, p the respective intersections of the two systems of diagonal planes, and as O (the *mid-centre*) the point where the axis joining the centres of gravity of the two triangular faces meets the plane containing the centres of gravity of the three quadrilateral faces, QO being joined and produced through O to G, so that $GO=\frac{QO}{4}$, G will be the conjectural position of the centre of gravity of the frustum. This is easily verified for the case where the frustum becomes an entire pyramid.

P.S.—Since the above was in press I have ascertained that each of the above two conjectural methods is erroneous. Apparently the *euristic* problem to be solved is to discover in the pyramidal frustum, the analogue to the cross-centre in the quadrilateral; this, there is every reason to believe, is closely connected with the points P and p above described; it is, however, certainly not the point mid-way between them.

[* Below, p. 342.]

65.

ON THE CENTRE OF GRAVITY OF A TRUNCATED TRIANGULAR PYRAMID, AND ON THE PRINCIPLES OF BARYCENTRIC PERSPECTIVE.

[*Philosophical Magazine,* XXVI. (1863), pp. 167—183.]

THERE is a well-known geometrical construction for finding the centre of gravity of a plane quadrilateral, which may be described as follows.

Let the intersection of the two diagonals (say Q) be called the *cross-centre*; the intersection of the lines bisecting the middle points of pairs of opposite sides (say O) the *mid-centre* (which, it may be observed, is the centre of gravity of the four angles viewed as equal weights); then the centre of gravity is in the line joining these two centres produced past the latter (the mid-centre), and at a distance from it equal to one-third of the distance between the two centres; in a word, if G be the centre of gravity of the quadrilateral, QOG will be in a right line, and $OG = \frac{1}{3}QO$.

The frustum of a pyramid is the nearest analogue in space to a quadrilateral *in plano*, since the latter may be regarded as the frustum of a triangle. The analogy, however, is not perfect, inasmuch as a quadrilateral may be regarded as a frustum of either of two triangles, but the pyramid to which a given frustum belongs is determinate. Hence *à priori* reasonable doubts might have been entertained as to the possibility of extending to the pyramidal frustum the geometrical method of centering the plane quadrilateral. The investigation subjoined dispels this doubt, and will be found to lead to the perfect satisfaction, under a somewhat unexpected form, of the hoped-for analogy.

Let abc, $\alpha\beta\gamma$ be the two triangular faces, $a\alpha$, $b\beta$, $c\gamma$ the edges of the quadrilateral faces of a pyramidal frustum. Then this frustum may be

resolved in six different ways into the sum total of three pyramids, as shown in the annexed double triad of schemes,

$$\begin{array}{lll} a,\ b,\ c,\ \alpha, & b,\ c,\ a,\ \beta, & c,\ a,\ b,\ \gamma, \\ b,\ c,\ \alpha,\ \beta, & c,\ a,\ \beta,\ \gamma, & a,\ b,\ \gamma,\ \alpha, \\ c,\ \alpha,\ \beta,\ \gamma, & a,\ \beta,\ \gamma,\ \alpha, & b,\ \gamma,\ \alpha,\ \beta, \\ & & \\ b,\ a,\ c,\ \beta, & a,\ c,\ b,\ \alpha, & c,\ b,\ a,\ \gamma, \\ a,\ c,\ \beta,\ \alpha, & c,\ b,\ \alpha,\ \gamma, & b,\ a,\ \gamma,\ \beta, \\ c,\ \beta,\ \alpha,\ \gamma, & b,\ \alpha,\ \gamma,\ \beta, & a,\ \gamma,\ \beta,\ \alpha. \end{array}$$

If, then, taking any one of the above schemes we draw a plane through the centres* of the three pyramids of which it is composed, the six planes thus drawn will meet in a point, which will be the centre of the frustum†.

Let the point in which αa, βb, γc meet when produced be the origin of coordinates, and $bc\beta\gamma$, $ca\gamma\alpha$, $ab\alpha\beta$ be taken as the planes of x, y, z; and let $4a, 0, 0$; $0, 4b, 0$; $0, 0, 4c$ be the coordinates of a, b, c, and $4\alpha, 0, 0$; $0, 4\beta, 0$; $0, 0, 4\gamma$ those of α, β, γ. Consider the first of the schemes above written.

$$\begin{array}{llll} a+\alpha, & b, & c & \text{will be the coordinates of the centre of } abc\alpha, \\ \alpha, & b+\beta, & c & \quad ,, \qquad ,, \qquad ,, \qquad bc\alpha\beta, \\ \alpha, & \beta, & c+\gamma & \quad ,, \qquad ,, \qquad ,, \qquad c\alpha\beta\gamma; \end{array}$$

because, as everyone knows, the centre of a pyramid is the same as that of its angles regarded as of equal weight. But again, if we define as the mid-centre the centre of the six angles of the frustum regarded as of equal weight, its coordinates will be

$$\frac{2a+2\alpha}{3}, \quad \frac{2b+2\beta}{3}, \quad \frac{2c+2\gamma}{3};$$

and if we substitute for each of the three centres last named points lying respectively in a right line with them and the mid-centre on the opposite side of the mid-centre and at distances from it double those of these centres themselves, these quasi-images of the centres in question will have for their coordinates

$$\begin{array}{lll} 0\ , & 2\beta, & 2\gamma, \\ 2a, & 0\ , & 2\gamma, \\ 2a, & 2b, & 0\ . \end{array}$$

These points are accordingly the centres of the lines $\beta\gamma$, γa, ab respectively.

And a similar conclusion will apply to each of the six schemes. Hence using in general (p, q) to mean the middle of the line p, q, and by the

* I shall throughout in future for greater brevity hold myself at liberty to use the word *centre* to mean *centre of gravity*.

† I shall hereafter show that these six planes all touch the same cone, of which, as also of its polar reciprocal, I have succeeded in obtaining the equations.

collocation of the symbols for three points understanding the plane passing through them, it is clear

1. That the six planes,

$$(\beta, \gamma);\ (\gamma, a);\ (a, b);\quad (\gamma, \alpha);\ (\alpha, b);\ (b, c);\quad (\alpha, \beta);\ (\beta, c);\ (c, a);$$
$$(\gamma, \beta);\ (\beta, a);\ (a, c);\quad (\alpha, \gamma);\ (\gamma, b);\ (b, a);\quad (\beta, \alpha);\ (\alpha, c);\ (c, b),$$

will meet *in a single point* which may be called the *cross-centre,* being the true analogue of the intersection of the two diagonals of a quadrilateral figure in the plane.

2. That if we join this cross-centre (say Q) with O the mid-centre, and produce QO to G making $OG = \frac{1}{2}QO$, G will be the centre of the frustum $abc\alpha\beta\gamma$.

It may be satisfactory to some of my readers to have a direct verification of the above.

Let, then,

$$A = \frac{a^2bc - \alpha^2\beta\gamma}{abc - \alpha\beta\gamma}, \quad B = \frac{ab^2c - \alpha\beta^2\gamma}{abc - \alpha\beta\gamma}, \quad C = \frac{abc^2 - \alpha\beta\gamma^2}{abc - \alpha\beta\gamma}.$$

A moment's reflection will serve to show that A, B, C are the coordinates of the centre of the frustum.

Again, the first three of the six planes last referred to will be found to have for their equations respectively,

$$\beta\gamma x + \gamma a y + abz = 2a\gamma(b + \beta),$$
$$bcx + \gamma\alpha y + \alpha bz = 2b\alpha(c + \gamma),$$
$$\beta cx + cay + \alpha\beta z = 2c\beta(a + \alpha).$$

The determinant

$$\begin{vmatrix} \beta\gamma, & \gamma a, & ab \\ bc, & \gamma\alpha, & \alpha b \\ \beta c, & ca, & \alpha\beta \end{vmatrix} = (abc - \alpha\beta\gamma)^2.$$

The determinant

$$\begin{vmatrix} \gamma a, & ab, & 2a\gamma(b + \beta) \\ \gamma\alpha, & \alpha b, & 2b\alpha(c + \gamma) \\ ca, & \alpha\beta, & 2c\beta(a + \alpha) \end{vmatrix}$$
$$= 2\alpha a(bc - \beta\gamma)(abc - \alpha\beta\gamma),$$
$$= 2\{(\alpha^2\beta\gamma - a^2bc)(abc - \alpha\beta\gamma) + (a + \alpha)(abc - \alpha\beta\gamma)^2\}.$$

Hence if x, y, z be the coordinates of the intersection of the above-mentioned three planes,

$$x = -2A + 2(a + \alpha),$$
$$y = -2B + 2(b + \beta),$$
$$z = -2C + 2(c + \gamma);$$

and the same will evidently be true of the other ternary system of planes; so that all six planes intersect in a single point Q, of which x, y, z above written are the coordinates. And the coordinates of O being

$$\frac{2a+2\alpha}{3}, \quad \frac{2b+2\beta}{3}, \quad \frac{2c+2\gamma}{3},$$

and those of G being

$$A, \quad B, \quad C,$$

it is obvious QOG is a right line, and $OG = \frac{1}{2}QO$, as was to be shown.

The analogy with the quadrilateral does not end here. There is a construction* for the centre of a quadrilateral still easier than that above cited, which may be expressed in general terms by aid of a simple definition. Agree to understand by the *opposite* to a point L on *a limited line* AB a point M, such that L and M are at equal distances from the *centre* of AB but on opposite sides of it; then we may affirm that the centre of a quadrilateral is the centre of the triangle whose apices are the intersection of its two diagonals (that is, the cross-centre), and the *opposites* of that intersection on those two diagonals respectively. So now if we agree to understand by opposite points on a limited triangle two points in a line with the centre of the triangle and at equal distances from it on opposite sides, and bear in mind that the cross-centre of a pyramidal frustum is the intersection of *either* of two distinct ternary systems of triangles which may be called the two systems of cross-triangles†, we may affirm that the centre of a pyramidal frustum is the centre of a pyramid whose apices are its cross-centre, and the opposites of that centre on the three components of either of its systems of cross-planes. This is easily seen; for if we take the first of the two systems, their respective centres will evidently be

$$\begin{array}{ccc} \dfrac{4a}{3}, & \dfrac{2b+2\beta}{3}, & \dfrac{4\gamma}{3}, \\[2ex] \dfrac{4\alpha}{3}, & \dfrac{4b}{3}, & \dfrac{2c+2\gamma}{3}, \\[2ex] \dfrac{2a+2\alpha}{3}, & \dfrac{4\beta}{3}, & \dfrac{4c}{3}. \end{array}$$

* This is the mode of statement (except that the important notion of opposite points was not explicitly contained in it) which, accidentally meeting my eye in a proof sheet of some Geometrical Notes (by an anonymous author) intended for insertion in the forthcoming (if not forthcome) Number of the *Quarterly Journal of Mathematics*, led to the long train of reflections embodied in this paper, which but for that casual glance would never have seen the light. The same construction, under another and somewhat less eligible form, is given in the *Mathematician* (a periodical now extinct, edited by Dr Rutherford and Mr Fenwick, both of the Royal Military Academy), 1847, Vol. II. p. 292, and is therein stated by the latter gentleman to have, "as he believes, first appeared in the *Mechanics' Magazine*, and subsequently in the *Lady's Diary* for 1830."

† From the description given previously, it will be seen that *a cross-triangle* of the frustum is one which has its apices at the centres of either diagonal of any quadrilateral face and of the two edges coterminous but not in the same face with that diagonal.

Thus the three opposites to the cross-centre whose coordinates are

$$-2A+2(a+\alpha),\quad -2B+2(b+\beta),\quad -2C+2(c+\gamma),$$

will have for their x coordinates

$$\frac{2a}{3}-2\alpha+2A,$$

$$-2a+\frac{2\alpha}{3}+2A,$$

$$-\frac{2a}{3}-\frac{2\alpha}{3}+2A\,;$$

for their y coordinates

$$\frac{2b}{3}-2\beta+2B,$$

$$-2b+\frac{2\beta}{3}+2B,$$

$$-\frac{2b}{3}-\frac{2\beta}{3}+2B\,;$$

and for their z coordinates

$$\frac{2c}{3}-2\gamma+2C,$$

$$-2c+\frac{2\gamma}{3}+2C,$$

$$-\frac{2c}{3}-\frac{2\gamma}{3}+2C\,;$$

and consequently the centre of the pyramid whose apices are the cross-centre and its three opposites will be A, B, C, that is, will be the centre of gravity of the frustum, as was to be shown*.

* I at one time supposed that a, b, c; α, β, γ formed two systems of *diagonal* planes, and that there were thus two cross-centres; and dreamed a dream of the construction for the centre of gravity of the pyramidal frustum based upon this analogy, inserted (it is true as a conjecture only) in the *Quarterly Journal of Mathematics*; but the nature of things is ever more wonderful than the imagination of men's minds, and her secrets may be won, but cannot be snatched from her. Who could have imagined *à priori* that for the purposes of this theory a diagonal of a quadrilateral was to be viewed as a line drawn through two opposite angles of the figure regarded, not as themselves, but as their *own centres of gravity*! Some of my readers may remember a signal case of a similar autometamorphism which occurred to myself in an algebraical inquiry, in which I was enabled to construct the canonical form of a six-degreed binary quantic from an analogy based on the same for a four-degreed one, by considering the *square* of a certain function which occurs in the known form as consisting of two factors, one the function itself, the other a function morphologically derived from, but happening for that particular case to coincide with the function. This parallelism is rendered more striking from the fact of 4 and 6 being the *numbers* concerned in each system of analogies, those numbers referring to degrees in the one theory and to angular points in the other. It is far from improbable that they have their origin in some common principle, and that so in like manner the parallelism will be found

It is clear that these results may be extended to space of higher dimensions. Thus in the corresponding figure in space of four dimensions bounded by the hyperplanar quadrilaterals *abcd*, $\alpha\beta\gamma\delta$, which will admit of being divided into four hyperpyramids in twenty-four different ways, all corresponding to the type

$$\begin{array}{ccccc} a, & b, & c, & d, & \alpha, \\ b, & c, & d, & \alpha, & \beta, \\ c, & d, & \alpha, & \beta, & \gamma, \\ d, & \alpha, & \beta, & \gamma, & \delta, \end{array}$$

there will be a cross-centre given by the intersection of any four out of twenty-four hyperplanes resoluble into six sets of four each,—one such set of four being given in the scheme subjoined, where in general *pqr* means the point which is the centre of (p, q, r) and the collocation of four points means the hyperplane passing through them, namely,

$$\begin{array}{cccc} \beta\gamma\delta, & \gamma\delta a, & \delta ab, & abc, \\ \gamma\delta\alpha, & \delta\alpha b, & \alpha bc, & bca, \\ \delta\alpha\beta, & \alpha\beta c, & \beta cd, & cdb, \\ \alpha\beta\gamma, & \beta\gamma d, & \gamma da, & dac. \end{array}$$

The mid-centre will mean the centre of the eight angles $a, b, c, d, \alpha, \beta, \gamma, \delta$, regarded as of equal weight; and to find the centre of the hyperpyramidal frustum, we may either produce the line joining the cross-centre with the mid-centre through the latter and measure off three-fifths of the distance of the joining line on the part produced (as in the preceding cases we measured off two-fourths and one-third of the analogous distance), or we may take the four opposites of the cross-centre on the four components of any one of the six systems of hyperplanar tetrahedrons of which it is the intersection, and find the centre of the hyperpyramid so formed. The point determined by either construction will be the centre of gravity of the hyperpyramidal frustum in question. And so on for space of any number of dimensions. It will of course be seen that a general theorem of determinants* is contained

to extend in general to any quantic of the degree $2n$, and the corresponding barycentric theory of the figure with $2n$ apices (n of them in one hyperplane and n in another), which is the problem of a hyperpyramid in space of n dimensions. The probability of this being so is heightened by the fact of the barycentric theory admitting, as is hereafter shown, of a *descriptive* generalization, descriptive properties being (as is well known) in the closest connexion with the theory of invariants. Much remains to be done in fixing the canonic forms of the higher even-degreed quantics; and this part of their theory may hereafter be found to draw important suggestions from the hyper-geometry above referred to, if the supposed alliance have a foundation in fact.

* We learn indirectly from this how to represent under the form of determinants of the ith order, and that in a certain number of different ways, the general expressions

$$(l_1 l_2 \ldots l_i - \lambda_1 \lambda_2 \ldots \lambda_i)^{i-1}$$

and
$$l_1 \lambda_1 (l_2 l_3 \ldots l_i - \lambda_2 \lambda_3 \ldots \lambda_i)(l_1 l_2 \ldots l_i - \lambda_1 \lambda_2 \ldots \lambda_i)^{i-2},$$

in the assertion that for space of n dimensions there will be $n!$ quasi-planes all intersecting in the same point, as also in the general relation connecting this point (the cross-centre) with the mid-centre and centre of gravity, of each of which it is easy to assign the value of the coordinates in the general case.

But returning to the case of the ordinary pyramidal frustum, the preceding results lead at once to an easy geometrical proof of the well-known analytical formula for finding the centre of gravity of a pyramidal frustum in the case where the base and its opposite plane are parallel.

As we know that the centre of gravity in this case is in the line joining the centres of the opposite faces, what is wanted here is merely the proportion of the segments into which this joining line is divided at the centre in question, or, in other words, the ratio to each other of the distances of the centre from the parallel faces.

Let $$ab : \alpha\beta = bc : \beta\gamma = ca : \gamma\alpha = l : \lambda.$$

Then obviously
$$\text{vol. } abc\alpha : \text{vol. } bc\alpha\beta = ab\alpha : b\alpha\beta = l : \lambda,$$
$$\text{vol. } bc\alpha\beta : \text{vol. } c\alpha\beta\gamma = bc\alpha : c\alpha\gamma = l : \lambda :$$

hence $$abc\alpha : bc\alpha\beta : c\alpha\beta\gamma = l^2 : l\lambda : \lambda^2;$$

also if h be the distance between abc, $\alpha\beta\gamma$, the distances of the centres of $abc\alpha$, $bc\alpha\beta$, $c\alpha\beta\gamma$ respectively from abc will be $\frac{h}{4}, \frac{h}{2}, \frac{3h}{4}$.

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a strange conclusion to be able to draw incidentally from a hyper-theory of centre of gravity! Thus, for example, on taking $i=4$, we shall find

$$\begin{vmatrix} bcd, & cda, & da\beta, & \alpha\beta\gamma \\ \beta\gamma\delta, & cda, & da\beta, & \alpha\beta\gamma \\ b\gamma\delta, & \gamma\delta a, & dab, & ab\gamma \\ bc\delta, & c\delta a, & \delta a\beta, & abc \end{vmatrix} = (abcd - \alpha\beta\gamma\delta)^3.$$

And again,

$$\begin{vmatrix} ad\,(bc+c\beta+\beta\gamma), & cda, & da\beta, & \alpha\beta\gamma \\ \beta a\,(cd+d\gamma+\gamma\delta), & cda, & da\beta, & \alpha\beta\gamma \\ \gamma b\,(da+a\delta+\delta a), & \gamma\delta a, & dab, & ab\gamma \\ \delta c\,(ab+ba+\alpha\beta), & c\delta a, & \delta a\beta, & abc \end{vmatrix} = a\alpha\,(bcd-\beta\gamma\delta)\,(abcd-\alpha\beta\gamma\delta)^2.$$

The number of these representations will not be twenty-four, that is, 4!, but only twelve, the half of that number, because it will easily be seen that the cycles $abcd$, $\alpha\beta\gamma\delta$ will lead to the same determinants, only differently arranged, as the cycles $bcda$, $\beta\gamma\delta a$. I believe the law is, that the number of varieties of such representations is $(i)!$, or $\frac{1}{2}(i)!$, according as i is odd or even. The expression $ab-\alpha\beta$ at once conjures up the idea of a determinant. We now see that there is an equally natural determinantive representation, or system of representations, of $(abc-\alpha\beta\gamma)^2$, $(abcd-\alpha\beta\gamma\delta)^3$, &c.

Hence the distance of the centre of the frustum from abc will be $\frac{h}{4}\left(\frac{l^2+2l\lambda+3\lambda^2}{l^2+l\lambda+\lambda^2}\right)$, and so from $\alpha\beta\gamma$ it will be $\frac{h}{4}\left(\frac{\lambda^2+2l\lambda+3l^2}{l^2+l\lambda+\lambda^2}\right)$, agreeing with the well-known formula applicable to this case*.

But I pass on to a subject of much deeper interest.

The geometrical constructions included in the preceding inquiry (such for instance as depend on the properties of *centres* and *opposites*), like those which occur in the more ordinary theory of the triangle and pyramid, at once suggest the existence of descriptive propositions in which harmonic centres and harmonic opposites, and in general harmonic multiplications and divisions, take the place of the corresponding arithmetical operations.

To make my meaning perfectly clear, let us conceive a fixed plane; and by a harmonic succession of points $A, B, C, D \ldots$ in a line meeting the fixed plane† (which we may term the plane of relation) in O, let us understand that $ABCO$, $BCDO$, &c. form so many harmonic systems of points; B may be then called a harmonic centre of AC, A and C opposites to B; also we may call AB, BC harmonic steps of the succession, so that by multiplying a line AB n times, or making AX equal to n times AB, we are constructing the point X to which A will be transferred by n harmonic steps, of which AB is the first; and by n-secting a line AX, we mean finding a point B in it such that a succession of n harmonic steps, commencing with AB, will carry A to X.

In all this there is of course nothing new: these principles are familiar to all geometers, and have received their fullest development at the hands of Professor Cayley. We know *à priori* that the descriptive properties included in the preceding (or similar) constructions, such, for example, as that the six cross-triangles of a frustum all meet in a point, will remain true when, adopting a fixed plane of relation, we substitute harmonic centres in respect to that plane in lieu of arithmetical centres‡. Or, again, we may affirm that

* If we agree to denote by a, b, c; α, β, γ, the planes $\alpha\beta\gamma$, $b\gamma a$, $ca\beta$; abc, βca, γab respectively, it may easily be shown that each quaternary system of planes a, b, α, β; b, c, β, γ; c, a, γ, α passes through a single point; we have thus given three points which determine a plane; the intersection of this plane with the line a, b, c; α, β, γ is a sort of centre to the frustum, and must possess properties deserving closer investigation.

† It will of course be understood that in dealing with figures lying in the same plane, a line of relation (namely, the intersection of the plane of relation with the plane of the figures) may be substituted instead of the former plane, since the distances from the one and the other are in an invariable ratio; and so for different segments in a right line, we may substitute a point of relation on the line itself instead of the plane. I deal with a plane of relation as comprising implicitly all the subordinate cases; were it required to go out into space of four or a higher number of dimensions, it would of course become necessary to deal with hyper-planes of relation.

‡ Geometers have long been familiar with the idea of the *pole* or *harmonic centre* of a triangle in respect to a line in its plane; the principles now about to be developed will enable us to attach a precise signification to the *pole* or *harmonic centre* of every geometrical figure of any form whatever.

the lines joining the harmonic centres of the opposite edges of a tetrahedron will all *intersect* and *harmonically* bisect each other, and so on. But what is further wanted, and what I will proceed to supply, is a firm quantitative basis to this enlarged theory, so formed as that we shall be able in the general case to follow step by step the reasoning used in the common theory where the plane of relation goes off to infinity, and to assign to every point determined in the general constructions as distinctive a character as it possesses in the special ones. This may be done by the aid of very elementary considerations, which I proceed to unfold, and which will be seen at once to bring the general or perspective theory under the dominion of the so-called integral calculus or calculus of continuity.

The arithmetical centre of two *points* A, B is the centre of gravity of two equal atoms at A and B; let us then so assign the weights of the atoms A, B in the general case as to make their centre of gravity fall on the harmonic centre: this may evidently be done by considering their weights as proportional to their inverse distances from the plane of relation, and accordingly we shall understand by the weight of an atom at any point a quantity proportional to its inverse distance from the plane of relation. But, moreover, the centre of gravity of the homogeneous line AB ought to fall at this same point, which we may if we please consider as an inference at the limit from the same thing being true for equal atoms at distances dividing the line into any even number of equal parts. Hence in the general analogical theory we must take the infinitesimal intervals of our atoms at points in *harmonic* succession.

Let P, Q, R be any three such points, and let x, $x + dx$, $x + 2dx + d^2x$ be their respective distances from the plane of relation; and let q be the *frequency* at P, that is a quantity proportional to the number of atoms which occur in a given infinitesimal space about P; then evidently qdx is constant, and $qd^2x + dxdq = 0$; but by virtue of the harmonic relation between P, Q, R, we have

$$(x + 2dx + d^2x)(dx) = x(dx + d^2x),$$

or

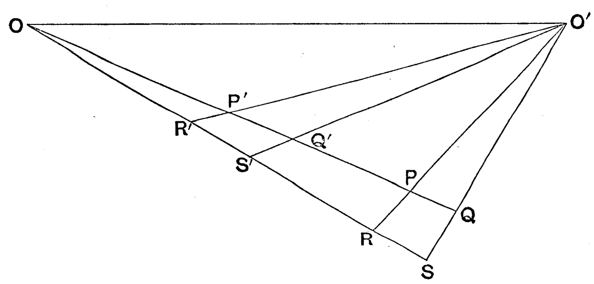
$$xd^2x = 2(dx)^2, \text{ or } -\frac{dq}{q} = 2\frac{dx}{x},$$

that is q varies as $$\frac{1}{x^2}.$$

Moreover the weight of each atom varies as $\frac{1}{x}$, hence the *density* of any element in a line must be taken to vary as the inverse cube of its distance from the plane of relation.

Let us now endeavour to obtain the law of density for any element of a *plane*. Let O, O' be any two points in the line in which the plane in

question meets the plane of relation, and let the plane be divided into infinitesimal elements similar to $PQSR$ in the figure by pencils whose rays are in harmonic succession proceeding from O and O'; then one atom belongs to



every such element, which will be the analogue of a rectangular element in the common theory; but the area of this element, as compared with any similar element, say $P'Q'S'R'$ in the infinite sector QOS, varies as

$$OP \,.\, RS + OR \,.\, PQ,$$

where PQ, RS, by what has been last shown, vary as the square of the distance of the element from the plane of relation, and OP, OR vary directly as the distance; hence the *frequency* of the atoms at any element in *either sector* will vary as the inverse cube of its distance from the plane of relation, and hence this will be the law of frequency for elements all over the plane, and is irrespective of the particular positions of O, O'; and consequently, the density being proportional to the product of the frequency of the atoms by their atomic weights, the law of density is that it varies about any point as the inverse *fourth* power of its distance from the plane of relation. In like manner, by taking three points O, O', O'' in the plane of relation and dividing space into solid elements by plane bundles passing through OO', OO'', $O'O''$ respectively, it may be proved that the law of density for a solid figure will be that it varies as the inverse *fifth* power of the distance from the plane of relation*.

Atoms whose weights vary inversely as their distances from the plane of relation may be termed *like* atoms; lines, areas, and solids whose elements vary in *density* inversely as the cubes, fourth powers and fifth powers respectively, may be termed *qualiform* figures, or figures of *qualiform* density, the terms *like* and *qualiform* being adopted as the closest analogues to *equal* and *uniform*. It now becomes true, and may easily be verified, that

* The law of density for a solid is the inverse fifth power, for an area the inverse fourth power, and for a line the inverse third power. Here we must stop, for a *point* is that which has no parts: we can speak of the law of atomic *weights* at a point, but not of density, for the latter implies the existence of *elements* which are wanting to the point. In a hyper-ontological sense there would be no objection to saying that for an element of a point the law of density in this theory is as the inverse square, always remembering that no such element exists.

the centres of gravity of a *qualiform* finite line, triangle, and tetrahedron are respectively identical with the centres of gravity of *like* atoms placed at their apices*; and so every known or discoverable theorem whatever relating to the centre of gravity of uniform figures bounded by right lines or planes becomes immediately transferable to that of *qualiform* figures of the same kind. Thus, to take a most simple example, since the centre of gravity of a parallelogram is at the intersection of its diagonals, it must be and is true that the centre of gravity of a quadrilateral whose density at any point varies as the inverse fourth power at that point from the line joining the intersections of its two pairs of opposite sides, will also be at the intersection of the diagonals of that figure. I am informed by Professor Cayley that a somewhat analogous consideration of altered density has been employed by our eminent friend Professor William Thomson in his theory of images, in reference to the distribution of electricity, given in *Liouville's Journal.*

* As regards the finite line, these results may be very easily verified by the integral calculus. For the triangle, it may be made to depend on the preceding case by drawing from the point where the direction of any side intersects the plane of relation, rays dividing the triangle into infinitesimal portions; the centre of gravity of every one such portion will easily be seen to be in the right line joining the harmonic centre of the intersecting side with the opposite angle; and an analogous method applies to the tetrahedron.

The same results may also be obtained analytically. Thus, for example, for a qualiform triangle whose apices are distant h, k, l from the opposite sides, and $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ from the plane of relation, the distances of the centre of gravity from the respective sides will be

$$\frac{h\alpha}{\alpha+\beta+\gamma}, \quad \frac{k\beta}{\alpha+\beta+\gamma}, \quad \frac{l\gamma}{\alpha+\beta+\gamma}.$$

The masses, say M, of a qualiform line, triangle, or tetrahedron, using α, β; α, β, γ; $\alpha, \beta, \gamma, \delta$ for the inverse distances of the apices from the plane of relation, and V for the length, area, or volume, in the three cases respectively become expressible under the very noticeable forms

$$\frac{\alpha+\beta}{2}\alpha\beta V, \quad \frac{\alpha+\beta+\gamma}{3}\alpha\beta\gamma V, \quad \frac{\alpha+\beta+\gamma+\delta}{4}\alpha\beta\gamma\delta V,$$

their moments in respect to the plane of relation being respectively

$$\alpha\beta V, \quad \alpha\beta\gamma V, \quad \alpha\beta\gamma\delta V;$$

so that the *mean* density $\frac{M}{V}$ is in each case a simple symmetric function of the atomic weights of the apices (it being of course understood that the *absolute* atomic weight and frequency are each taken as unity). As the same figure may be variously partitioned, and the sum of the component areas and of their moments is unaffected by the mode of partition, the preceding formulæ obviously give rise to, or imply the existence of, a class of purely geometrical theorems relating to systems of points. It may be here observed that the moment of a qualiform figure in respect to its plane of relation represents the size, so to say, of (that is, the number of atoms contained in) the single molecule which, placed at the centre of gravity, will be the statical equivalent of such figure; for if n be this number, and d the distance of the centre from the plane of relation, and w the weight of the figure, since the atomic weight is $\frac{1}{d}$, we must have $\frac{n}{d}=w$, or

$$n=dw=\text{moment of } w \text{ in respect to the plane of relation.}$$

So in like manner, wherever the plane of relation is situated, two molecules A and B, placed at two *points*, will be equivalent to the molecule $A+B$ placed at their centre of gravity.

It is an easy inference* from what has been established concerning the law of *frequency*, that if in the perspective of any plane figure, by tinting or relief, we express the degree of crowding of any element, and proportion the tint or elevation to the inverse *cube* of its distance from the vanishing line, then any portion of the picture will accurately represent (and indeed if we use relief, the *volume* or weight of such portion will be strictly proportional to) the area (or its weight) of the corresponding part in the object plane. Supposing different object planes to be represented in perspective on the same picture plane, with liberty for the position of the eye to vary, it may be shown without difficulty† that if the *absolute* intensity of tint or relief for any object plane varies as the square of the distance of its trace upon the picture plane from its vanishing line, and as the first power of the distance of the eye from the same line, the ratio between corresponding portions of object and picture will be alike for every plane.

In the corresponding problem for right lines, the relief or tint of any element in the perspective of a given right line must vary as the inverse square of the distance from the vanishing point, and the absolute intensity for different lines must vary as the product of the distance between the trace and the vanishing point into the distance of the eye from that point. In *barycentric* perspective we have seen the further consideration of atomic weight enters, so that the density follows the law of the inverse fourth and third powers for planes and lines respectively, instead of third and second powers as in geometrical perspective; in fact in the geometrical theory the quantities visibly represented correspond to the *moments*‡ in respect to

* It may here also incidentally be noticed that the area of the primitive of any perspective projection of a figure in a given plane is proportional to the *attraction* exercised upon it by the object plane indefinitely extended, the force of attraction between any two elements being supposed to vary inversely as the fifth power of the distance.

† For if we take T the trace of an object line, V its vanishing point, and through O (the eye) draw OPp meeting TV in P and the object line in p, Tp the quantity of $TP = \dfrac{\mu TP}{TV.PV}$, so that $\mu = TV\dfrac{Tp}{TP}PV = TV.OV$; and again, if tTt' be the trace of an object plane, V the foot of the perpendicular from O on the vanishing line VT perpendicular to tTt', P a point in VT, and p the point where OP meets the object plane, we have tpt' (the quantity of tPt') $= \mu\dfrac{tPt'}{TV.TV.PV}$, or

$$\mu = TV^2.\frac{tpt'}{tPt'}.PV = TV^2.\frac{Tp}{TP}.PV = TV^2.OV.$$

The preceding calculations assume the expressions $\mu\alpha\beta$, $\mu\alpha\beta\gamma$ applicable to a linear and triangular space, given in a preceding footnote.

‡ And consequently if, in the pictorial representation of any plane surface, there is taken a triangular patch of given area, the quantity in the object corresponding thereto will vary inversely as the product of the distances of the three angles of the patch from the vanishing line,—a proposition in perspective which I imagine to be new, and at all events is certainly little known. This may be applied to determine instantaneously the area of an ellipse of which the perspective projection is a circle of radius r, and whose centre is at the distance h from the vanishing line. Writing μ equal to the distance of the vanishing line from the eye, multiplied by the square of its

the vanishing line of the quantities visibly represented in the barycentric theory *.

I have termed this a theory of barycentric perspective, because it includes a method whereby the centre of gravity of a plane figure is retained in perspective with the centre of gravity of its projection; by what has pre-

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distance from the trace of the ellipse upon the plane of the circle, the area of the ellipse (regarded as made up of infinitesimal sectors with the centre of the projection for their common vertex) becomes

$$\int_0^{2\pi} d\theta \frac{\frac{1}{2}\mu r^2}{h(h - r\sin\theta)^2} = \frac{\mu\pi r^2}{h^3\left[1 - \left(\frac{r}{h}\right)^2\right]^{\frac{3}{2}}} = \frac{\mu\pi r^2}{(h^2 - r^2)^{\frac{3}{2}}};$$

so that the area of any ellipse in a *given* plane, the perspective representation of which ellipse is a circle, will vary directly as the area of the circle, and inversely as the cube of the tangent drawn to meet it from the orthogonal projection of its centre on the vanishing line. More generally, if the figure in the plane of projection be an ellipse with semiaxes a, b, eccentricity e, inclination of minor axis to vanishing line α, and distance of one of its foci from that line h, then calling V the area of the primitive and μ the absolute ratio between a primitive element and its projection, we shall have

$$V = \frac{\mu}{2h}\int_0^{2\pi} d\theta \frac{r^2}{(h - r\sin\theta)^2}, \text{ where } r = \frac{a(1 - e^2)}{1 + e\sin\theta}.$$

This integration may be performed with extreme facility, and gives

$$V = \mu\pi ab\,[h^2 + 2hea\cos\alpha - a^2(1 - e^2)]^{-\frac{3}{2}},$$

say

$$\frac{\mu}{D^3}\pi ab,$$

where to find D we may use the following construction:—*Draw a circle in the plane of, and concentric with, the projection, and such that a common tangent to the two shall be parallel to the vanishing line, and from the foot of the perpendicular upon that line from the centre draw a tangent to the circle, the length of the tangent so drawn will be* D; so that the area of any ellipse will be to the area of its perspective projection as the product of the square of the distance of the trace into that of the eye from the vanishing line is to the cube of the tangent just described,—a very remarkable proposition in perspective, if new. By varying the origin of our polar coordinates, as by taking it, for instance, at the centre of the projection or any other point, we may obtain a new class of definite integrals of known values, and which it might be exceedingly difficult to determine by any direct method. It may be added that all ellipses in the same plane will bear a constant ratio to their projections if these latter have a common tangent parallel to the vanishing line, and their centres be in another line also parallel to the same.

* The above statements, combined with the varying law of *frequency*, amount to the following propositions in perspective:—

1. If O be a *linear* element, P its perspective representation, H, h the distances of the eye and P from the line of O, and d of the eye from the line of P, then

$$O : P :: dH : (H - h)^2.$$

2. If O be a *plane* element, P its perspective, H, h the distances of the eye and P from the plane of O, and d the distance of the eye from the plane of P, then

$$O : P :: dH^2 : (H - h)^3.$$

These formulæ would become necessary in applying (as might be done perhaps advantageously) in some cases the integral calculus to the quantification of curved lines and surfaces by a perspective method more general than the one in ordinary use, which is essentially a method of orthogonal projection.

ceded, it appears that this may be effected by regarding its projection, not as of uniform density, but of a density following the law of the inverse cube of the distance. From this it follows that the distance of the perspective position in the picture of the centre of gravity of the primitive from the vanishing line becomes immediately known by a process of differentiation when the area of the primitive is expressed as a function of the distance of any arbitrarily fixed point in the plane of projection from the vanishing line. For if this area, which is the moment of the qualiform projection in respect to the vanishing line, be called M, and the mass of the same be termed Q, and if h, d be the distances of the origin and of the centre of gravity from the vanishing line, we have $d = \frac{M}{Q}$, where

$$M = \mu \int_0^{2\pi} \frac{r^2 d\theta}{(h - r\sin\theta)^2 h},$$

$$Q = \frac{1}{3}\mu \int_0^{2\pi} \frac{r^2 d\theta}{(h - r\sin\theta)^2 h}\left(\frac{1}{h - r\sin\theta} + \frac{1}{h - r\sin\theta} + \frac{1}{h}\right);$$

hence $$Q = -\frac{1}{3}\frac{dM}{dh},$$

and $$d = \frac{\frac{1}{3}M}{\frac{dM}{dh}}.$$

Thus, for example, if we wish to find the perspective position of the centre of gravity of the primitive of a given elliptic projection, we have found in a preceding footnote,

$$M = \mu (h^2 + 2hae\cos\alpha + a^2e^2 - a^2)^{-\frac{3}{2}};$$

hence $$d = \frac{h^2 + 2hae\cos\theta + a^2e^2 - a^2}{h + ae\cos\alpha};$$

or, calling R the radius of the circle concentric with the given projection, and having with it a common tangent parallel to the vanishing line, and H the distance of the centre of this circle from that line, $d = \frac{H^2 - R^2}{H}$, an equation the geometrical interpretation whereof is readily obtained.

More generally, if we take $x\cos\alpha + y\sin\alpha - h = 0$ as the equation to the vanishing line, using, as before, M to denote the moment of the qualiform projection in respect to that line (well worthy in this theory of being termed the principal moment), or, which is the same thing, the area of the primitive, and take M_x for the moment of the same in respect to the axis of y, we shall have

$$M = \iint \frac{dx\,dy}{(x\cos\alpha + y\sin\alpha - h)^3},$$

$$M_x = \iint \frac{dx\,dy\,x}{(x\cos\alpha + y\sin\alpha - h)^4},$$

from which it is easy to deduce

$$M_x = \cos\alpha\left(M + \frac{1}{3}h\frac{d}{dh}M\right) + \frac{1}{3}\sin\alpha\frac{d}{d\alpha}M;$$

and consequently $\dfrac{M_x}{Q} - h\cos\alpha$, which is the distance of the perspective of the centre of gravity of the primitive in the direction of x from the foot of the perpendicular from the assumed origin upon the vanishing line, will be

$$\frac{3\cos\alpha\,.\,M + \sin\alpha\dfrac{dM}{d\alpha}}{\dfrac{dM}{dh}}.$$

And thus we are led to the remarkable proposition, that when we know the area of the primitive in terms of the parameters of its vanishing line, we can completely determine the perspective position of its centre of gravity by means of processes of differentiation only; so that a method closely akin to (if not identical with) that of potentials in the theory of attraction has a necessary place also in the theory of perspective.

If, as is most convenient, we fix the perspective of the centre of gravity of the object figure by its distance from the vanishing line and its distance from the line through the origin perpendicular to the vanishing line, we see, by making α successively zero and $\frac{1}{2}\pi$ in the above formula, that these distances are $\dfrac{3M}{\dfrac{d}{dh}M}$ and $\dfrac{\dfrac{d}{d\alpha}M}{\dfrac{d}{dh}M}$ respectively*. Analogous results may be obtained for

* In the case of the ellipse, we have found in a preceding footnote,

$$M = \mu\,(h^2 + 2aeh\cos\alpha + a^2e^2 - a^2)^{\frac{3}{2}},$$

so that

$$\frac{3M}{\dfrac{d}{dh}M} = \frac{h^2 + 2eah\cos\alpha + a^2e^2 - a^2}{h + ea\cos\alpha} = y,$$

$$\frac{\dfrac{dM}{d\alpha}}{\dfrac{dM}{dh}} = \frac{ea\sin ah}{h + ea\cos\alpha} = x,$$

where y and x are the coordinates of the point referred to in the text, if we take the vanishing line and a line perpendicular thereto from the focus for the axes of x and y. Consequently, if we remove the origin of coordinates to the centre of the ellipse, preserving the directions of the axes, and call x', y' the new coordinates, we shall have

$$x' = ae\sin\alpha - x = \frac{a^2e^2\sin\alpha\cos\alpha}{h + ae\cos\alpha},$$

$$y' = h + ae\cos\alpha - y = \frac{a^2[1 - e^2(\sin\alpha)^2]}{h + ae\cos\alpha},$$

$$\frac{y'}{x'} = \frac{1 - e^2(\sin\alpha)^2}{e^2\sin\alpha\cos\alpha},$$

solid figures, substituting the more general notion of homography for that of perspective, as will more fully appear in the sequel.

Remembering that M is the area of the primitive plane object, it seems to result as an indirect inference from the preceding theory, that whenever we can determine the area of an oval section (whether the bounding curve be the whole or a part of the curve of section) of an algebraical cone, then we can determine the position of the centre of gravity of that oval in its own plane by processes of differentiation only; and, *mutatis mutandis*, the same conclusion will admit of extension to solids bounded by algebraical surfaces; so that $\iint dx\,dy$ or $\iiint dx\,dy\,dz$ being given, subject to certain conditions of limit, $\iint (ax+by)\,dx\,dy$, $\iiint (ax+by+cz)\,dx\,dy\,dz$, subject to the same conditions, become known by algebraical and differentiation processes only, and so obviously for any number of variables*.

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which may easily be shown to be the equation to the diameter drawn to the point of the ellipse where the tangent is parallel to the vanishing line; and consequently the perspective of the centre of gravity of the original lies in this diameter, as evidently it ought to do, since every infinitesimal slice of the *qualiform* area contained between parallels to the vanishing line is of uniform density throughout, and is bisected by the diameter conjugate to the direction of that line.

* The inference made hesitatingly in the text, upon further reflection appears to me perfectly clear, and will become so, I think, to the reader with the aid of a few words of explanation.

Let Q be a closed curve of the kind supposed lying in a plane which will be treated as a constant plane of projection; and for greater simplicity, and in order to steady the ideas, imagine that the vanishing plane (meaning thereby the plane passing through the eye and the vanishing line), and the plane of the object to be put in perspective, are retained at a constant distance from each other and always perpendicular to the picture plane, and also that the height of the eye above the vanishing line is invariable. Take any fixed line and point in the picture Q, and determine the equation to the curve boundary of its primitive O corresponding to a given distance h between the fixed point and the variable vanishing line and to a given angle of inclination α between the fixed line and this variable line. Then by hypothesis the area of O, say M, is known in terms of its coefficients, which will be known functions of α and h; hence $\frac{dM}{d\alpha}$ and $\frac{dM}{dh}$ are known, and consequently the position of the perspective of the centre of gravity of O on the picture is known; and from this the position of that centre in its own plane can be constructed, and therefore will have been found by aid of algebraical and differentiation processes only, as was to be shown.

The above explanation may be made still more distinct if we suppose that we *begin* with an object Ω (the curve for which is expressed by an equation in its most general form), wherein we have, say, $\alpha=0$ and $h=1$; that from this we deduce the equation of P in the preceding investigation, and from P pass to O as before; then, having found the coordinates of the perspective of the centre of gravity of O as functions of h and α, make $\alpha=0$, $h=1$, and pass back to the coordinates of the centre of gravity in Ω, of which the centre of gravity last named then becomes the perspective.

66.

NOTE ON A THEOREM OF THE INTEGRAL CALCULUS.

[*Philosophical Magazine,* XXVI. (1863), pp. 293, 294.]

I PROPOSE briefly to lay before the mathematical readers of the Magazine a wide generalization, and at the same time a more precise statement, of the theorem contained at the close of my paper in the last Number. The theorem, as therein enunciated, was drawn from geometrical considerations, it having first manifested itself dimly to the author by a sort of indirect reflection from a metrical theory of perspective. I have since obtained a very easy proof of it in its extended form, which in spirit amounts to a free algebraical paraphrase of the method indicated in the final footnote of the paper in question. The ultimate form of the perfected theorem is particularly interesting from its simplicity of application, and from its connexion with the grand and growing theory of invariants. The proof of it will appear in its proper place in the continuation of the paper in which, in its incipient state, it first came to light*.

Theorem.—Let a figure, whether plane, solid, or hyperspatial, be supposed to be limited by a locus or loci defined by one or more algebraical equations, not necessarily the most general of their respective degrees, but each at least the most general of its degree and kind†, and let the density at any point of the figure be any *homogeneous* function of the coordinates, and let the mass of such figure be supposed to be known in terms of the constants which enter into the defining equations; next let the density at each point of the mass be multiplied by a new factor, which may be any rational integral homogeneous function of the coordinates. Then the theorem affirms that the expression

* Strange cradle this for the inception of a quasi-invariantive theory of integration, "A geometrical construction of the centre of gravity of a truncated pyramid"! Où la vérité va-t-elle se nicher?

† By *kind* I mean descriptive character, that is such character as is not affected by perspective or homographical deformation. Thus, for example, the case of a cone may be treated apart from the more general case of a surface of the second degree. So, again, a curve of the third degree with a multiple point, or having one or both of its fundamental invariants zero, may be treated apart from the case of a general cubic curve.

for the new mass may be obtained by operating upon the expression for the original one with differential operators precisely identical with combinations of certain of those which serve to define an invariant of the given system of equations, and which will be found set forth in my paper "On the Calculus of Forms," in* the *Cambridge and Dublin Mathematical Journal*†. Thus, for example, by means of the known expressions for the area or content of a triangle, ellipse, pyramid, ellipsoid, or cone, this theorem enables us by differentiation and algebraical processes alone to obtain the parameters which define the centres of gravity, moments of inertia, principal axes, &c., of such figures.

I must add an important observation, namely, that the theorem remains true when one of the defining equations (supposing there to be more than one), instead of being the most general of a certain degree and kind, is affected with arbitrary numerical coefficients (zeros or others), provided only that it be *homogeneous* in the variables. Again, the theorem continues to hold when the original density, instead of being a homogeneous function of the variables, is such function multiplied by any Covariant of the defining equations taken separately or in groups—using the word covariant in its most extended sense, so as to comprehend fractional and irrational as well as integral forms,—the only effect of the introduction of such new factor into the density being to modify the form of the differential operators. There are certain very special cases, to which it is not necessary to allude here in detail, in which the theorem becomes illusory: such will be the case, for example, for a plane area when the given density is a homogeneous function in the variables of the negative degree 3, and for a solid content when that density is of the negative degree 4‡.

* [Volume I. of this Reprint, p. 356.]

† The partial differential equations for invariants, covariants, and contravariants will be found therein stated with absolute generality for any number of functions and any number of variables. Dr Aronhold, in the last Number of *Crelle's Journal*, states erroneously that these equations were given by me for binary functions only, and subsequently generalized by Cayley and Clebsch.

‡ A similar method applied to *extents* (as curves, surfaces, &c.) gives rise to curious theorems. Thus I find that the mass of a plane curve affected with a density varying at each point as the square of the cosine of the inclination of the curve to a fixed line, is a differential derivative of the length of the curve. So, again, the moment of inertia of a curve in respect to any axis perpendicular to its plane, is a differential derivative of its moment in respect to an arbitrary line in its plane.

67.

THÉORÈME SUR LA LIMITE DU NOMBRE DES RACINES RÉELLES D'UNE CLASSE D'ÉQUATIONS ALGÉBRIQUES.

[*Comptes Rendus de l'Académie des Sciences*, LVIII. (1864), pp. 494, 495.]

SOIENT $u_1, u_2, \ldots, u_n$ des fonctions linéaires d'une seule variable, à coefficients réels, et supposons qu'on ait l'équation

$$\lambda_1 u_1^{2i} + \lambda_2 u_2^{2i} + \ldots + \lambda_n u_n^{2i} = 0\,;$$

il est évident que si tous les coefficients $\lambda_1, \lambda_2, \ldots, \lambda_n$ portent les mêmes signes, le nombre des racines réelles est nul.

En général, supposons que le nombre des signes de même nom soit r, et de nom opposé soit s. Si r est égal ou moindre de s, on peut parler de r comme étant le nombre inférieur des signes semblables de la série $\lambda_1, \lambda_2, \ldots, \lambda_n$; et alors on peut affirmer que le nombre des racines réelles dans l'équation donnée ne peut jamais excéder le double du nombre inférieur de signes semblables dans ses coefficients λ.

Je crois que cette proposition est nouvelle, mais elle n'est qu'une conséquence très-particulière du théorème plus spécifique que voici:

Soient $c_1, c_2, \ldots, c_n$ une série croissante ou décroissante composée avec des quantités réelles, et soit donnée l'équation

$$\lambda_1 (x + c_1)^m + \lambda_2 (x + c_2)^m + \ldots + \lambda_n (x + c_n)^m = 0.$$

Formons la suite $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}, \lambda_n, (-1)^m \lambda_1$: je dis que le nombre des racines réelles dans l'équation donnée ne peut pas excéder *le nombre de variations de signe* dans cette suite, et comme corollaire on déduit aisément que ce nombre dans tous les cas ne peut pas excéder le double du nombre inférieur de signes semblables quand m est pair, ni ce double augmenté de l'unité quand m est impair.

Il est bon de remarquer que le maximum spécifique du nombre des racines réelles donné par la suite *déterminée* $\lambda_1, \lambda_2, \ldots, \lambda_n, (-1)^m \lambda_1$ *ne change pas* quand on transforme l'équation donnée en effectuant une substitution homographique réelle quelconque sur la variable x, de sorte qu'on peut dire que chaque *maximum spécifique* est un nombre jouant le rôle d'*invariant*, ce qui n'a pas lieu quand on se sert de la méthode ordinaire pour limiter le nombre des racines réelles de $fx = 0$, en considérant le nombre des racines imaginaires de $f'x = 0$.

68.

SUR UNE EXTENSION DE LA THÉORIE DES ÉQUATIONS ALGÉBRIQUES.

[*Comptes Rendus de l'Académie des Sciences,* LVIII. (1864), pp. 689—691.]

QUELQUES recherches que j'ai faites tout récemment sur la règle donnée sans démonstration par Newton dans l'*Arithmetica universalis* (voir le chapitre *De resolutione æquationum*), pour trouver une limite inférieure au nombre de racines imaginaires d'une équation, m'ont conduit forcément à reconnaître l'existence d'un nouveau et très-intéressant genre d'équations algébriques qui ont exactement le même degré de généralité que les équations ordinaires et jouissent de propriétés parfaitement analogues à celles de ces dernières.

Ce sont les équations pour lesquelles, en partant des deux extrémités de la fonction égalée à zéro, les coefficients se composent, deux à deux, de quantités conjuguées de la forme

$$\lambda + i\mu, \quad \lambda - i\mu$$

respectivement, sauf (pour les équations de degré pair) le coefficient central qui reste seul et nécessairement réel.

Une telle équation peut se mettre sous la forme

$$U + iV = 0,$$

et, en supposant que tout facteur algébrique commun à U et V a été préalablement chassé, elle peut être nommée équation conjuguée. Les équations conjuguées ainsi définies ne peuvent contenir ni racines réelles ni paires de racines imaginaires de la forme

$$\rho e^{i\theta}, \quad \rho e^{-i\theta};$$

mais néanmoins leurs racines, comme celles des équations ordinaires, se diviseront en deux classes, c'est-à-dire classe de racines solitaires et classe de racines associées. Ces deux classes seront chacune du même ordre de généralité. Les racines solitaires seront quantités complexes avec l'unité

pour module, c'est-à-dire de la forme $e^{i\theta}$; les racines associées seront quantités complexes dont le rapport est réel et les modules réciproques, c'est-à-dire de la forme

$$\rho e^{i\theta}, \quad \frac{1}{\rho} e^{i\theta}.$$

Il va sans dire que les racines solitaires sont les analogues aux racines réelles, et que les racines associées sont les analogues aux racines imaginaires des équations ordinaires. Dans une forme conjuguée du degré n, comme dans une forme ordinaire du même degré, le nombre de paramètres sera évidemment $n+1$. Tous leurs invariants (sauf le facteur i pour quelques-uns) seront réels, et toutes leurs formes, invariants des dérivées, covariants, contre-variants, etc., seront, elles aussi, des formes conjuguées.

Les théorèmes et les propriétés fondamentales des équations ordinaires se reproduisent (sans exception) sous une forme convenablement modifiée dans la théorie des équations conjuguées; je cite comme exemples la règle pour connaître si le nombre des racines réelles renfermées entre deux quantités réelles est pair ou impair, la liaison de position entre les racines réelles des équations et celles de leurs dérivées différentielles, les théorèmes pour reconnaître le nombre ou une limite au nombre des racines réelles, et en particulier la règle de Sturm et la règle merveilleuse et jusqu' aujourd'hui non démontrée de Newton. Je dois ajouter comme auxiliaire à ce genre de recherches un théorème qui donne une loi d'inertie pour les formes quadratiques (à un nombre quelconque de variables) assujetties à subir des substitutions qui peuvent être qualifiées comme étant substitutions conjuguées au lieu de réelles.

Il n'est pas sans intérêt de faire remarquer que, de même que les racines des équations ordinaires peuvent être représentées géométriquement au moyen de points solitaires situés sur une ligne droite, et par des points associés en couples qui se trouvent deux à deux et à distances égales sur les deux côtés de cette ligne, de sorte que ces derniers points constituent, pour ainsi dire, des images optiques les uns aux autres par rapport à la ligne, de même les racines géométriquement représentées des équations conjuguées se divisent en des points simples situés sur la circonférence d'un cercle dont le rayon est l'unité, et des points qui se trouvent deux à deux à des distances réciproques du centre sur les mêmes rayons, et qui constituent ainsi, pour me servir du langage de M. William Thomson, des images électriques les uns des autres. Ces principes auront prochainement leur développement dans un supplément au Mémoire sur le théorème de Newton déjà cité, que j'ai lu récemment devant la Société Royale de Londres.

69.

SUR UNE EXTENSION DE LA THÉORIE DES RÉSULTANTS ALGÉBRIQUES.

[*Comptes Rendus de l'Académie des Sciences,* LVIII. (1864), pp. 1074—1079.]

JE me propose de dire quelques mots sur une nouvelle classe très-bien définie d'invariants appartenant à l'ordre des combinants et admettant des applications importantes pour la Géométrie. Pour fixer les idées, imaginons un système de surfaces de degré quelconque chacune. Commençons avec le cas de quatre surfaces. En général, elles ne se rencontreront pas: pour que cela ait lieu, une condition doit être satisfaite entre les coefficients, ou, si l'on veut bien, une certaine fonction des coefficients des équations qui représentent ces surfaces doit s'évanouir.

Passons au cas de trois surfaces: ces surfaces s'entrecouperont dans un système de points qui en général seront tous-distincts. Mais il peut arriver que deux de ces points se confondent, c'est-à-dire que les trois surfaces se rencontreront en deux points consécutifs, ou, si l'on veut bien, seront toutes trois touchées par la même ligne droite; pour que cela ait lieu, une certaine fonction des coefficients doit s'évanouir, laquelle, pour le moment, manque de nom. Continuons en supprimant encore une surface. Les deux surfaces qui restent se couperont dans une courbe qui, en général, ne possédera aucune singularité. Mais il peut arriver que cette courbe possède un point double, dans lequel cas les deux surfaces seront touchées par le même plan. Pour que cela arrive, une certaine fonction des coefficients doit s'évanouir, à laquelle, comme exprimant la condition de tangence, notre grand géomètre M. Cayley a proposé de donner le nom de *tact-invariant.*

On peut exprimer sous une forme générale la nature des conditions analytiques qui doivent être satisfaites dans tous ces cas, et dans le cas le plus général où il y aura i fonctions $U_1, U_2, \ldots, U_i$ de n variables $x_1, x_2, \ldots, x_n$. Ecrivons

$$U_1=0, \quad U_2=0, \quad \ldots, \quad U_i=0,$$
$$\lambda_1\delta U_1+\lambda_2\delta U_2+\ldots+\lambda_i\delta U_i=0,$$

$\lambda_1, \lambda_2, \ldots, \lambda_i$ étant des quantités indéterminées. Puisque

$$\delta U = \frac{dU}{dx_1}\delta x_1 + \frac{dU}{dx_2}\delta x_2 + \ldots + \frac{dU}{dx_n}\delta x_n,$$

cette dernière équation donne lieu à $(n-i+1)$ équations indépendantes: donc le nombre total des équations homogènes à satisfaire avec les n variables sera

$$(n-i+1)+i=n+1\,;$$

pour que cela soit possible dans le cas général d'un tel nombre d'équations avec un tel nombre de variables, deux conditions entre les coefficients devraient être satisfaites; mais dans le cas actuel une seule sera suffisante, car il existera toujours un rapport syzygétique entre les équations. Dans le cas où il n'y a qu'une seule fonction U, l'équation $U=0$ devient tout à fait superflue, et dans le cas où $i=n$, l'équation $\Sigma\lambda\delta U=0$, qui exprime que la jacobienne des n fonctions est égale à zéro, devient également superflue. Mais dans tout autre cas, quoique en vertu de l'identité

$$U = \Sigma\left(x_1\frac{dU}{dx_1}\right)$$

il existe un rapport syzygétique entre les équations, il n'est pas permis de se passer d'une quelconque d'entre elles, sous peine d'introduire des facteurs étrangers dans l'expression finale. J'espère ne pas trop encourir l'indignation de mon très-honoré confrère M. Poncelet, en donnant un nom spécifique à la fonction dont l'évanouissement exprime la condition suffisante et nécessaire pour que ce système d'équations soit simultanément satisfait, et je propose de lui donner le nom, qui n'est pas tout à fait étranger à la Géométrie, d'*osculant*; ainsi on peut partir de l'osculant d'un système de i fonctions homogènes quelconques de n variables, et on voit que les discriminants, les *tact-invariants* de M. Cayley et les résultants ne sont que des espèces particulières des osculants: pour les discriminants $i=1$, pour les *tact-invariants* $i=2$, pour les résultants $i=n$.

Il importe beaucoup au développement de cette théorie de bien fixer le degré des osculants par rapport à chaque système de coefficients contenu dans les fonctions auxquelles ils appartiennent.

Pour les deux extrémités de l'échelle d'osculants, c'est-à-dire les discriminants et les résultants, les expressions pour ce degré sont très-simples et bien connues. Pour les *tact-invariants* le degré n'a été trouvé (je crois par M. Cayley) que pour le seul cas où $n=3$, c'est-à-dire pour les contacts des courbes. Le théorème suivant donne l'expression absolument générale pour les osculants de chaque ordre n et de chaque classe i.

Soient $m_1, m_2, \ldots, m_i$ les degrés des variables des i fonctions, et pour plus de simplicité écrivons

$$m_1 = 1+\mu_1, \quad m_2 = 1+\mu_2, \quad \ldots, \quad m_i = 1+\mu_i.$$

En général, soit $H_\omega(\mu_2, \mu_3, \ldots, \mu_i)$ la somme des puissances et des produits homogènes de $\mu_2, \mu_3, \ldots, \mu_i$, et soit G_k le degré de l'osculant du système par rapport aux coefficients de la fonction U_k. Alors je dis que

$$\frac{1}{m_2 m_3 \ldots m_i} G_1 = H_{n-i}(\mu_2, \mu_3, \ldots, \mu_i) + 2H_{n-i-1}(\mu_2, \mu_3, \ldots, \mu_i)\,\mu_1$$
$$+ 3H_{n-i-2}(\mu_2, \mu_3, \ldots, \mu_i)\,\mu_1^2 + \ldots$$
$$+ (n-i)\,H_1(\mu_2, \mu_3, \ldots, \mu_i)\,\mu_1^{n-i-1} + (n-i+1)\,\mu_1^{n-i},$$

et on trouve de même les valeurs de $G_2, G_3, \ldots, G_i$.

Pour les *tact-invariants* $i = 2$, et le théorème devient

$$G_1 = m_2[\mu_2^{n-2} + 2\mu_2^{n-3}\mu_1 + 3\mu_2^{n-4}\mu_1^2 + \ldots + (n-1)\,\mu_1^{n-2}],$$
$$G_2 = m_1[\mu_1^{n-2} + 2\mu_1^{n-3}\mu_2 + 3\mu_1^{n-4}\mu_2^2 + \ldots + (n-1)\,\mu_2^{n-2}],$$

ou, si l'on veut,

$$G_1 = m_2 \frac{\mu_2^n - n\mu_2\mu_1^{n-1} + (n-1)\,\mu_1^n}{(\mu_1 - \mu_2)^2},$$

$$G_2 = m_1 \frac{\mu_1^n - n\mu_1\mu_2^{n-1} + (n-1)\,\mu_2^n}{(\mu_1 - \mu_2)^2}.$$

Si $n = 3$,

$$G_1 = m_2[(m_2 - 1) + 2(m_1 - 1)] = m_2(m_2 + 2m_1 - 3), \qquad G_2 = m_1(m_1 + 2m_2 - 3):$$

c'est le cas du contact de deux courbes. Quand $n = 4$, c'est-à-dire qu'on veut trouver le degré de la condition pour le contact de deux surfaces, on trouve

$$G_1 = m_2(m_2^2 + 2m_1m_2 + 3m_1^2 - 4m_2 - 8m_1 + 4).$$

Pour trouver les degrés de la condition de rencontre en deux points consécutifs de trois surfaces, il faut prendre $i = 3$, $n = 4$; alors on trouve

$$G_1 = m_2m_3(m_2 + m_3 + 2m_1 - 4).$$

Pour le cas des polaires réciproques, on a

$$i = 2, \quad m_1 = m, \quad m_2 = 1,$$

et on retombe sur les résultats connus pour ce cas. Si on suppose dans le cas général $m_1 = m_2 = \ldots = m_i$, on obtient pour le degré de l'osculant, dans un système quelconque de coefficients,

$$\frac{n(n-1)\ldots(n-i+1)}{1\,.\,2\ldots i} m^{i-1}(m-1)^{n-i}.$$

Pour mettre en plein jour la véritable identité de nature de ce genre compréhensif des osculants, je ferai l'extension à une classe de ces fonctions d'un théorème bien connu pour le discriminant de deux fonctions.

On sait bien que le discriminant du produit de deux fonctions homogènes en x et y est égal au produit de leurs discriminants multiplié par le carré de leur résultant. Ainsi, en se servant de Ω comme le symbole universel de l'osculation et supposant F et F' ces deux fonctions, on peut écrire

$$\Omega (FF') = \Omega F \times \Omega F' \times [\Omega (F, F')]^2.$$

Remarquons bien qu'on ne peut pas étendre ce théorème dans sa forme actuelle à des fonctions de plus de deux variables, car quand F, F' sont des fonctions de 3 ou un plus grand nombre de variables, on a identiquement

$$\Omega (FF') = 0.$$

Or, considérons $F_1, F_2, \ldots, F_i, F_i', (i+1)$ fonctions de $(i+1)$ variables; j'énonce le théorème suivant:

$$\begin{aligned} &\Omega (F_1 F_2 \ldots F_{i-1} F_i F_i') \\ &= \Omega (F_1, F_2, \ldots, F_{i-1}, F_i) \times \Omega (F_1, F_2, \ldots, F_{i-1}, F_i') \\ &\times [\Omega (F_1, F_2, \ldots, F_i, F_i')]^2, \end{aligned}$$

où on peut remarquer que le dernier des trois facteurs est le carré d'un résultant. De plus, j'affirme que si les F deviennent fonctions de plus de $(i+1)$ variables, la quantité

$$\Omega (F_1 F_2 \ldots F_{i-1} F_i F_i')$$

s'évanouit identiquement. Mais je passe outre à un autre théorème sur les discriminants d'une fonction vue comme un *quantic* de quantics dont j'ai eu occasion de me servir dans quelques recherches récentes sur le théorème de Newton pour la découverte des racines imaginaires.

Soit F une fonction rationnelle homogène et entière du degré m en ϕ et ψ, ϕ et ψ étant elles-mêmes fonctions rationnelles homogènes et entières du degré μ en x et y. Servons-nous du symbole D pour désigner *discrimination* par rapport à x, y, et de D' pour désigner la même chose par rapport à ϕ, ψ; R sera le symbole du résultant par rapport à x, y, et J représentera la fonction *jacobienne*

$$\frac{d\phi}{dx}\frac{d\psi}{dy} - \frac{d\phi}{dy}\frac{d\psi}{dx}.$$

Alors je trouve que

$$D(F) = [R(\phi, \psi)]^{m^2-2m} [D'(F)]^{\mu} R(F, J).$$

Dans le cas où ϕ, ψ sont des fonctions linéaires de x, y, $R(F, J)$ devient égale à $[R(\phi, \psi)]^m$, et on retombe sur la formule connue pour les transformations linéaires

$$D_{x,y} F = [R(\phi, \psi)]^{m^2-m} D_{\phi,\psi} F.$$

Quand F est une fonction symétrique par rapport à x, y, F sera une fonction homogène et entière de (x^2+y^2) et de xy, dont la jacobienne a pour racines $\frac{x}{y}=\pm 1$, et conséquemment on voit que son discriminant prend la forme

$$I^2F(1,1).F(1,-1).$$

Or, pour généraliser le théorème, soient F_1, F_2, ..., F_{i-1}, des fonctions homogènes et entières des degrés m_1, m_2, ..., m_{i-1} des i quantités ϕ_1, ϕ_2, ..., ϕ_i, dont chacune est une fonction homogène et entière de degré μ en x_1, x_2, ..., x_i. Servons-nous de Ω pour exprimer osculation par rapport à $x_1, x_2, \ldots, x_i$, Ω' pour exprimer la même chose par rapport à ϕ_1, ϕ_2, ..., ϕ_i, de J pour exprimer la jacobienne de ϕ_1, ϕ_2, ..., ϕ_i par rapport à x_1, x_2, ..., x_i, et soit

$$M=(m_1+m_2+\ldots+m_{i-1}-i)(m_1m_2\ldots m_{i-1}).$$

Alors on aura l'identité suivante:

$$\Omega(F_1, F_2, \ldots, F_{i-1})$$
$$=[\Omega(\phi_1, \phi_2, \ldots, \phi_i)]^M.[\Omega'(F_1, F_2, \ldots, F_{i-1})]^{\mu^{i-1}}.\Omega(F_1, F_2, \ldots, F_{i-1}, J).$$

Il me semble qu'on peut reconnaître ici l'approche de la véritable aurore de cette science des formes dont on ne voit qu'une phase bornée et passagère dans la théorie des transformations linéaires. Les actions mutuelles des formes, les unes sur les autres, constituant une espèce de chimie algébrique, me paraît le vrai but de cette science naissante.

P.S. Il n'est pas inutile de remarquer qu'on peut donner une définition des osculants qui montre d'une manière immédiate leur identité avec les discriminants. Soient U_1, U_2, ..., U_i, i fonctions homogènes rationnelles et entières de n variables, et soit R le résultat de l'élimination de $(i-1)$ quelconques des variables entre les équations

$$U_1=0,\ U_2=0,\ \ldots,\ U_i=0.$$

Alors l'osculant du système donné de fonctions U sera contenu comme facteur dans le discriminant de R. De même on peut démontrer que si on combine ensemble $i-k$ des équations $U=0$ et si on prend $(k+1)$ de telles combinaisons, et si pour chaque combinaison on forme un résultant en éliminant les mêmes $i-k-1$ variables, l'osculant du système donné sera contenu comme facteur dans l'osculant de ces $(k+1)$ résultants.

70.

ADDITION À UNE NOTE INSÉRÉE DANS LE *COMPTE RENDU* DE LA SÉANCE PRÉCÉDENTE.

[*Comptes Rendus de l'Académie des Sciences*, LVIII. (1864), p. 1130.]

DANS la Note que j'ai eu l'honneur de soumettre à l'Académie sur une extension de la théorie des résultants algébriques, on trouve la formule générale pour le degré de l'*osculant* d'un système d'un nombre quelconque i de fonctions d'un nombre n quelconque de variables, et j'ai cité comme déjà connu le degré pour le cas de $n=3$, $i=2$, qui correspond à la condition de contact de deux courbes [p. 364].

Je dois citer en même temps comme également connus les degrés de l'osculant pour les cas de $n=4$, $i=2$, et de $n=4$, $i=3$, c'est-à-dire les cas qui correspondent à deux surfaces qui se touchent et à trois surfaces qui se rencontrent en deux points consécutifs.

Les degrés des conditions pour ces cas ont été donnés dans un excellent article par M. Th. Moutard, dans les *Nouvelles Annales de Mathématiques*, t. XIX. ce que j'ignorais au moment où j'ai écrit la Note en question.

71.

ADDITION À LA NOTE SUR UNE EXTENSION DE LA THÉORIE DES RÉSULTANTS ALGÉBRIQUES.

[*Comptes Rendus de l'Académie des Sciences*, LVIII. (1864), pp. 1178—1180.]

ON peut mettre la formule pour exprimer le degré d'un osculant de r fonctions homogènes de n variables sous une forme très-simple qu'il importe de signaler.

En sous-entendant toujours par $H_k(a, b, c, \ldots, l)$ la somme des puissances et des produits homogènes du degré k de $a, b, c, \ldots, l$, c'est-à-dire le coefficient de τ^k dans le développement en série de

$$\frac{1}{(1-a\tau)(1-b\tau)\ldots(1-l\tau)},$$

on verra sans aucune difficulté que la série donnée [p. 365 above] dans les *Comptes rendus* du 13 juin pour $\dfrac{1}{m_2 m_3 \ldots m_i} G_1$ n'est autre chose que la quantité

$$H_{n-i}(\mu_1, \mu_1, \mu_2, \ldots, \mu_i),$$

de sorte qu'on aura en général

$$m_\omega G_\omega = \Pi(m) \,.\, H_{n-i}[(m_1-1), (m_2-1), \ldots, (m_i-1), (m_\omega-1)],$$

où, dans la série écrite entre les crochets, $m_\omega - 1$ sera deux fois rencontré.

Pour les résultants, $(n-i)$ étant zéro, on trouve

$$m_\omega G_\omega = \Pi(m).$$

Pour les discriminants, $n-i=n-1$ et G devient égal à

$$H_{n-1}[(m-1), (m-1)] = n(m-1)^{n-1}.$$

J'apprends de la part de M. Salmon qu'il y a grand nombre d'années qu'il a trouvé le degré des osculants pour le cas de deux courbes; il paraît donc que je me trompais en attribuant cette détermination (qui de plus n'offre aucune difficulté) à M. Cayley.

72.

SUR LA THÉORIE DES RACINES RÉELLES ET IMAGINAIRES DES ÉQUATIONS DU CINQUIÈME DEGRÉ.

[*Comptes Rendus de l'Académie des Sciences*, LIX. (1864), pp. 749—753.]

On sait la découverte faite par M. Hermite et insérée dans le tome IX. du *Journal de Mathématiques de Cambridge et Dublin*. C'est là que M. Hermite a fait la belle observation, qu'aux conditions fournies par le théorème de Sturm on peut substituer des fonctions des invariants d'une forme binaire de degré impair quelconque, pour déterminer le nombre de ses racines réelles et imaginaires. De plus, M. Hermite, en suivant une marche toute particulière, a donné les *criteria* actuels, qui servent à peu près pour distinguer entre les trois cas qui se présentent dans la considération des formes du cinquième degré, c'est-à-dire le cas où toutes les racines sont réelles, celui où trois seulement sont réelles et le cas où il n'y a qu'une seule réelle. Cependant ce grand travail avait laissé quelque chose à désirer; car pour remplir cet objet, M. Hermite a été conduit à se servir de cinq invariants, un du degré 4, un (le discriminant) du degré 8 et trois chacun du degré 12, tandis que la méthode de M. Sturm n'exige que l'emploi de quatre *criteria*. De plus, le système de conditions donné par M. Hermite n'est pas absolument complet, mais laisse une certaine lacune à combler: je veux dire qu'il y a de certaines combinaisons de ses *criteria* pour lesquelles il reste douteux si la forme possède cinq ou bien une seule racine réelle; c'était une omission dont M. Hermite avait conscience et qu'il aurait sans doute trouvé le moyen de remplir. En me pénétrant de l'esprit de la méthode de M. Hermite, mais en suivant une tout autre voie d'application, je suis parvenu à trouver la solution la plus générale de ce problème important sous une forme d'une simplicité qui ne laisse rien à désirer, et à laquelle aucun cas n'échappe. Dans cette solution, au lieu d'excéder le nombre des *criteria* donnés par la méthode générale de M. Sturm, on se sert d'un de moins; en effet, en outre du discriminant, on n'a besoin que d'un invariant (le seul qui existe) du quatrième ordre et un du douzième ordre. Nommons D le discriminant de la forme

proposée, J le discriminant de son covariant quadratique le plus simple multiplié par -4, L le discriminant de son covariant cubique le plus simple multiplié par -27, et de plus écrivons

$$\Lambda = J^3 - 2^{11}L;$$

J, D, Λ suffisent pour déterminer le caractère des racines selon la règle suivante:

Quand D est négatif, trois racines sont réelles, deux imaginaires.

Quand D est positif, si J et $\Lambda + \mu JD$ sont tous les deux négatifs, les racines seront toutes réelles; dans le cas contraire, une seule sera réelle.

μ est un paramètre numérique variable à volonté entre certaines limites que j'ai trouvées, mais que je n'ose rapporter, n'ayant pas les calculs sous mes yeux. Je crois cependant pouvoir affirmer en toute sûreté que ces limites sont ou 1, -2, ou bien -1, 2. Avec ces mêmes *criteria* on peut aussi déterminer le caractère des racines dans le cas où D devient zéro, mais je n'entrerai pas ici dans ce détail.

La valeur $\mu = -\frac{4}{5}$ ne sort pas des limites permises, et on trouvera que $\Lambda - \frac{4}{5}JD$ s'exprime facilement en fonction des racines. Nommons-les a, b, c, d, e en désignant par K un certain coefficient numérique et positif, on aura

$$\tfrac{4}{5}JD - \Lambda$$
$$= K\Sigma\,[(a-b)^2(a-c)^2(b-c)^2(a-d)^4(a-e)^4(b-d)^4(b-e)^4(c-d)^4(c-e)^4].$$

De plus, en nommant q un autre multiplicateur numérique et positif, on aura

$$-J = q\Sigma\,[(a-b)^2(a-c)^2(b-c)^2(d-e)^4].$$

Posons $$Q(d, e) = (a-b)^2(a-c)^2(b-c)^2(d-e)^4.$$

Alors, pour distinguer entre le cas où il n'y a pas de racines imaginaires et le cas où il y en a quatre (les seuls qui se présentent quand D est positif), la règle donnée ci-dessus conduit à l'observation que si les racines ne sont pas toutes réelles et si D est positif, $\Sigma Q(d, e)$ et $\Sigma\,\dfrac{1}{Q(d, e)}$ ne peuvent pas rester tous les deux positifs. Dans le cas contraire il est évident que ΣQ et $\Sigma\,\dfrac{1}{Q}$ sont tous les deux positifs. J'ajouterai quelques mots sur la marche que j'ai suivie pour obtenir ces résultats. Je démontre qu'en général la forme $(x, y)^5$ peut être réduite par des substitutions linéaires et réelles à l'expression $au^5 + bv^5 + cw^5$, où w est une fonction linéaire et réelle de x, y; u, v des fonctions linéaires, mais pas nécessairement réelles, et où de plus $u + v + w = 0$. Le cas d'exception, c'est celui où le covariant cubique du troisième ordre par rapport aux coefficients (dit le *canonisant*)

contient des racines égales ou bien s'évanouit. Dans ce cas, sauf la supposition de trois racines égales et quand, conséquemment, tous les invariants s'évanouissent, la proposée se réduit par des substitutions linéaires à la forme de Jerrard $ax^5 + exy^4 + fy^5$. De là on conclut facilement que, étant donnés J, D, L (pourvu qu'on n'ait pas en même temps $J=0$, $D=0$, $L=0$), le caractère des racines, quant à la distinction entre le réel et l'imaginaire, est absolument déterminé, et de plus que J, D, L, non-seulement doivent être réels, mais encore (comme l'a remarqué le premier mon devancier M. Hermite) doivent satisfaire à une certaine condition d'inégalité, c'est-à-dire qu'une certaine fonction (nommons-la G) de J, D, L doit rester toujours positive. Je prends J, D, L pour coordonnées d'un point dans l'espace. Alors la surface $G=0$ divisera l'espace en deux portions pour l'une desquelles (qu'on peut nommer *la portion facultative*) tous les points correspondront à des familles d'équations avec des coefficients réels et dans l'autre (qu'on peut nommer *la portion non facultative*) tous les points correspondront à des familles d'équations avec des coefficients conjugués. Ces deux portions d'espace sont exactement égales et contraires, étant disposées symétriquement par rapport à l'axe de D. Cela étant, je trouve que la première (en faisant pour le moment abstraction du plan de D) se divise en trois régions. Toute la portion facultative au-dessous du plan de D constitue une seule région, tandis que la portion facultative au-dessus de ce plan se divise en deux régions qui se rencontrent dans la ligne où la surface G touche le plan de D, c'est-à-dire la ligne parabolique

$$\Lambda = 0, \ D = 0.$$

La condition qui fixe les limites de ces trois régions ou, si l'on veut, de ces trois circonscriptions limitrophes, c'est qu'on doit pouvoir passer dans une région donnée d'un point à un autre sans percer ni toucher le plan de D. Cela étant ainsi, on démontre facilement que pour chaque région les familles des formes représentées par un point qui y est renfermé appartiennent à la même catégorie, quant au nombre de leurs racines réelles et imaginaires, et on assigne sans aucune difficulté son propre caractère radical à chaque région. En exprimant dans la langue de l'analyse les conditions qui servent pour déterminer à quelle région répond un système donné de valeurs de J, D, L, on établit la règle donnée ci-dessus pour fixer le caractère des racines de la forme à laquelle ces trois invariants appartiennent. On devinera facilement comment le paramètre μ vient s'offrir dans ces conditions: en effet,

$$\Lambda + \mu JD = 0$$

représente une surface qui, passant par la ligne limitrophe aux deux régions supérieures, ne passe par aucun point facultatif au-dessus du plan de D, c'est-à-dire ne rencontre nulle part la surface $G=0$ au-dessus de ce plan.

Le perfectionnement que j'ai eu le bonheur d'ajouter ainsi à la découverte de mon confrère s'est offert à moi comme une conséquence (dans l'ordre subjectif des idées) du théorème que j'ai eu l'honneur déjà de publier dans les *Comptes rendus* de cette année*, et qui se rapporte à la limite du nombre des racines réelles de l'équation

$$\lambda_1 (x+c_1)^m + \lambda_2 (x+c_2)^m + \ldots + \lambda_i (x+c_i)^m = 0.$$

Dans cette équation, en supposant $c_1, c_2, \ldots, c_m$ arrangés en ordre de leurs grandeurs et en écrivant la suite $\lambda_1, \lambda_2, \ldots, \lambda_i, (-1)^m \lambda_1$, le théorème consiste en ce que le nombre des racines réelles ne peut pas dépasser le nombre de changements de signe dans la suite; mais j'avais imposé la condition que m doit être un nombre entier et positif; cette dernière restriction au moins est superflue; le théorème reste vrai quand m est un nombre négatif, tout aussi bien comme quand il est positif. Cette extension suit comme conséquence immédiate d'un théorème algébrique qu'on peut établir sans aucune difficulté, mais que je ne me rappelle pas d'avoir jamais rencontré.

Soient† $f(x, y)$, $\phi(x, y)$ deux fonctions homogènes quelconques en x, y; J la jacobienne de f, ϕ, c'est-à-dire $\frac{df}{dx}\frac{d\phi}{dy} - \frac{df}{dy}\frac{d\phi}{dx}$. Alors je dis qu'un nombre impair des racines de J sera compris entre chaque paire de racines réelles et consécutives de f, comme évidemment aussi entre chaque paire de racines réelles et consécutives de ϕ, de sorte que le nombre des racines réelles de f ni de ϕ ne peut excéder de plus d'une unité le nombre des racines réelles de J. Si on prend $\phi(x, y) = y$, on retombe sur le théorème d'algèbre élémentaire qui donne la disposition des racines réelles de $f'x$ par rapport aux racines réelles de fx.

* [p. 360 above.]

† [See p. 375 below.]

73.

CORRECTION DE LA NOTE INSÉRÉE DANS LES *COMPTES RENDUS* POUR LA SÉANCE DU 7 NOVEMBRE.

[*Comptes Rendus de l'Académie des Sciences,* LIX. (1864), pp. 944, 945.]

UNE erreur assez grave, mais n'ayant nul rapport à l'objet principal de la communication mentionnée ci-dessus, s'est glissée dans le théorème donné vers sa fin [p. 374]. En supposant ϕ et ψ deux fonctions homogènes et entières en x, y et J leur jacobienne, j'ai affirmé qu'entre deux racines consécutives quelconques de ϕ (comme aussi de ψ) se trouvera une racine ou un nombre impair de racines de J. J'aurais dû dire qu'entre deux telles racines de ψ se rencontrera une racine ou un nombre impair de racines de ϕJ, et pareillement pour ϕ, ψJ. En conséquence, l'extension que je m'imaginais avoir faite du théorème pour les équations de la forme $\Sigma\lambda_i(x+c_i)^m$ au cas de m négatif n'aura pas lieu.

74.

ALGEBRAICAL RESEARCHES, CONTAINING A DISQUISITION ON NEWTON'S RULE FOR THE DISCOVERY OF IMAGINARY ROOTS, AND AN ALLIED RULE APPLICABLE TO A PARTICULAR CLASS OF EQUATIONS, TOGETHER WITH A COMPLETE INVARIANTIVE DETERMINATION OF THE CHARACTER OF THE ROOTS OF THE GENERAL EQUATION OF THE FIFTH DEGREE, &c.

[*Philosophical Transactions of the Royal Society of London*, CLIV. (1864), pp. 579—666.]

AN INQUIRY INTO NEWTON'S RULE FOR THE DISCOVERY OF IMAGINARY ROOTS.

[*Proceedings of the Royal Society of London*, XIII. (1863–4), pp. 179—183.]

(*Abstract.*)

In the *Arithmetica Universalis*, in the chapter *De Resolutione Equationum*, Newton has laid down a rule, admirable for its simplicity and generality, for the discovery of imaginary roots in algebraical equations, and for assigning an inferior limit to their number. He has given no clue towards the ascertainment of the grounds upon which this rule is based, and has stated it in such terms as to leave it quite an open question whether or not he had obtained a demonstration of it. Maclaurin, Campbell, and others have made attempts at supplying a demonstration, but their efforts, so far as regards the more important part of the rule, that namely by which the limit to the number of imaginary roots is fixed, have completely failed in their object. Thus hitherto any opinion as to the truth of the rule rests on the purely empirical ground of its being found to lead to correct results in particular arithmetical instances. Persuaded of the insufficiency of such a mode of verification, the author has applied himself to obtaining a rigorous demonstration of the rule for equations of specified degrees. For the second degree no demonstration is necessary. or cubic equations a proof is found without difficulty. For biquadratic

equations the author proceeds as follows. He supposes the equation to be expressed homogeneously in x, y, and then, instituting a series of infinitesimal linear transformations obtained by writing $x + hy$ for x, or $y + hx$ for y, where h is an infinitesimal quantity, shows that the truth of Newton's rule for this case depends on its being capable of being shown that the discriminant of the function $(1, \pm e, e^2, \pm e, 1 \between x, y)^4$ is necessarily positive for all values of e greater than unity, which is easily proved. He then proceeds to consider the case of equations of the 5th degree, and, following a similar process, arrives at the conclusion that the truth of the rule depends on its being capable of being shown that the discriminant, say D, of the function $(1, \epsilon, \epsilon^2, \eta^2, \eta, 1 \between x, y)^5$, which for facility of reference may be termed "the (ϵ, η) function," is necessarily positive when $\epsilon^4 - \epsilon\eta^2$ and $\eta^4 - \eta\epsilon^2$ are both positive. This discriminant is of the 12th degree in ϵ, η. But on writing $x = \epsilon\eta$, $y = \epsilon^5 + \eta^5$, it becomes a rational integral function of the 6th degree in x, and of the second degree in y, and such that, on making $D = 0$, the equation represents a sextic curve, of which x, y are the abscissa and ordinate, which will consist of a single close. It is then easily demonstrated that all values of ϵ, η which cause the variable point x, y to lie inside this curve, will cause D to be negative (in which case the function ϵ, η has only two imaginary factors), and that such values as cause the variable point to lie outside the curve, will make D positive, in which case the ϵ, η function has four imaginary factors. When the conditions concerning ϵ, η above stated are verified, it is proved that the variable point must be exterior to the curve, and thus the theorem is demonstrated for equations of the 5th degree.

The question here naturally arises as to the significance of the sign of D when such a position is assigned to the variable point as gives rise to *imaginary* values of ϵ, η, which in such case will be conjugate quantities of the form $\lambda + i\mu$, $\lambda - i\mu$ respectively.

The curve D will be divided by another sextic curve into two portions, for one of which the couple ϵ, η corresponding to any point in its interior is *real*, and for the other *conjugate*. This brings to view the necessity of there being in general a theory for equations with conjugate coefficients, which for greater brevity may be termed conjugate equations, analogous to that for real equations in respect of the distinction between real and imaginary roots in the latter. A conjugate equation is one in which the coefficients, reckoning from the two ends of the equation, go in pairs of the form $p \pm iq$, with the obvious condition that when there is a middle coefficient this must be real. Such an equation may be supposed to be so prepared that, when thrown into the form $P + iQ$, P and Q shall have no common algebraical factor; and when this is effected, it may easily be shown that the conjugate equation can neither have real roots nor roots paired together of the form $\lambda + i\mu$, $\lambda - i\mu$ respectively. How, then, it may be asked, is the

analogy previously referred to possible? On investigation it will be found that the roots divide themselves into two categories, each of exactly the same order of generality,—namely *solitary* roots of the form $\epsilon^{i\theta}$, and *associated* roots which go in pairs, the two roots of each pair being of the form $\rho\epsilon^{i\theta}$, $\frac{1}{\rho}\epsilon^{i\theta}$ respectively; so that, following the ordinary mode of geometrical representation of imaginary quantities, the roots of a conjugate equation may be denoted by points lying on the circumference of a circle of radius unity (corresponding to solitary roots), and points (corresponding to the associated roots) lying in couples on different radii of the circle at reciprocal distances from the centre, each couple in fact constituting, according to Prof. W. Thomson's definition, electrical images of each other in respect to the circle. If the circle be taken with radius infinity instead of unity (so as to become a straight line), then we have the geometrical *eidolon* of the roots of an ordinary equation, the solitary roots lying on a straight line, and the associated or paired (imaginary) roots on each side of, and at equal distances from, the line.

In the inquiry before us, whether the variable point belong to the real or conjugate part of the plane of the D curve, it is shown to remain true that the number of *associated* roots will be two, if it lie inside the curve, and four if it lie outside. The author then suggests a probable extension of Newton's rule to conjugate equations of any degree. In conclusion, he deals with a question in close connexion with, and arising out of the investigation of this rule, relating to equations of the form $\Sigma \pm (ax+b)^m = 0$, to which, for convenience, he gives the provisional name of "superlinear equations" (denoting the function equated to zero as a superlinear form), and establishes a rule for limiting the number of real roots which they can contain, which is, that if such equation be thrown under the form

$$\lambda_1(x+c_1)^m + \lambda_2(x+c_2)^m + \dots + \lambda_n(x+c_n)^m = 0,$$

and $c_1, c_2, \dots c_n$ be an ascending or descending order of magnitudes, the equation cannot have more real roots than there were variations of sign in the sequence $\lambda_1, \lambda_2, \dots, \lambda_n, (-)^m\lambda_1$.

This theorem was published by the author, but without proof, in the *Comptes Rendus* for the month of March in this year*.

But the method of demonstration now supplied is deserving of particular attention in itself; for it brings to light a new order of purely tactical considerations, and establishes a previously unsuspected kind of, so to say, algebraical polarity. The proof essentially depends upon the character of every superlinear form being associated with, and capable of definition by means of a pencil of rays, which may be called the type pencil, subject to

* [Above, p. 360.]

a species of circulation of a different nature according as the degree of the form is even or odd, which he describes by the terms "per-rotatory" in the one case, and "trans-rotatory" in the other; so that the types themselves may be conveniently distinguished by the names "per-rotatory" and "trans-rotatory." By per-rotatory circulation is to be understood that species in which, commencing with any element of the type, passage is made from it to the next, from that to the one following, from the last but one to the last, from the last to the first, and so on, until the final passage is to the element commenced with, from the one immediately preceding. By trans-rotatory circulation, on the other hand, is understood that species in which, commencing with any element and proceeding in the same manner as before to the end element, passage is made from that, not to the first element itself, but to its polar opposite, from that to the polar opposite of the next, and so on, until the final passage is made to the polar opposite of the element commenced with, from the polar opposite of its immediate antecedent. The number of changes of sign in effecting such passages, whether in a per-rotatory or a trans-rotatory type, is independent of the place of the element with which the circulation is made to commence, and may be termed the variation-index of the type, which is always an even number for per-rotatory, and an odd number for trans-rotatory types. A theorem is given whereby a relation is established between the variation-index of a per-rotatory or trans-rotatory and that of a certain trans-rotatory or per-rotatory type capable of being derived from them respectively; and this purely tactical theorem, combined with the algebraical one, that the form $f(x, y)$ cannot have fewer imaginary factors than any linear combination of $\frac{df}{dx}$, $\frac{df}{dy}$, leads by successive steps of induction to the theorem in question, but under a more general form, which serves to show intuitively that the limit to the number of real roots of a superlinear equation which the theorem furnishes must be independent of any homographic transformation operated upon the form. The author believes that, whilst it is highly desirable that a simple and general method should be discovered for the proof of Newton's rule as applicable to equations of any degree, and that the strenuous efforts of the cultivators of the New Algebra should be directed to the attainment of this object, his labours in establishing a proof applicable as far as equations of the 5th degree inclusive will not have been unproductive of good, as well on account of the confirmation they afford of the truth of the rule, towards the establishment of which on scientific grounds they constitute the first serious step yet made, as also, and still more, by reason of the accessions to the existing field of algebraical speculation to which they have incidentally led.

"Turns them to shapes and gives to airy nothing
A local habitation and a name."

(1) THIS memoir* in its present form is of the nature of a trilogy; it is divided into three parts, of which each has its action complete within itself, but the same general cycle of ideas pervades all three, and weaves them into a sort of complex unity. In the first is established the validity of Newton's rule for finding an inferior limit to the number of imaginary roots of algebraical equations as far as the fifth degree inclusive. In the second is obtained a rule for assigning a like limit applicable to equations of the form $\Sigma(ax+b)^m = 0$, m being any positive integer, and the coefficients a, b real. In the third are determined the absolute invariantive criteria for fixing unequivocally the character of the roots of an equation of the fifth degree, that is to say, for ascertaining the exact number of real and imaginary roots which it contains. This last part has been added since the original paper was presented to the Society. It has grown out of a foot-note appended to the second, itself an independent offshoot from the first part, but may be studied in a great measure independently of what precedes, and constitutes, in the author's opinion, by far the most valuable portion of the memoir, containing as it does a complete solution of one of the most interesting and fruitful algebraical questions which has ever yet engaged the attention of mathematicians (1). I propose in a subsequent addition to the memoir to resume and extend some of the investigations which incidentally arise in this part. The foot-notes are numbered and lettered for facility of reference, and will be found in many instances of equal value with the matter in the text, to which they serve as a kind of free running accompaniment and commentary.

PART I.—ON NEWTON'S RULE FOR THE DISCOVERY OF IMAGINARY ROOTS.

(2) In the *Arithmetica Universalis*, in the first chapter on equations, Newton has given a rule for discovering an inferior limit to the number of imaginary roots in an equation of any degree, without proof or indication of the method by which he arrived at it, or the evidence upon which it rests (2). Maclaurin, in vol. XXXIV. [1726–7], p. 104, and vol. XXXVI. p. 59 of the

* [The Author's Table of Contents is given on p. 477.]

(1) I owe my thanks to my eminent friend Professor De Morgan for bringing under my notice, in a marked manner, the original question from which all the rest has proceeded. As all roads are said to lead to Rome, so I find, in my own case at least, that all algebraical inquiries sooner or later end at that Capitol of Modern Algebra over whose shining portal is inscribed "Theory of Invariants."

(2) It appears to be the prevalent belief among mathematicians who have considered the question, that Newton was not in possession of other than empirical evidence in support of his rule.

Philosophical Transactions, Campbell [3] in vol. xxxv. p. 515 of the same, and other authors of reputation have sought in vain for a demonstration of this marvellous and mysterious rule [4]. Unwilling to rest my belief in it on mere empirical evidence, I have investigated and obtained a demonstration of its truth as far as the fifth degree inclusive, which, although presenting only a small instalment of the desired result, I am induced to offer for insertion in the *Transactions* in the hope of exciting renewed attention

(3) Campbell's memoir is rather on an analogous rule to Newton's than on the rule itself, to which he refers only by way of comparison with his own. In it the same singular error of reasoning is committed as in the notes of the French edition of the *Arithmetica*, namely, of assuming, without a shadow of proof, that if each of a set of criteria indicates the existence of some imaginary roots, a succession of sets of such criteria must indicate the existence of at least as many distinct imaginary pairs of roots as there are such sets (see paragraph at foot of p. 528, *Phil. Trans.*, vol. xxxv.)—much as if, supposing a number of dogs to be making a point in the same field, the existence could be assumed of as many birds as pointers.

(4) Mr Archibald Smith has obligingly called my attention to Waring's treatment of the question of Newton's rule in the *Meditationes Analyticæ*. On superficial examination the reader might be induced to suppose that in part 9, p. 68, ed. 1782, Waring had deduced a proof of the rule from the preceding propositions; but on looking into the case will find that there is not the slightest vestige of proof, the rule being stated, but without any demonstration whatever being either adduced or alleged. In fact, on turning to the preface of this (the last) edition of the *Meditationes*, the reader will find at p. 11 an explicit avowal of the demonstration being wanting. After referring in order to Campbell's, Maclaurin's, and Newton's rules, as well as his own, for discovering the existence of impossible roots, he adds these words :

"At omnes hæ regulæ prædictæ perraro invenerunt verum numerum impossibilium radicum in æquationibus multarum dimensionum *et adhuc demonstratione egent*; vulgares enim demonstrationes solummodo probant impossibiles radices in data æquatione contineri, non vero quod *saltem tot sunt quot invenit regula*."

"Vera resolutio problematis est perdifficilis et valde laboriosa; cognitum est radices ex possibilitate per æqualitatem transire ad impossibilitatem ; ergo in generali resolutione hujusce problematis necesse est invenire casum in quo radices datæ æquationis evadunt æquales; resolutio autem hujus casus valde laboriosa est; et consequenter resolutio generalis prædicti problematis magis *erit* laboriosa."

Written in Latin, and when the proper language of algebra was yet unformed, it is frequently a work of much labour to follow Waring's demonstrations and deductions, and to distinguish his assertions from his proofs. I find he agrees with the opinion expressed by myself, that Newton's rule will *not* "pene," as stated by Newton, but only "perraro," give the true number of imaginary roots. Like myself, too, in the body of the memoir Waring has given theorems of probability in connexion with rules of this kind, but without any clue to his method of arriving at them. Their correctness may legitimately be doubted.

[Since the above was sent to press, I have been enabled to ascertain that the great name of Euler is to be added to the long list of those who have fallen into error in their treatment of this question: see *Institutiones Calculi Differentialis*, vol. II. cap. XIII. He says (p. 555, edition of Prony), "videndum est utrum hæc duo criteria (meaning Newton's criteria of imaginariness) sint contigua necne; priori casu numerus radicum imaginarium non augebitur; posteriori vero *quia criteria litteras prorsus diversas involvunt*, unumquodque binas radices imaginarias monstrabit."

The force of the supposed argument is contained in the words in italics. It is sufficiently met by the question, why or how the conclusion follows from them? Moreover the letters of two non-contiguous criteria are *not* necessarily *prorsus diversæ* ; for two criteria with but a single other intervening between them will contain one letter in common.]

to a subject so intimately bound up with the fundamental principles of algebra.

Before commencing the inquiry I ought to state that, in addition to the rule for detecting the existence of a certain number of imaginary roots, Newton has given a remarkable subsidiary method for dividing this number into two parts, representing respectively how many of the positive and how many of the negative roots indicated by Descartes's rule are, so to say, absorbed, and thereby obtains two distinct limits to the number of positive and the number of negative roots separately: of the grounds of this method, as far as I am aware, no one has even attempted an explanation, nor do I propose here to enter upon it; the rule, as I treat it, may be stated, not in Newton's own words, but most simply as follows:—

If the literal parts of the coefficients of an equation affected with the usual binomial coefficients be a, b, c, d, $e \ldots h$, k, l, and if we form the successive criteria $b^2 - ac$; $c^2 - bd$; $d^2 - ce$; $\ldots$; $k^2 - hl$, or, which is the same thing differently expressed, if we write down the determinants[5] *of all the successive quadratic derivatives of the given equation, then as many sequences as there are of negative signs in the arithmetical values of these criteria, so many pairs of imaginary roots at least there will be in the given equation.* If we choose to consider a^2 and l^2 also as criteria, appearing at the beginning and end of the series, then we may vary the expression of the rule by saying that there will be at least as many imaginary roots as there are variations of sign in the complete series so formed.

It will, however, be found more convenient for our present purpose to confine the designation of criteria to the determinants above alluded to.

(3) I shall deal with the homogeneous equation $f(x, y) = 0$ so that the question of the reality of the roots is that of the reality of the ratios $\frac{x}{y}$ or $\frac{y}{x}$. It is obvious, from known principles, that f cannot have fewer imaginary roots than exist in $\frac{df}{dx}$ or $\frac{df}{dy}$[6], or, more generally, than in

(5) To avoid the possibility of misapprehension, I state here once for all, that in the *discriminant* of a form of any degree I suppose the sign to be so taken as to render *positive* the term which is a power of the product of the first and last coefficients; and it may be well to remember that with this definition the number of real roots in any equation $\equiv 0$ or 1 to modulus 4 when the discriminant is positive, and $\equiv 2$ or 3 when the discriminant is negative; whereas the Determinant of a Quadratic form is to be taken in the same sense as that in which it is used by Gauss, and is the same for such form as the Discriminant with the sign changed.

(6) This rule I find merges in the following more general and symmetrical one. Let f, ϕ be any two quantics in x, y; call J the Jacobian of f, ϕ; then the difference between the number of real roots in f and the like number in ϕ, taken positively and augmented by unity, cannot exceed the number of real roots in J. When ϕ is made equal to y, this theorem recurs to the familiar one alluded to in the text.

$\left(\dfrac{d}{dx} + \lambda \dfrac{d}{dy}\right) f$; from which it immediately follows[7] that if f have all its roots real, and the quadratic derivatives of f be called $Q_1, Q_2, \ldots Q_{n-1}$, and the coefficients of any function F of two degrees lower than f, whose roots are also *all* real, be $p_1, p_2, \ldots p_{n-1}$, the quadratic function

$$p_1 Q_1 + p_2 Q_2 + \ldots + p_{n-1} Q_{n-1}$$

must have its roots real, that is its discriminant must be positive: a particular consequence of this is, that by causing F to consist successively of the single terms x^{n-2}, $x^{n-3}y$, $\ldots xy^{n-3}$, y^{n-2}, we see that the determinants of $Q_1, Q_2, \ldots Q_{n-1}$ must each of them be positive; or, in other words, if any of the Newtonian criteria of an equation are negative, it must have *some* imaginary roots, which is all that Maclaurin, Campbell, and others have succeeded in proving.

(4) The labour of proof of the cases hereinafter considered will be much lightened by the following rule of induction, namely, granting Newton's rule to be true for the degree $n-1$, it must be true for all those cases appertaining to the degree n in which the series of the signs of the criteria does not commence with $-+$ and end with $+-$: to prove this, we have only to remember that f must have at least as many imaginary roots as $\dfrac{df}{dx}$ or $\dfrac{df}{dy}$, and that the criterion-series corresponding to $\dfrac{df}{dx}$ and to $\dfrac{df}{dy}$ will be found by cutting off from the series of f one term to the right and left respectively[8]. If, now, the series for f begins with $++$ or $--$ or $+-$, the number of negative *sequences* is the same as when the left-hand sign is removed; so that it is only necessary to prove that the number of imaginary roots in f is not less than the number of negative sequences in $\dfrac{df}{dx}$; but this, by hypothesis, is not greater than the number of pairs of imaginary roots in $\dfrac{df}{dx}$, and, *à fortiori*, not greater than the number of such in f. In like manner, if the two *last* criteria of f are not $+-$, it may be shown that the truth of the rule for such form of f is implied in what is supposed to be known to be true for $\dfrac{df}{dy}$.

(7) By operating upon f successively with *any* $(n-2)$ distinct factors each of the form

$$\left(\frac{d}{dx} + \lambda_i \frac{d}{dy}\right).$$

(8) For
$$\frac{d}{dx}(a, b, \ldots k, l \between x, y)^n = n(a, b, \ldots k \between x, y)^{n-1},$$
and
$$\frac{d}{dy}(a, b, \ldots k, l \between x, y)^n = n(\quad b, \ldots k, l \between x, y)^{n-1}.$$

We may therefore limit our attention, as we ascend in the scale of proof, to those forms of f in which the criterion-series begins with $-+$ and ends with $+-$. Accordingly, since the rule is a truism for $n=2$, it is at once proved, by virtue of the above considerations, for $n=3$ [(9)].

If all the criteria are zero, it is evident that, whatever n may be, all the roots are real. In every other case we shall find that *zero* may be made positive or negative at will. Thus in the case before us, if the two criteria are $0+$ or $0-$, there will be a pair of imaginary roots, as the first may be read as $-+$ and the second as $+-$.

To prove this, we have only to observe that in either case $\dfrac{df}{dx}$ will have two equal roots; so that f will be of the form $(ax+by)^3+cy^3$, which obviously, for any real values of a, b, c, has only one real root.

(5) We may now pass to the case of $n=4$, and excluding for the moment the consideration of *zeros*, limit our attention to the criterion-series $-+-$.

Let $ax^4+4bx^3y+6cx^2y^2+4dxy^3+ey^4=0$ be the equation for which the signs of the criteria b^2-ac, c^2-bd, d^2-ce are $-+-$. Call these criteria

(9) The theorem for the case of cubic equations may be also proved directly as follows:

Writing the equation $ax^3+3bx^2y+3cxy^2+dy^3=0$, the two criteria are $L=b^2-ac$, $M=c^2-bd$; and the discriminant is $a^2d^2+4ac^3+4db^3-3b^2c^2-6abcd=\Delta$.

1. Let L and M be of opposite signs, so that one and only one of them is negative. Then

$$\Delta=(ad-bc)^2-4(b^2-ac)(c^2-bd)=(ad-bc)^2-4LM,$$

and is therefore positive.

2. Let L and M be both negative. The equation may evidently, by writing x and y for $a^{\frac{1}{3}}x$, $d^{\frac{1}{3}}y$, be brought under the form

$$x^3+3\epsilon x^2y+3\eta xy^2+y^3=0,$$

with the conditions $\epsilon^2<\eta$, $\eta^2<\epsilon$; from which we may deduce that ϵ and η are both positive, and $\epsilon\eta<1$ and >0.

Also we have

$$\begin{aligned}\Delta &= 1+4(\epsilon^3+\eta^3)-6\epsilon\eta-3\epsilon^2\eta^2\\ &> 1+4(\epsilon+\eta)\epsilon\eta-6\epsilon\eta-3\epsilon^2\eta^2\\ &> 1-6\epsilon\eta+8(\epsilon\eta)^{\frac{3}{2}}-3\epsilon^2\eta^2;\end{aligned}$$

or, writing $\epsilon\eta=q^2$,

$$\begin{aligned}\Delta &> 1-6q^2+8q^3-3q^4,\\ &> (1-q)^3(1+3q);\end{aligned}$$

but $1>q>0$. Hence Δ is positive.

Hence in either case two of the roots of the cubic are impossible. Or the same thing may be shown more immediately from the identities

$$a^2\Delta=(a^2d+2b^3-3abc)^2+4(ac-b^2)^3,$$
$$d^2\Delta=(ad^2+2c^3-3bcd)^2+4(bd-c^2)^3,$$

so that Δ must be positive, and therefore two roots imaginary, if either $bd>c^2$ or $ca>b^2$. It may be noticed that the square and cube in these identities are semi-invariants, being in the first of them unaffected by the change of x into $x+hy$, and in the second by the change of y into $y+hx$.

L, M, N respectively. It has to be proved that all four roots are imaginary, since there are two distinct negative sequences, each sequence consisting of a single $-$. Let x become $x + \epsilon y$ (10), where ϵ is an infinitesimal quantity, and the equation transformed into one between x and y; then we have obviously,

$$\delta a = 0, \quad \delta b = a\epsilon, \quad \delta c = 2b\epsilon, \quad \delta d = 3c\epsilon, \quad \delta e = 4d\epsilon,$$

$$\delta L = 2b\delta b - a\delta c = 0, \quad \delta M = 2c\delta c - b\delta d - d\delta b = (bc - ad)\,\epsilon,$$

$$\delta^2 M = (b\delta c + c\delta b - a\delta d)\,\epsilon = 2\,(b^2 - ac)\,\epsilon^2 = 2L\epsilon^2;$$

so that $\delta^2 M$ is essentially negative, since L is so.

Hence, by continually augmenting x by an infinitesimal variation, we may, leaving L unaltered, so choose the sign of ϵ as to decrease M: nor can this process stop when $bc - ad$ becomes zero, by reason that $\delta^2 M$ is *negative.* Hence we may reduce M to zero. Now, in the course of this reduction, either N retains its sign or changes it; and if the latter is the case, N must have passed through zero. If when M becomes zero N is still negative, the criteria of the linearly transformed equation become $-\,0\,-$; and it may be noticed that its first, middle, and last coefficients must have the same sign, by virtue of the negativity of the two last criteria, and the second and fourth the same signs, by virtue of the zero middle criterion; consequently the equation will take the form

$$(\lambda^2 + e^4)\,x^4 \pm 4e^3\epsilon x^3 y + 6e^2\epsilon^2 x^2 y^2 \pm 4e\epsilon^3 xy^3 + (\mu^2 + \epsilon^4)\,y^4 = 0,$$

or

$$\lambda^2 x^4 + \mu^2 y^4 + (ex \pm \epsilon y)^4 = 0,$$

which obviously has all its roots impossible. This being true of the transformed equation, will also, on the suppositions made, be equally so of the original equation.

Let us next suppose that N changes its sign either at the instant when, or before M becomes zero. If M and N both become zero together, so that the criteria of the transformed equation bear the signs $-\,0\,0$, calling the transformed equation $F = 0$, $\dfrac{dF}{dy}$ will have all its roots equal, and F will therefore be of the form $(ax + by)^4 + kx^4$, with the condition

$$(a^3 b)^2 - (a^4 + k)\,(a^2 b^2) < 0.$$

Hence k is positive, and consequently $F = 0$ has all its roots imaginary; and the same, as before, must hold good of the original equation $f = 0$.

(10) This method of infinitesimal substitution is that which I applied* in my memoir "On the Theory of Forms," in the *Cambridge and Dublin Mathematical Journal*, to obtain the partial differential equations to every possible species of invariants (including covariants and contravariants) of forms, or systems of forms, with a single set or various sets of variables, proceeding upon the pregnant principle that every finite linear substitution may be regarded as the result of an indefinite number of *simple* and *separate* infinitesimal variations impressed upon the variables. M. Aronhold has erroneously ascribed to others the priority of the publication of these equations.

[* Volume I. of this Reprint, p. 356.]

It remains then only to consider the case when N becomes zero before M vanishes. When this is the case, as soon as N is reduced to zero, in lieu of the substitution of $x+\epsilon y$ for x, we must leave x unaltered, and continue substituting $y+\epsilon x$ for y. We thus start from the sequence $-+0$; N will then always remain zero, and we must either come to the series $-\,0\,0$, which we know, from what has been shown above, corresponds to four imaginary roots, or to the sequence $0+0$, which I shall proceed to consider.

Since the first and last coefficients must have the same sign, we may, by giving either variable a proper multiple (11), make these two coefficients alike,

(11) (*a*) The form $(1, e, e^2, e, 1\between x, y)^4$ may be regarded as a new and, for many purposes, useful canonical form of a binary quartic. It may be made to comprise within its sphere of representation all forms corresponding to two or four imaginary factors, but excludes the case of four real factors. The ordinary canonical form $(1, 0, 6m, 0, 1\between x, y)^4$ comprises within its spheres of representation those forms for which the factors are all real or all imaginary, but, so far as real transformations are concerned, excludes the case of two real and two imaginary factors [that case is met by the form $(1, 0, 6m, 0, -1\between x, y)^4$], as may easily be established either by decomposing the form first named into its factors, or by the consideration that its discriminant Δ is $(1-9m^2)^2$, and is therefore always positive; whereas if a form which it is used to represent have two real and two unreal factors, its discriminant is negative. If now the determinant of transformation be D, and the discriminant corresponding thereto be called Δ', we have $\Delta'=D^6\Delta$, showing that D^2 is negative, and the transformation therefore unreal.

(*b*) The *reality* of m for each of these cases (usually assumed without proof) may be demonstrated as follows: Calling the cubic invariant and the discriminant of any quartic form T, D, we shall have, using the *ordinary* canonical form, $\dfrac{(m-m^3)^2}{(1-9m^2)^2}=\dfrac{T^2}{D}$, showing that when D is positive, which is the case of four real or unreal factors, there will be one real value of m, and when D is negative, a real value of im. The former case possesses over the latter a striking distinction, which is that *all* the roots of m will be real; for, as I have shown elsewhere*, if m is one root the complete system of roots will be $\pm m$, $\pm\dfrac{1-m}{1+3m}$, $\pm\dfrac{1+m}{1-3m}$: in the latter case the reality of the two values $\pm im$ does not seem necessarily to imply the reality of the other four values of the system.

(*c*) Analogy suggests the establishment of an analogous canonical form or forms for ternary cubics, of which, as is well known and is even dimly foreshadowed in Newton's Enumeration of Lines of the Third Order, the theory runs closely parallel to that of binary quartics. This will be effected by assuming the form

$$F(x, y, z)=\Sigma x^3+3e\Sigma x^2y+6gxyz,$$

and assuming g so as to make the discriminants of

$$\frac{dF}{dx},\ \frac{dF}{dy},\ \frac{dF}{dz}$$

all zero. This gives rise to a quadratic equation in g, of which the roots are $g=e$, $g=2e^2-e$. When $g=e$, I find

$$S=e\,(1-e)^3,\quad T=(1-e)^4(1+4e-8e^2),\quad \Delta=T^2+64S^3=(1+8e)\,(1-e)^8.$$

When $g=2e^2-e$, I find $\Delta=(1-e)^i\,(1-4e)^j\,(1+2e)^k$, where i, j, k are integers to be determined. These forms will, I think, be found important in the future perspective discussion of curves of the third degree. Whilst I yield to no one in admiration of the surpassing genius with which Newton has handled these curves, I cannot withhold the expression of my opinion that every theory of forms in which invariants are ignored must labour under an inherent imperfection, and that Newton, from want of acquaintance with the indelible characters which their invariants

[* Volume I. of this Reprint, p. 600.]

and with the first, second, and third, as well as the third, fourth, and fifth coefficients form geometrical series; hence it is obvious that the transformed equation may be reduced to one or the other of the two following forms, namely

$$x^4 + 4ex^3y + 6e^2x^2y^2 - 4exy^3 + y^4 = 0, \qquad (a)$$

or

$$x^4 + 4ex^3y + 6e^2x^2y^2 + 4exy^3 + y^4 = 0, \qquad (b)$$

with the condition in the latter case that $e^4 - e^2$ is positive, that is $e^2 > 1$.

It must be remembered that we know, from the form of the criteria-series to the derivatives in respect to either x or y (indifferently), that the equation must have *some* imaginary roots; and the question therefore lies between its having two or four. If the discriminant is negative, the former will be the case, if positive, the latter. I shall show that in each equation the discriminant is positive.

Let s, t represent in general the quartic invariants, then we have to show that $s^3 - 27t^2$ is positive.

In case (a),

$$\begin{aligned} s &= 1 + 4e^2 + 3e^4 \\ &= (1+e^2)(1+3e^2), \end{aligned} \qquad t = \begin{vmatrix} 1 & e & e^2 \\ e & e^2 & -e \\ e^2 & -e & 1 \end{vmatrix} \begin{aligned} &= e^2 - e^4 - e^4 - e^2 - e^6 - e^2 \\ &= \quad -e^2 - 2e^4 - e^6 \\ &= \quad -e^2(1+e^2)^2, \end{aligned}$$

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stamp upon curves, has in the parallel which he has drawn between the generation by shadows of all conics from a common type, and of all cubic curves from a limited number of forms, either himself fallen into error of conception, or at least used language which could scarcely fail to lead others into such error. For no species whatever of cubic curve can be formed for which an infinite number of individuals cannot be found which defy linear or perspective transformation into each other; whereas all conics proper may be propagated as shadows from a single individual. It should be noticed in connexion with this subject, that the *indelible* characters of quartic binary, and cubic ternary forms are two in number, namely, the value of $\frac{s^3}{t^2}$ (where s, t are the two fundamental invariants in either case) and the *sign* of t. The indelibility of the sign of s being implied in the invariability of the value of $\frac{s^3}{t^2}$, does not constitute a distinct character. Of course all symmetrical invariants have an invariable sign; but this is not the case with skew invariants, as for example, M. Hermite's octodecimal invariant of a binary quintic, which will change its sign with that of the determinant of transformation.

(d) Whilst upon this subject of invariants, I may allow myself to make a remark, bearing upon what will be noticed further on in the text, about a case of equality between roots not necessarily being a mark of transition from real to imaginary roots. If a, b, c, d being the roots of a binary quartic we form a secondary cubic, of which the roots are $(a-b)(c-d)$, $(a-c)(d-b)$, $(a-d)(b-c)$, it may be easily shown that two of these quantities become equal, or, in other words, the roots of the original equation mark out a harmonic group of points, when t (the cubinvariant) is zero. Notwithstanding which a change of sign in t will not command a change of character in the above three roots of the secondary (nor consequently of the original equation), because it is not an odd but an even power of t, namely, t^2, which enters into the discriminant of the secondary.

so that

$$s^3 - 27t^2 = (1+e^2)^3\{(1+3e^2)^3 - 27e^4(1+e^2)\} = (1+e^2)^3(1+9e^2),$$

and is positive.

In case (*b*),

$$s = (1-4e^2+3e^4) = (1-e^2)(1-3e^2),$$

$$t = \begin{vmatrix} 1 & e & e^2 \\ e & e^2 & e \\ e^2 & e & 1 \end{vmatrix} \begin{aligned} &= e^2+e^4+e^4-e^2-e^6-e^2 \\ &= -e^2+2e^4-e^6 = -e^2(1-e^2)^2, \end{aligned}$$

and
$$\begin{aligned} s^3 - 27t^2 &= (1-e^2)^3\{(1-3e^2)^3 - 27e^4(1-e^2)\} \\ &= (1-e^2)^3(1-9e^2). \end{aligned}$$

The above can only be negative when e^2 lies between 1 and $\frac{1}{9}$; but in the case supposed $e > 1$. Hence the discriminant is positive, and the roots are all imaginary (12). Thus, then, the theorem is established for $n = 4$, as well as for the cases where the criteria are zero (as will have been observed in the course of the demonstration), as for those where they are *plus* or *minus*; and it should be observed that the demonstration proceeds upon our being able to show that the quartic, in the case where it resists reduction to the case of the cubic, namely where the criteria are negative at the two extremes and positive in the middle, may by real linear transformations be changed into a form where either the middle criterion is zero and the two extremes negative, or the two extremes zero, and the middle one positive.

Observation.—To make the foregoing demonstration quite exact, it should be noticed that when the criteria L, M, N have been brought to the form $-+0$, and the series of substitutions of $y+\epsilon x$ for y has set in, we have

$$N=0, \quad \delta N=0, \quad \delta M=(cd-be)\epsilon, \quad \delta^2 M = N\epsilon = 0, \quad \delta^3 M = 0.$$

Consequently if $cd-be$ should become *zero*, we can no longer go on decreasing M. But as soon as $cd-be=0$, since we have also $d^2=ce$, b, c, d, e come to be in geometrical progression, and the transformed equation takes the form

$$ax^4 + 4\omega x^3y + 6\omega^2x^2y^2 + 4\omega^3xy^3 + \omega^4x^4 = 0,$$

(12) The reader conversant only with ordinary algebra may easily verify this result. For writing $\frac{x}{y}+\frac{y}{x}=z$, the equation becomes $z^2+4ez+6e^2-2=0$, and this will have its roots impossible unless $4e^2>6e^2-2$, or $2e^2-2$ negative, which it cannot be, since $e^2>1$, and consequently $x:y$ has all its roots impossible. Moreover the same conclusion would (as before shown) hold good unless e^2 lay between 1 and $\frac{1}{9}$; for on making $z=2$, the function above written in z becomes $2+8e+6e^2$, or $2(1+e)(1+3e)$; and making $z=-2$, it becomes $2-8e+6e^2$, or $2(1-e)(1-3e)$, which two quantities evidently have both positive signs unless e lies between 1 and $\frac{1}{3}$, or between -1 and $-\frac{1}{3}$; so that the first and third Sturmian functions are (except on that supposition) respectively positive and negative for $z=2$, and also for $z=-2$, showing that no root of z can lie between 2 and -2, and consequently that all the roots of $x:y$ remain impossible.

with the condition $\omega^2 - a\omega^2$ negative, or $a > 1$. Hence we have

$$q^2x^4 + (x + \omega y)^4 = 0,$$

which obviously has all its roots impossible [13].

(6) We may now pass on to equations of the fifth degree, in which the case resisting reduction will be that where the criterion-series bears the signs

$$- + + -.$$

Let the criteria be called L, M, N, P, so that writing the equation

$$ax^5 + 5bx^4y + 10cx^3y^2 + 10dx^2y^3 + 5exy^4 + fy^5 = 0,$$

$$L = b^2 - ac, \quad M = c^2 - bd, \quad N = d^2 - ce, \quad P = e^2 - df,$$

and writing for x, $x + \epsilon y$, we have, as before,

$$\delta L = 0, \quad \delta M = (bc - ad)\,\epsilon, \quad \delta^2 M = L\epsilon^2,$$

so that M may be continually diminished.

If M becomes zero before either N or P changes its sign, the criterion-series for the transformed equation becomes $- 0 + -$, and for its derivative in respect to x, the series is $0 + -$, which proves the existence of four imaginary roots in the transformed, and consequently also in the given equation. In like manner, if N becomes zero before M or P have changed their signs, the criterion-series becomes $- + 0 -$, which obviously leads to the same result. So likewise the same inference may be drawn if L and M, or M and N, or L, M, N become zeros all at the same time, and we have only to consider the case when, L and M retaining their signs, N becomes zero. At this moment the order of the substitutions must be reversed, and for y must be written $y + \epsilon x$; we shall then have

$$P = 0, \quad \delta P = 0, \quad \delta N = (de - cf)\,\epsilon \ldots\ldots;$$

and reasoning as in the preceding case for $n = 4$ (with the sole difference, that if δN vanishes by virtue of $de - cf$ vanishing, we should have $P = 0$,

(13) From the first and third criteria it follows that in the form $(a, b, c, d, e \between x, y)^4$, a, c, e have the same sign and may be regarded as all positive; so that writing $a - \frac{b^2}{c} = h^2$, $e - \frac{d^2}{c} = k^2$, the form becomes $h^2x^2 + F + k^2y^2$, where

$$F = \frac{b^2}{c}x^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + \frac{d^2}{c}y^4,$$

and consequently the given form will have all its roots imaginary when this is true for F, so that we might have proceeded at once to deal with the forms marked (a), (b) at p. [387]; but as the method of homographic transformation by infinitesimal substitutions appears to be necessary in passing to the corresponding forms in the case of the fifth degree, and as in treating that case reference is made to what appears above, I have thought that no object would be gained by altering the text.

$N=0$, and the criterion-series $-+0\,0$, which at once indicates the existence of four imaginary roots), we see that there remains only to consider the case where the criterion-series takes the form $0++0$. It is scarcely necessary to observe that all the criteria can never vanish simultaneously; for that would indicate the equality of all the roots in the transformed, and therefore in the given equation, whose own criteria, contrary to hypothesis, would also be all zero. The zero values of the two extreme criteria indicate that the three first and the three last literal parts of the coefficients are in geometrical progression, from which it will immediately be seen that the equation to be considered may be thrown (by substituting in lieu of x and y suitable multiples of x and y, which will not affect the characters of the criteria) into the convenient form

$$x^5 + 5\epsilon x^4 y + 10\epsilon^2 x^3 y^2 + 10\eta^2 x^2 y^3 + 5\eta x y^4 + y^5 = 0,$$

with the two conditions $\epsilon^4 - \epsilon\eta^2$ *positive*, $\eta^4 - \eta\epsilon^2$ *positive.*

The form of the criterion-series, apocopated from either end, shows that two of the roots must be imaginary; and consequently, in order to establish the existence of two imaginary pairs of roots, it is only necessary to show that the discriminant of the above equation, subject to the above conditions, must remain always positive. That discriminant I proceed to determine; but as a guide to the form under which it is to be expressed, the following observation is important. Let us take the more general form

$$ax^5 + bx^4y + cx^3y^2 + dx^2y^3 + exy^4 + fy^5 = 0,$$

where $\quad a=1, \quad b=\lambda\epsilon, \quad c=\mu\epsilon^2, \quad d=\mu\eta^2, \quad e=\lambda\eta, \quad f=1,$

λ, μ being any numerical quantities.

The discriminant will evidently be a symmetrical function of ϵ and η.

Let $a^p b^q c^r d^s e^t$ be the literal part of a term in the discriminant. By the *law of weight* we must have

$$q + 2r + 3s + 4t = 5 \times 4 = 20.$$

But in the equation before us, $a^p b^q c^r d^s e^t$ (to a numerical factor *près*) is $\epsilon^{q+2r}\eta^{2s+t}$, and

$$\begin{aligned}(q+2r) - (2s+t) &= (q+2r+3s+4t) - 5(s+t)\\ &= 5(4-s-t).\end{aligned}$$

Hence the difference between the indices of ϵ and η in each term is a multiple of 5, and consequently, since the discriminant is a symmetrical function in ϵ and η, it will be a rational integral function of $\epsilon^5 + \eta^5$ and $\epsilon\eta$. Moreover, as no such term as c^4d^4 can figure in the discriminant, which, as we know, must in all cases contain one or the other of the two final and of the two initial coefficients, we see that no term can be of higher than the

14th degree in ϵ, η, nor yet so high, for the only terms that could be of that degree would be bc^3d^3e; but making a and f each zero in the original form, it becomes obvious that all the terms free from a and f contain b^2e^2 as a factor [14]. Hence, in fact, the discriminant will be only of the twelfth degree in ϵ, η, and being therefore of only the second degree in $\epsilon^5+\eta^5$, will admit of comparatively easy treatment.

(7) Before proceeding to the calculation of this discriminant, it will be useful to investigate, as a Lemma ancillary to the subsequent discussion, under what conditions four of the roots of the supposed equation will become imaginary when $\epsilon=\eta$.

In this case writing $\frac{x}{y}+\frac{y}{x}=z$, the equation

$$\frac{1}{x+y}(1,\ \epsilon,\ \epsilon^2,\ \epsilon^2,\ \epsilon,\ 1\between x,\ y)^5=0$$

becomes

$$z^2-2-z+1+5\epsilon(z-1)+10\epsilon^2=z^2+(5\epsilon-1)z+10\epsilon^2-5\epsilon-1=0,$$

or say $f(z)=0$.

The determinant of $f(z)$ is thus

$$(5\epsilon-1)^2-40\epsilon^2+20\epsilon+4,$$

that is $5(1-\epsilon)(1+3\epsilon)$; and all the roots of z, and consequently of (x, y), will be impossible, unless z lies between 1 and $-\frac{1}{3}$.

Now $$f(2)=1+5\epsilon+10\epsilon^2,$$
$$f'(2)=3+5\epsilon;$$

so that when z has any real roots, that is when ϵ lies between 1 and $-\frac{1}{3}$, $f(2)$, $f'(2)$ are both positive, and the Sturmian functions are of the signs $+\ +\ +$.

Again, $$f(-2)=5-15\epsilon+10\epsilon^2=5(1-\epsilon)(1-2\epsilon),$$
$$f'(-2)=-5+5\epsilon;$$

so that, on the same supposition as before, the Sturmian functions are $\pm-+$, namely

$$+-+ \text{ when } \tfrac{1}{2}>\epsilon>-\tfrac{1}{3},$$
$$--+ \text{ when } 1>\epsilon>\tfrac{1}{2}.$$

In the former case two real roots, in the latter one real root of z lies between $2, -2$. Hence in the former case no real roots of z lie between the limits ∞, 2, and the limits $-2, -\infty$, and in the latter case one real root lies between those limits. Hence x, y will have four imaginary roots, unless ϵ lies between 1 and $\frac{1}{2}$, and two such roots in every other case.

[14] For the discriminant of $xy\phi(x, y)$ = the discriminant of $\phi(x, y)$ multiplied by the square of the product of the resultant of (x, ϕ) and of (y, ϕ).

Thus the discriminant of $(1, \epsilon, \epsilon^2, \eta^2, \eta, 1 \between x, y)^5$, when $\epsilon = \eta$, is negative when ϵ lies between 1 and $\frac{1}{2}$, but for every other value of ϵ is positive, save that it vanishes when

$$\epsilon = 1, \text{ or } \epsilon = \tfrac{1}{2}\,^{(15)}, \text{ or } \epsilon = -\tfrac{1}{3}.$$

(8) I now proceed to calculate the discriminant of the form

$$x^5 + 5\epsilon x^4 y + 10\epsilon^2 x^3 y^2 + 10\eta^2 x^2 y^3 + 5\eta x y^4 + y^5$$

for general values of ϵ, η. This will be accomplished most expeditiously by taking the resultant of the two derivatives of the above form, say U and V, where

$$U = x^4 + 4\epsilon x^3 y + 6\epsilon^2 x^2 y^2 + 4\eta^2 x y^3 + \eta y^4,$$
$$V = \epsilon x^4 + 4\epsilon^2 x^3 y + 6\eta^2 x^2 y^2 + 4\eta x y^3 + y^4;$$

so that $\quad \epsilon U - V = 6(\epsilon^3 - \eta^2) x^2 y^2 + 4(\epsilon\eta^2 - \eta) x y^3 + (\epsilon\eta - 1) y^4 = y^2 P,$

$$-U + \eta V = (\epsilon\eta - 1) x^4 + 4(\eta\epsilon^2 - \epsilon) x^3 y + 6(\eta^3 - \epsilon^2) x^2 y^2 = x^2 Q.$$

Hence

Resultant of $(U, V) = \dfrac{1}{(\epsilon\eta - 1)^4} \times$ *Resultant* of $(y^2 P, x^2 Q) =$ *Resultant* of (P, Q);

where $\quad P = 6(\epsilon^3 - \eta^2) x^2 + 4(\epsilon\eta^2 - \eta) xy + (\epsilon\eta - 1) y^2,$

$$Q = (\epsilon\eta - 1) x^2 + 4(\eta\epsilon^2 - \epsilon) xy + 6(\eta^3 - \epsilon^2) y^2.$$

Hence, calling Δ the discriminant of the original form, we obtain by the well-known formula for the resultant of two binary quadratics, writing for the moment

$$P = (B, 2\eta A, A \between x, y)^2, \quad Q = (A, 2\epsilon A, B' \between x, y)^2,$$

$$\begin{aligned}\Delta &= -(4\epsilon A^2 - 4\eta AB')(4\eta A^2 - 4\epsilon AB) + (A^2 - BB')^2 \\ &= (1 - 16\epsilon\eta) A^4 + 16(\epsilon^2 B + \eta^2 B') A^3 - 16\epsilon\eta BB'A^2 - 2BB'A^2 + B^2 B'^2.\end{aligned}$$

Hence, writing $\epsilon\eta = q$, $\epsilon^5 + \eta^5 = S$,

$$\Delta = (1 - 16q)(q - 1)^4 + 96(S - 2q^2)(q - 1)^3 - 72(8q + 1)(q^3 + q^2 - S)(q - 1)^2 + 36^2(q^3 + q^2 - S)^2.$$

Let $S - q^2 - q^3 = \sigma$, $q - 1 = p$, so that

$$S - 2q^2 = \sigma - q^2 + q^3 = \sigma + (p + 1)^2 p.$$

Then

$$\begin{aligned}\Delta &= 36^2\sigma^2 + 72(8p + 9)p^2\sigma + 96p^3\sigma + 96(p + 1)^2 p^4 - (16p + 15)p \\ &= 1296\sigma^2 + (648p^2 + 672p^3)\sigma + 96p^6 + 176p^5 + 81p^4 \\ &= \tfrac{1}{9}\{(108\sigma + 27p^2 + 28p^3)^2 + 729p^4 + 1584p^5 + 864p^6 - (27p^2 + 28p^3)^2\},\end{aligned}$$

or

$$9\Delta = (108\sigma + 27p^2 + 28p^3)^2 + 72p^5 + 80p^6.$$

(15) When $\epsilon = \frac{1}{2}$ the discriminant of $f(z)$ does not vanish, but $z = -2$ satisfies the equation in z, and consequently $\dfrac{x}{y}$ has two equal roots -1, so that the discriminant of the original equation vanishes.

(9) Hence we see at once that Δ can be negative only when p lies between 0 and $-\frac{9}{10}$, that is when $\epsilon\eta$ (which is $p+1$) lies between 1 and $\frac{1}{10}$. Accordingly when Δ is negative, ϵ and η must be both positive or both negative. The latter supposition may easily be disproved as follows: treating the equation $\Delta = 0$ as a quadratic equation in σ, in order that Δ may be capable of becoming negative, its discriminant in respect to σ must be negative, and its value when $\sigma = -\infty$ is positive. Now

$$S = \epsilon^5 + \eta^5, \quad p + 1 = \epsilon\eta, \quad \sigma = S - (p+1)^2 - (p+1)^3;$$

so that when ϵ and η are real we have

$$S > 2(p+1)^{\frac{5}{2}}\,{}^{(16)}, \text{ that is } \sigma > -(p+1)^2 + 2(p+1)^{\frac{5}{2}} - (p+1)^3$$

when ϵ, η are both positive, and

$$S < -2(p+1)^{\frac{5}{2}}\,{}^{(16\,\text{bis})}, \text{ that is } \sigma < (p+1)^2 + (p+1)^3 - 2(p+1)^{\frac{5}{2}}$$

when ϵ, η are both negative.

If now we substitute $(p+1)^2 + (p+1)^3 - 2(p+1)^{\frac{5}{2}}$ for σ in Δ, I say that the resulting value will be positive whatever positive value be given to $(p+1)$; in fact, if we write $p = \nu^2 - 1$, and make $\sigma = -\nu^4 + 2\nu^5 - \nu^6$, so that Δ becomes a function of the twelfth degree in ν, this function is what the discriminant of the equation in x, y becomes when we have $\epsilon = \eta = \nu$; but in the antecedent Lemma it has been shown that this discriminant is only negative when the two equal quantities ϵ or η, or, which is the same thing, when ν lies between 1 and $\frac{1}{2}$; hence Δ is positive when ν is negative, and consequently when

$$\sigma = (p+1)^2 + (p+1)^3 - 2(p+1)^{\frac{5}{2}}.$$

Thus Δ, a quadratic function in σ, and its discriminant are respectively + and − for this value of σ, as well as for $\sigma = -\infty$. Hence no real root of σ lies between such value of σ and $-\infty$, and consequently Δ must be always positive when ϵ and η are both negative. Hence, if Δ is negative, we must have $1 > \epsilon\eta > \frac{1}{10}$; $\epsilon > 0$; $\eta > 0$. But our *criteria* give $\epsilon^4 - \epsilon\eta^2 > 0$, $\eta^4 - \eta\epsilon^2 > 0$, which, when $\epsilon > 0$, $\eta > 0$, imply $\epsilon^3 > \eta^2$, $\eta^3 > \epsilon^2$, and consequently $\epsilon\eta > 1$, which is in contradiction to the inequality $1 > \epsilon\eta$. Hence when these criteria are satisfied the determinant is necessarily *positive*, and all the roots are imaginary, which completes the proof of Newton's rule for equations of the fifth degree.

(10) It follows as a corollary to the Lemma employed in the preceding investigation, that if in Δ we write $\sigma = -(\nu^2 - \nu^3)^2$ and $p = \nu^2 - 1$, and distinguish this particular value by the symbol (Δ), then (Δ) ought to break up into the product of odd powers of $\nu - 1$, $\nu - \frac{1}{2}$, of some even power of $(\nu + \frac{1}{3})$, and of a factor incapable of changing its sign, and remaining always positive. This may be easily verified; for dividing (Δ) by $(\nu - 1)^4$, we obtain

$$1296\nu^8\,[648(\nu+1)^2 + 24(\nu^2-1)(\nu+1)^2]\,\nu^4 + 96(\nu^2-1)^2(\nu+1)^4$$
$$+ 176(\nu^2-1)(\nu+1)^4 + 81(\nu+1)^4;$$

(16) It is of course understood that $(p+1)^{\frac{5}{2}}$ is to be taken *positive*.

and collecting the terms $1296\nu^8 - 648\nu^6(\nu+1)^2 + 81(\nu+1)^4$ whose sum contains the factor $(\nu-1)$, we have

$$\begin{aligned}
\frac{(\Delta)}{(\nu-1)^5} = &\ \ 648(\nu^7+\nu^6+\nu^5+\nu^4+\nu^3+\nu^2+\nu+1)\\
&-1296(\nu^6+\nu^5+\nu^4+\nu^3+\nu^2+\nu+1)\\
&-\ \ 648(\nu^5+\nu^4+\nu^3+\nu^2+\nu+1)\\
&+\ \ \ 81(\nu^3+5\nu^2+11\nu+15)\\
&-\ \ \ 24(\nu^7+3\nu^6+3\nu^5+\nu^4)\\
&+\ \ \ 96(\nu^7+5\nu^6+9\nu^5+5\nu^4-5\nu^3-9\nu^2-5\nu-1)\\
&+\ \ 176(\nu^5+5\nu^4+10\nu^3+10\nu^2+5\nu+1)\\
= &\ 720\nu^7-240\nu^6-328\nu^5+40\nu^4+65\nu^3+5\nu^2-5\nu-1.
\end{aligned}$$

Hence

$$\begin{aligned}
(\Delta) &= (\nu-1)^5(2\nu-1)^3\{90\nu^4+105\nu^3+49\nu^2+11\nu+1\}\\
&= (\nu-1)^5(2\nu-1)^3(3\nu+1)^2\{10\nu^2+5\nu+1\};
\end{aligned}$$

showing, agreeably with what was seen in the Lemma, that the discriminant of

$$(1,\ \epsilon,\ \epsilon^2,\ \epsilon^2,\ \epsilon,\ 1 \between x,\ y)^5$$

vanishes then, and then only, when

$$\epsilon = 1, \text{ or } \epsilon = \tfrac{1}{2}, \text{ or } \epsilon = -\tfrac{1}{3},$$

but does not *change its sign*, except as ϵ passes through the limits 1 and $\frac{1}{2}$, and only within those limits can become negative [17].

(11) Although the theory of the possibility of the roots of

$$(1,\ \epsilon,\ \epsilon^2,\ \eta^2,\ \eta,\ 1 \between x,\ y)^5 = 0$$

has now been completely investigated, so far as is necessary for the proof of Newton's theorem applied to equations of the fifth degree, it will be found that the labour will not be ill spent of considering more closely the real

(17) In *general* the case of equal roots of an equation is the state of transition of two real roots into imaginary, or *vice versâ*. But we see by the above instance that this is not necessarily the case *always*, for Δ vanishes on making $\epsilon = -\frac{1}{3}$, and two roots become equal without any change in the nature of the roots when ϵ passes from being greater to being less than $-\frac{1}{3}$. In such case, however, there is a sort of unstable equilibrium in the form of the equation, by which I mean that the effect of any general infinitesimal change performed upon the coefficients of the equation would be either to cause the real roots in the neighbourhood of $\epsilon = -\frac{1}{3}$ to disappear by the factor $(\epsilon+\frac{1}{3})^2$ becoming superseded by a quadratic function of ϵ with impossible roots, or else a region in the neighbourhood of $\epsilon = -\frac{1}{3}$ would reappear, for which the equation would acquire two real roots, owing to $(\epsilon+\frac{1}{3})^2$ becoming superseded by a quadratic function of ϵ with real roots, in which case there would be two values in the neighbourhood of $\epsilon = -\frac{1}{3}$, for *each* of which there would be a pair of equal roots in the equation. The above is probably the first instance distinctly noticed of this singular obliteration of the usual effect upon real and imaginary roots of a passage through equality, owing to the appearance of a square factor in the discriminant.

nature of the criteria which separate the case of one pair from that of two pairs of impossible roots in the above equation. Newton's *criteria* being constructed so as to cover every possible case for equations of every degree, will always be found to fit loosely, so to speak, upon each case treated *per se*; so that more precise conditions can be assigned in each particular case than those which are furnished by his rule. So, for example, it may be remembered that in the equation $(1, e, e^2, e, 1 \between x, y)^4 = 0$, Newton's rule implies only that when $e > 1$, the roots are all impossible; but we have found further that unless $1 > e > \frac{1}{3}$ (a much closer condition), the same thing takes place.

It is obvious from what has been demonstrated above, that if we treat p and σ, which are respectively $\epsilon\eta - 1$ and $\epsilon^5 + \eta^5 - \epsilon^2\eta^2 - \epsilon^3\eta^3$, as the abscissa and ordinate of a variable point in a plane, the curve $\Delta = 0$, that is $(108\sigma + 27p^2 + 28p^3)^2 + 72p^5 + 80p^6 = 0$ will be the line of demarcation between those values of ϵ, η which correspond to one pair, and those which correspond to two pairs of imaginary roots.

For all values of ϵ, η corresponding to internal points of the curve Δ there will be two imaginary and three distinct real roots; for all such as correspond to external points there will be four imaginary roots, and for points *on* the curve two imaginary and two equal roots.

The curve Δ is a curve of the 6th degree whose form* will presently be discussed. But there is an important remark to be made in the first instance. Not all the points within the curve Δ will correspond to *real* values of ϵ, η. In order that these quantities may be real, we must have

$$\epsilon^5 + \eta^5 > 2(\epsilon\eta)^{\frac{5}{2}},$$

that is
$$\sigma + q^2 + q^3 > 2q^{\frac{5}{2}}, \text{ where } q = p + 1,$$

or
$$\sigma^2 + 2(q^2 + q^3)\sigma + q^4 - 4q^5 + q^6 > 0.$$

Writing this inequality under the form $R > 0$, we see that the curve $R = 0$ will represent a second sextic curve intersecting the former. Δ may be called the curve of the discriminant or *discriminatrix*, and will be a closed curve, and R the curve of equal parameters or *equatrix*, and will consist of a single infinite branch. All points on the latter correspond to equal values of ϵ, η, those on one side of it to real values of ϵ, η, and those on the other side of it to conjugate values of the form $\lambda + i\mu$, $\lambda - i\mu$ respectively. Thus the area confined within the curve Δ will be divided into two portions by the equatrix, and it is impossible to shut one's eyes to the inquiry as to the meaning of the variable point lying in that portion which gives conjugate values to ϵ, η. It becomes clear by analogy that some kind of distinction must be capable of being drawn between the nature of the roots of the equation $(1, \epsilon, \epsilon^2, \eta^2, \eta, 1 \between x, y)^5 = 0$ when ϵ, η are conjugate, in some sense similar or parallel to that which we know to exist between them when ϵ, η are real; and obviously this inference

[* See the Figure, p. 478 below.]

cannot be confined to equations of the particular form and degree of that above written; in a word, equations whose coefficients are not real but conjugate, must have roots of two kinds, one analogous to the real, the other to the imaginary roots of equations with real coefficients. This inference will be justified in the sequel; but in the meanwhile it will be desirable to complete the investigation of the special equation under consideration, by a discussion of the forms and relations of the two curves Δ and R. These curves we know *à priori*, from what has been already demonstrated, can only meet in the three points corresponding to

$$\epsilon = \eta = 1, \quad \epsilon = \eta = \tfrac{1}{2}, \quad \epsilon = \eta = -\tfrac{1}{3};$$

and since $p = \epsilon\eta - 1$, the abscissæ of these three points will be $0, -\frac{1}{4}, -\frac{8}{9}$.

Moreover the 3rd point will be distinguished from the other two by the circumstance that Δ does not change its sign as p passes through the value $-\frac{8}{9}$. Consequently the two curves must touch each other at this point.

Since when $\Delta = 0$ p lies between 0 and $-\frac{9}{10}$, the curve Δ is confined to the negative side of the axis of σ. It is also confined to the negative side of the axis of p.

For between the limits $p = 0$, $p = -\frac{9}{10}$,

$648p^2 + 672p^3$, that is $24(27p^2 + 28p^3)$ is obviously positive,

and $\qquad 96p^6 + 176p^5 + 81p^4 = \dfrac{p^4}{6}\{(24p + 22)^2 + 2\}$ is always positive.

Hence the two values of σ are both negative throughout the extent of the curve Δ.

Thus $\epsilon^5 + \eta^5 - \epsilon^2\eta^2 - \epsilon^3\eta^3$ being negative, $\epsilon^3 - \eta^2$ and $\eta^3 - \epsilon^2$ have the same signs when ϵ, η are *real*, as should be the case; for in order that Δ may be capable of vanishing, $\epsilon(\epsilon^3 - \eta^2)$ and $\eta(\eta^3 - \epsilon^2)$ must, by Newton's rule, be *both* negative, which could not be the case if either ϵ or η were negative; so that $\epsilon^3 - \eta^2$ and $\eta^3 - \epsilon^2$ must have the same signs, in fact each must be negative.

The curve Δ under consideration has a multiple point of the 4th order of multiplicity at the origin, where it is touched by the axis of p. Its distance from the axis for the extreme value of p, namely $p = -\frac{9}{10}$, is $\frac{27}{2000}$.

It has three real maxima and minima, two belonging to its upper portion and one to the lower portion at the points, for which p has the *approximate* values $-\frac{9}{16}$, $-\frac{15}{22}$, and $-\frac{7}{8}$ [18].

[18] The large numbers which enter into Δ may be usefully reduced, and the equation $\Delta = 0$ made more manageable, by aid of the simple substitutions $\sigma = -\dfrac{27v}{64}$, $p = -\dfrac{9u}{4}$. The equation $\Delta = 0$ then becomes

$$(v - 3u^2 + 7u^3)^2 = 2u^5 - 5u^6,$$

The curve R, that is $\sigma=((p+1)\pm(p+1)^{\frac{3}{2}})^2$, has the values 0 and -4 at the origin, a cusp at its extremity corresponding to $p=-1$, where both of its branches meet and touch the axis of p, and a negative maximum in its upper branch at the point where $p=-\frac{5}{9}$.

At all points within the curve R, ϵ and η are conjugate, and for the points outside real. Its lower branch will meet and touch the lower portion of Δ at the point where $p=-\frac{8}{9}$, and its upper branch will intersect and pass out of the upper branch of Δ at the point where $p=-\frac{3}{4}$. The only part of the area Δ therefore which corresponds to real values of ϵ, η, is that which is included between the upper segment of Δ and the upper branch of R, and extends only from $p=0$ to $p=-\frac{3}{4}$, that is from $\epsilon\eta=1$ to $\epsilon\eta=\frac{3}{4}$. Hence we may easily find an inferior limit to the values of ϵ and η when the equation $(\epsilon, \eta)=0$ has two real roots; for we have in that case ϵ, η, $\eta^2-\epsilon^3$, $\epsilon^2-\eta^3$ all positive. Hence

$$\eta^5 > \epsilon^3\eta^3 > q^3, \quad \eta^5 < \epsilon^2\eta^2 < q^2.$$

Consequently ϵ, η must each of them always lie between $q^{\frac{2}{5}}$, $q^{\frac{3}{5}}$; and since the least value of q is $\frac{1}{4}$, ϵ, η must each be always greater than $(\frac{1}{4})^{\frac{3}{5}}$, that is than ·33499 (19).

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whose maxima and minima will be given by the equation

$$(v-3u^2+7u^3)(-6u+21u^2)=5u^4-15u^5;$$

which, making $1-3u=\omega$, becomes

$$270\omega^3-46\omega^2-9\omega+1=0,$$

whose roots are all real, and are one just a little greater than $-\frac{1}{6}$, another a little less than $\frac{1}{4}$, and the third a very little less than $\frac{1}{11}$ respectively; whence $p=\frac{3}{4}(\omega-1)$ will have the approximate values given in the text.

(19) $\epsilon:\eta$ will have a maximum value, which can be found by writing $\delta\epsilon:\delta\eta::\epsilon:\eta$; and consequently, remembering that $q=p+1$, $S=\epsilon^5+\eta^5$, $\sigma=S-q^2-q^3$,

$$\delta S:\delta q::5S:2q,$$

and therefore $$\delta\sigma:\delta p::5\sigma+q^2-q^3:2q::5\sigma+p(p+1)^2:2(p+1).$$

Substituting the values of $\delta\sigma:\delta p$ in $\delta\Delta=0$, and combining the result with the equation $\Delta=0$, p and σ may be found by the solution of a numerical equation of the 5th degree, and then ϵ and η may be found by the solution of a quadratic and the extraction of 5th roots. To find the maxima and minima values of ϵ and η themselves exactly would lead to the solution of an equation of a degree quite unmanageable.

But we may first find the greatest maximum and least minimum values of S, that is, $\epsilon^5+\eta^5$, by making $\delta\sigma=(2q+3q^2)\delta q$ in $\delta\Delta=0$, which leads to an equation (I forget whether) of the 3rd or 5th degree (it is one of the two): calling this maximum and minimum m, μ respectively, and naming ρ (which of course must exceed unity) the greatest quotient of $\frac{\epsilon}{\eta}$ or $\frac{\eta}{\epsilon}$, we shall have

$$\sqrt[5]{\left(\frac{\rho^5}{1+\rho^5}\right)}m>\epsilon; \quad \eta>\sqrt[5]{\left(\frac{1}{1+\rho^5}\right)}\mu.$$

These limits will be tolerably near to the absolute maximum and minimum values of ϵ or η. It may be noticed that we know, from what has gone before, that ρ can never exceed $\left(\frac{1}{q}\right)^{\frac{1}{5}}$; and consequently ρ^5 cannot exceed 4, since q is always $>\frac{1}{4}$.

There is a third curve not undeserving of notice, of only the 3rd degree, which embodies the joint effect of the two middle criteria (the two extremes being supposed to be each zero) in the two cases where Newton's rule will prove all the roots of the equation under consideration to be impossible. These criteria are $c_1 = \epsilon^4 - \epsilon\eta^3$, $c_2 = \eta^4 - \eta\epsilon^3$. But

$$c_1\eta^4 + c_2\epsilon^4 = q(2q^3 - S) = q(2q^3 - q^3 - q^2 - \sigma) = q(q^3 - q^2 - \sigma),$$

which for all values of q on the positive side of the line $p = -1$ (that is $q = 0$) will have the same sign as $q^3 - q^2 - \sigma$, which we may call K[20]; and K positive will evidently imply that c_1, c_2 are one or both of them positive. The whole plane will be divided by the curve K into an *upper* region (commencing at $\sigma = \infty$), for which K is negative, and a lower region, in which K is positive. For any point of the curve K, $\sigma = q^3 - q^2$, which within the limits of q with which we are concerned, namely those within which Δ lies, is negative; for any point of the curve R, the smaller absolute value of σ is

$$-q^3 - q^2 + 2q^{\frac{5}{2}} = q^3 - q^2 + 2(q^{\frac{5}{2}} - q^3),$$

which $< q^3 - q^2$ within the limits in question. So that, remembering that each of these values of σ is negative, we see that the portion of the area Δ corresponding to real values of ϵ, η will be completely above the curve K, that is in the negative region of K, and that accordingly Δ for *real values* of ϵ, η can never vanish when K is positive, as should be the case. This remark does not, however, apply to the conjugate region of Δ; for the curve K will *pass through*[21] the lower or conjugate portion of the area Δ.

(12) I may now say a few words on the signification of that portion of Δ in which ϵ and η are conjugate imaginary quantities.

(20) I call K the Indicatrix, as exhibiting the joint effect of the *indicia* or criteria of the Rule.

(21) This may easily be verified; for at the point $p = -\frac{3}{4}$ it will be found that the ordinate in K and the lower ordinate in Δ are equal, and at the point $p = -\frac{9}{10}$ the lower ordinate in Δ is $-\frac{27}{2000}$, and in K is $-\frac{18}{2000}$; which shows that the curve K entering the area Δ when at the lower half of the curve, at a point where $p = -\frac{3}{4}$, must pass through its upper contour in order to cut the line $p = -\frac{9}{10}$ as it does above the point where Δ is touched by that line.

The curve K has its negative maximum at the point $q = \frac{2}{3}$, that is, $p = -\frac{1}{3}$. It passes through the origin, and begins with sweeping under the curve Δ, which it enters exactly under the point where R quits Δ, and passes through Δ at a point very close indeed to the horizontal extremity of Δ. It may be noticed that when $p = -\frac{3}{4}$, the smaller ordinates of R and Δ are each $-\frac{1}{64}$, the ordinate of K and the larger ordinate of Δ being each $-\frac{3}{64}$.

I have found the points of contact of K with Δ by actually substituting $q^3 - q^2$, that is, $p(p+1)^2$ for σ in $\Delta = 0$. This gives the equation

$$2064p^4 + 7352p^3 + 9823p^2 + 5832p + 1296 = 0,$$

one factor of which is $4p+3$, dividing out which we have

$$516p^3 + 1451p^2 + 1368p + 432 = 0.$$

The Newtonian criterion applied to the three first coefficients of the above gives $-1362\frac{5}{9}$, showing that two of the roots are impossible; the remaining real root I find to be ·8946, &c. It does not appear to be a rational number.

In general, let

$$(a + i\alpha,\ b + i\beta,\ c + i\gamma, \ldots\ldots, c - i\gamma,\ b - i\beta,\ a - i\alpha \between x,\ y)^n = 0$$

be an equation in which all the coefficients, reckoning simultaneously from the two ends, are conjugate to one another, and the central coefficient, if there is one, which can only be when n is even, *real.*

Let $\frac{x}{y} = p + iq$ satisfy this equation. Then evidently $\frac{y}{x} = p - iq$ will also satisfy it; or, which is the same thing, $\frac{x}{y} = \frac{p + iq}{p^2 + q^2}$ will satisfy it.

Now either this root will be identical with the former one, or a distinct root; in the former case we must have $p^2 + q^2 = 1$, and the root will be of the form $\cos\alpha + i\sin\alpha$; in the second case $p^2 + q^2$ will differ from unity, and there will be a pair of imaginary roots of the form $\rho(\cos\alpha + i\sin\alpha)$, $\frac{1}{\rho}(\cos\alpha + i\sin\alpha)$, in which the real parts ρ, $\frac{1}{\rho}$ are reciprocal to one another, and the directive parts $e^{i\alpha}$ identical. Moreover, if we write the given equation under the form $U + iV = 0$, and suppose, as can always be done, that U and V have been divested of any algebraical common factor, it may easily be shown that the equation so prepared, and which may be called a Conjugate Equation *proper*, can have no real roots and no *pairs of imaginary roots* in the sense in which that term is employed in the theory of equations with real coefficients; but the distinction between *simple* or solitary and *twin* or associated roots reappears in the theory of conjugate equations, under a different form. It will of course be understood that the class of simple roots for which the modulus is unity is quite as general as that of twin roots, for each of which the modulus may be anything different from unity, just as in the ordinary theory the case of real is quite as general as that of imaginary roots, although the former may be represented by points on a fixed straight line, whilst the points representing the latter may be anywhere in the plane, this liberty of displacement being balanced, so to say, by the constraint of coupling. The general geometrical representation of the roots of a real equation is a system of points in a line, and a system of pairs of points at equal distances on opposite sides of the line. So the general geometrical representation of the roots of a conjugate equation will be a system of points in the circumference of a circle of radius unity, and of points situated in pairs in the same radii at reciprocal distances from the centre. In a word, in each case we may say that the roots can be geometrically represented by points on a circle, and pairs of points electrical images of each other in respect to the circle, but the radius of the circle in the one case will be infinity, in the other unity. Conjugate like real equations will have all their invariants of an even degree real, and those of an odd degree will be pure imaginaries, or real quantities affected with the multiplier i.

Their morphological derivatives (covariants, contravariants, &c.) will be also conjugate forms. The whole doctrine of equations, as regards the separation of real from imaginary roots, and the determination of the limits within which the former lie, will reproduce itself with suitable modifications in the theory of conjugate equations, in which simple, on the one hand, and coupled or twin roots, on the other, will correspond respectively as analogues to the real and imaginary roots of the ordinary theory. Thus the following theorem may be demonstrated without difficulty, namely, in any conjugate equation the number of coupled roots is congruent to 0 in respect to the modulus 4 when the discriminant is positive, and to 2 in respect to the same modulus when the discriminant is negative [22]. We see now how to interpret the effect of the variable point whose coordinates are $\epsilon^5+\eta^5$ and $\epsilon\eta$ lying within the area Δ, in that portion of it for which ϵ, η become imaginary; namely it is that in such case the equation $(\epsilon, \eta)=0$, which then becomes of a conjugate form, will have three simple and two twin roots; and thus the unity of the interpretation is restored if we choose, as we very well may, to extend the use of these terms to the real roots and the paired imaginary roots of ordinary

[22] (*a*) A very simple linear transformation shows the immediate connexion between the solitary and associated roots of conjugate with the real and paired imaginary roots of ordinary equations. For if $f(x, y)=0$ be a conjugate equation, writing

$$y=v+iu, \quad x=v-iu,$$

$f(x, y)$ becomes $F(u, v)$, a real form in u, v.

When u, v are real, we have

$$\frac{y}{x}=\frac{v+iu}{v-iu}=\cos\left(2\tan^{-1}\frac{u}{v}\right)+i\sin\left(2\tan^{-1}\frac{u}{v}\right);$$

when $\frac{v}{u}=c\pm i\gamma$, the two values correspond to

$$\frac{y}{x}=\frac{c+i\gamma+i}{c+i\gamma-i}, \quad \left(\frac{y}{x}\right)'=\frac{c-i\gamma+i}{c-i\gamma-i}.$$

Thus
$$\frac{y}{x}:\left(\frac{y}{x}\right)'::c^2+(\gamma+1)^2:c^2+(\gamma-1)^2;$$

also
$$\frac{y}{x}\times\left(\frac{y}{x}\right)'=\frac{c^2-1+\gamma^2+2ci}{c^2-1+\gamma^2-2ci},$$

of which the modulus is obviously unity.

(*b*) Now it is known that if t be the number of real, and τ of imaginary roots in the real form, $(u, v)^n$, its discriminant, bears the sign $(-)^{\frac{t(t-1)}{2}}$. Hence the sign of the discriminant of the conjugate form $(x, y)^n$ (since the determinant of $v+iu$, $v-iu$ is $2i$) will be $(-)^q$, where

$$q=\frac{n(n-1)}{2}+\frac{t(t-1)}{2}=\frac{(t+\tau)(t-1+\tau)+t(t-1)}{2}=t(t-1)+t\tau+\frac{\tau(\tau-1)}{2}.$$

Hence since τ and $t(t-1)$ are both even, $(-)^q=(-)^{\frac{\tau(\tau-1)}{2}}$, and the sign of the discriminant of a conjugate form is + or − according as the number of imaginary roots does or does not contain 4 as a factor.

It must be remembered that the sign of the discriminant is not in general the same as that of the *zeta* or squared product of differences of the roots. The sign of the *zeta* for real equations follows precisely the same law as the sign of the *discriminant* for conjugate ones.

equations. We may neglect the curve of reality R altogether, and affirm that all over the area Δ, ϵ, η will have such values as will give rise to three simple and two coupled roots.

(13) That part of the theorem of Newton which had received a demonstration from Maclaurin and Campbell in the generalized form in which I have enunciated it in this paper, may be easily extended to the case of conjugate equations. It will, as applied to them, read thus: If the $(n-1)$ quadratic derivatives of a conjugate form of the nth degree, all whose roots are simple, be multiplied respectively by the coefficients of any other conjugate form, all whose roots are also *simple*, of the degree $(n-2)$, and the sum of these products be taken as a new quadratic form, the discriminant of this latter must be positive, or, which is the same thing, its determinant must be negative.

(14) So much for the case of $n=5$. If we were to proceed to the consideration of equations of the 6th degree, *two* cases of resistance would present themselves in the demonstration of Newton's rule, namely one in which the signs of the criteria are $-+++-$, the other $-+-+-$. In the latter it would only be necessary to show that the discriminant is necessarily negative, since we know from the derivatives that the equation must have four imaginary roots, and the choice would lie between the alternatives of there being four or six. In the former case the derivatives only indicate the necessary existence of two real roots, and it would become requisite to prove that there must be four or six—an alternative which depends not on the sign of one function of the coefficients, but on the nature of the signs of two such functions given by Sturm's or any equivalent theorem. It would thus become requisite to prove that two functions of the coefficients, say L, M, could not *both* be negative; and this might be shown by demonstrating the existence of two quantities, L', M', other functions of the coefficients incapable of assuming any but the positive sign such that $L'L+M'M$ would be necessarily positive.

Part II.—On the limit to the number of real roots in equations of the form $\Sigma(ax+b)^n$.

(15) I shall now proceed to the consideration of a theorem relating to a particular class of ordinary equations, which occurred to me in the course of and in connexion with the preceding investigations. The theorem itself, but unaccompanied by proof, has appeared in the *Comptes Rendus* of the Academy for the month of March 1864 [above, p. 360].

Both as regards its nature and the processes involved in the proof, it stands in close relation to Newton's rule, my study of which in fact led me to its discovery. It will therefore take its place most appropriately in this paper.

Certain preliminary properties of circulation introducing some new notions of polarity must be first established, by way of Lemmas to the proof in question.

By a *type* let us understand a succession of symbols of any subject matter whatever susceptible of receiving the signs $+-$, or any suchlike indications of opposite polarity.

Let $a, b, c, \ldots i, k, l$ be any such type, where the *elements* $a, b, c, \ldots$ may be regarded either as points in a line or rays in a pencil affected respectively with the signs of $+$ and $-$.

Then by a *per-rotatory* circulation of such type, I mean the act of passing from the first element to the second, from the second to the third, &c., from the last but one to the last, and from the last to the first.

By a *trans-rotatory* circulation of the same, I mean the act of passing from the first to the second, the second to the third, &c., from the last but one to the last, and from the last to the first, *with its sign reversed.*

A type considered subject to per-rotatory circulation may be termed a Per-rotatory Type; one subject to the other sort of circulation, a Trans-rotatory Type.

If a, b, c, d, e be a per-rotatory type, its direct *phases* are

$$\begin{array}{ccccc} a, & b, & c, & d, & e, \\ b, & c, & d, & e, & a, \\ c, & d, & e, & a, & b, \\ d, & e, & a, & b, & c, \\ e, & a, & b, & c, & d, \end{array}$$

and its retrograde phases

$$\begin{array}{ccccc} a, & e, & d, & c, & b, \\ e, & d, & c, & b, & a, \\ d, & c, & b, & a, & e, \\ c, & b, & a, & e, & d, \\ b, & a, & e, & d, & c. \end{array}$$

If, on the other hand, a, b, c, d, e be a trans-rotatory type, its direct *phases* will be

$$\begin{array}{ccccc} a, & b, & c, & d, & \bar{e}, \\ b, & c, & d, & \bar{e}, & \bar{a}, \\ c, & d, & \bar{e}, & \bar{a}, & \bar{b}, \\ d, & \bar{e}, & \bar{a}, & \bar{b}, & \bar{c}, \\ \bar{e}, & \bar{a}, & \bar{b}, & \bar{c}, & \bar{d}, \end{array}$$

and its retrograde phases

$$\begin{array}{ccccc} a, & \bar{e}, & \bar{d}, & \bar{c}, & \bar{b}, \\ \bar{e}, & \bar{d}, & \bar{c}, & \bar{b}, & \bar{a}, \\ \bar{d}, & \bar{c}, & \bar{b}, & \bar{a}, & e, \\ \bar{c}, & \bar{b}, & \bar{a}, & e, & d, \\ \bar{b}, & \bar{a}, & e, & d, & c, \end{array}$$

where the sign $(-)$ is, for greater convenience of writing, placed over instead of before the elements which it affects; and so on in general a type of n elements, whether per-rotatory or trans-rotatory, will admit of n direct and n retrograde phases.

If we count the number of variations of sign in the circulations of any phase of a per-rotatory type, this number will be the same for all the phases, and will be an even number; this even number may be termed the variation-index of the type.

So, again if, whatever be the original signs of the element in a trans-rotatory type, we count the number of variations in the circulation of any of its phases, this number also will be constant and will be odd, and this odd number may then be termed the variation-index of the type.

(16) Let any phase be taken of a per-rotatory type, and out of such phase let any element be *suppressed*; then we obtain a type one degree lower in the elements, which, if we please, we may consider as a trans-rotatory type, and such trans-rotatory type may be termed a derivative of the original per-rotatory one.

In like manner any phase being taken of a trans-rotatory type, one element may be suppressed, and the reduced type treated as a per-rotatory one, and termed a derivative of the original trans-rotatory one.

We may now enunciate the following important general proposition, namely:

Any trans-rotatory type or any per-rotatory type whose variation-index is different from zero being given, a per-rotatory derivative of the one and a trans-rotatory derivative of the other may be found such that the variation-index of the derived types in either case shall be less by a unit than the variation-index of the types from which they are derived.

Case (1). Let the given type be per-rotatory. Then by hypothesis, since it has some variations, we may find a phase of it beginning with $+$ and ending with $-$, by which I mean beginning with an element that is positive and ending with one that is negative. This gives rise to two sub-cases.

T, the phase in question, will be $+ \ldots\ldots + -$

Θ, the phase in question, will be $+ \ldots\ldots - -$.

In either sub-case let the last sign be suppressed, and the result treated as a trans-rotatory type; then T, Θ become respectively T', Θ', where

$$T' \text{ is } + \ldots\ldots +$$

and

$$\Theta' \text{ is } + \ldots\ldots -$$

and evidently the variation-index of $T -$ variation-index of $T' =$ number of changes of sign in $+ - +$ less changes of sign in $+ - = 2 - 1 = 1$; and again variation-index of $\Theta -$ variation-index of $\Theta' =$ number of changes of sign in $- - +$ less changes of sign in $- - = 1 - 0 = 1$. Hence the theorem is proved for the case where the given type is per-rotatory.

Case (2). Let the given type be *trans-rotatory*.

Then, again, there must either be a phase of the form P, or one of the form Φ, where P represents a *continual succession* of signs of the same name as $+ + \ldots +$ or $- - \ldots -$, and Φ represents a succession beginning with one sign as $+$ and ending with one or more signs $-$, or else beginning with $-$ and ending with a succession of signs $+$. Essentially, then, as a change of signs throughout a whole succession does not affect the variation-index, we may suppose

$$P = + \ldots\ldots + +,$$

$$\Phi = - \ldots - + \ldots +,$$

the signs intervening between the two expressed signs $-$ in Φ being filled up in any manner whatever, and those between the two signs $+$ with signs exclusively $+$.

Let now that phase of Φ be taken which commences with the first sign of the final succession of $+$. Then Φ becomes

$$(\Phi) = + \ldots + + \ldots . +,$$

which is of the form

$$+ \ldots . . + +,$$

so that P is only a particular case of (Φ). If the last sign in (Φ) be suppressed and the result treated as a per-rotatory type be called $(\Phi)'$, so that $(\Phi)' = + \ldots\ldots +$, we have variation-index in $(\Phi) -$ variation-index in $(\Phi)' =$ changes of sign in $- +$ less changes of sign in $+ + = 1 - 0 = 1$.

Hence the proposition is established for both cases.

(17) The theorem to which this Lemma-proposition is to be applied concerns equations of the form

$$\epsilon_1 u_1{}^m + \epsilon_2 u_2{}^m + \ldots + \epsilon_n u_n{}^m = 0,$$

where $u_1, u_2, \ldots, u_n$ are any linear functions of x, y; m is any positive integer, and $\epsilon_1, \epsilon_2, \ldots \epsilon_n$ are each respectively and separately, either *plus unity* or *minus unity*.

Such an equation for convenience of reference may be termed a superlinear equation, and the function equated to zero a superlinear function.

Every superlinear function may be conceived as having attached to it a pencil of rays constructed in a manner about to be explained.

We may conceive the function to be prepared in such a manner, that, supposing $ax+by$ to be any one of the n linear elements u, every b shall be positive. If m is even, this can be effected by writing when required for $ax+by$, $-ax-by$ without further change. If m is odd, we may write when required $-ax-by$ in place of $ax+by$, changing at the same time the factor ϵ, which appertains to $(ax+by)^m$ from $+1$ to -1, or, *vice versâ*, from -1 to $+1$.

Now take in a plane any two axes of coordinates $O\xi$, $O\eta$, and consider a, b as the ξ and η coordinates of a point. All the n points thus obtained, on account of every b being positive, will lie on the same side of the axis $O\xi$, and thus the entire n linear functions will be represented by a pencil of n rays, the two extreme rays of which make an angle less than two right angles with each other; but each term of the superlinear function contains, besides $(ax+by)^n$, a definite multiple $+1$, or -1, and we must accordingly, to completely express such term, conceive every ray affected with a distinct sign $+$ or $-$. A pencil thus drawn with its rays so polarized will give a complete representation of any given superlinear function, and may be called its type-pencil [(23)].

I am now able to state the following proposition:

(18) *The number of real roots in a superlinear equation cannot exceed the variation-index of its type-pencil, regarded as a per-rotatory type, if the degree of the equation be even, and as a trans-rotatory type if the degree of the equation be odd.* I prove this inductively as follows.

[(23)] Let a circle be imagined pierced by a pencil containing any number of rays protracted in both directions, say in the opposite points a, α; b, β; c, γ; d, δ; and let these points, taken in order of natural succession from left to right, or right to left, be $a, b, c, d, \alpha, \beta, \gamma, \delta$. Then, commencing with any point c, a *complete* circulation will be represented by the succession of transits

$$c \text{ to } d,\quad d \text{ to } \alpha,\quad \alpha \text{ to } \beta,\quad \beta \text{ to } \gamma,\quad \gamma \text{ to } \delta,\quad \delta \text{ to } a,\quad a \text{ to } b,\quad b \text{ to } c.$$

But whether $\alpha, \beta, \gamma, \delta$ bear respectively the same signs or signs contrary to those of a, b, c, d, the transit between any two points β to γ will be of the same nature, as regards continuance or change of sign, as the transit from b to c, and thus we see that the complete cycle or total revolution above indicated is only a reduplication of, and may be fully designated by the hemicyclic succession c to d, d to α, α to β, β to γ, for which the number of variations therefore will be the same as for any similar succession obtained by commencing with any other element in the original system of points instead of c. If the opposite points bear like signs, the above succession of transits may be indicated by the order c, d, a, b, c; if they bear contrary signs by the order $c, d, \bar{a}, \bar{b}, \bar{c}$, and thus it is that the idea arises of the two kinds of so-called circulation, but which are in fact only more or less disguised species of semicirculation.

Suppose the theorem to be true when the variation-index of the type-pencil is not greater than the even number ν, and consider an equation of the odd degree $(2i+1)$, for which the type-pencil viewed as trans-rotatory has the variation-index $\nu+1$.

Let a *phase* of this type be taken, say corresponding to the rays $\rho_n, \rho_{n-1} \dots \rho_2, \rho_1$, such that the per-rotatory type obtained by striking out the term ρ_1 has the variation-index ν (as we know may be done by virtue of the Lemma).

Take for new axes $O\xi'$, $O\eta'$, when $O\xi'$ coincides with ρ_1; then it is clear that the pencil $\rho_n, \rho_{n-1} \dots \rho_2, \rho_1$ will still serve as a type-pencil to the given function, the only change being that some of the rays, namely those that did lie on one side of ρ_1, have been inverted in direction and changed in sign (corresponding to a change in the coefficients a, b, accompanied with a change in the sign of the corresponding ϵ), whilst the rays on the other side of ρ_1 have been left unaltered.

The points (a_1, b_1), $(a_2, b_2) \dots (a_n, b_n)$ corresponding to the rays $\rho_1, \rho_2, \dots \rho_n$ will, with respect to the new axes, change their values, becoming converted into $(\alpha_1, 0)$, (α_2, β_2), $(\alpha_3, \beta_3), \dots (\alpha_n, \beta_n)$, where $\beta_2, \beta_3, \dots \beta_n$ will still all be positive, the angle between ρ_1 and ρ_n being the same as between the two extreme rays in the original figure of the type-pencil, and the superlinear equation may now be written in the form

$$F(u, v) = \epsilon_1(\alpha_1 u)^{2i+1} + \epsilon_2(\alpha_2 u + \beta_2 v)^{2i+1} + \epsilon_3(\alpha_3 u + \beta_3 v)^{2i+1} \dots + \epsilon_n(\alpha_n u + \beta_n v)^{2i+1} = 0,$$

where u, v are real linear functions of x, y.

Let the derivative of this function be taken in regard to v, and we have

$$\frac{1}{2i+1} F'(u, v) = \beta_2\epsilon_2(\alpha_2 u + \beta_2 v)^{2i} + \beta_3\epsilon_3(\alpha_3 u + \beta_3 v)^{2i} \dots + \beta_n\epsilon_n(\alpha_n u + \beta_n v)^{2i},$$

where $\beta_2\epsilon_2, \beta_3\epsilon_3 \dots \beta_n\epsilon_n$ have the same signs as $\epsilon_2, \epsilon_3, \dots \epsilon_n$ respectively.

Now the pencil-type of $F'(u, v)$ will be the per-rotatory type $\rho_n, \rho_{n-1}, \dots \rho_2$, of which by construction the variation-index is ν. Hence by hypothesis $F'(u, v)$ has not more than ν real roots, that is, at least $2i - \nu$ imaginary roots. Hence $F(u, v)$ has at least that number of imaginary roots, that is, at most $(2i+1) - (2i-\nu)$, that is, $\nu+1$ real roots. Hence if the theorem is true for ν an even number, it is true for $\nu+1$.

In like manner let us proceed to show that when it is true for ν an odd number, it would remain true for $\nu+1$.

The reasoning will be precisely similar to that followed in the antecedent case. We must find a phase of the *per-rotatory* type $\rho_n, \rho_{n-1}, \dots \rho_2, \rho_1$ having the variation-index $\nu+1$ such that the trans-rotatory reduced type $\rho_n, \rho_{n-1}, \dots \rho_2$ shall have the variation-index ν; the new pencil will still continue to be a type-pencil of the given superlinear function, the change of direction in the

bunch of rays on one side of ρ_1 being now unaccompanied with change of sign, such change corresponding to $\epsilon(ax+by)^{2i}$ becoming changed into $\epsilon(-ax-by)^{2i}$ without ϵ undergoing a change of sign.

As before, the axes of coordinates are transformed from ξ, η into ξ', η', and we obtain

$$F(u, v) = \epsilon_1(\alpha_1 u)^{2i} + \epsilon_2(\alpha_2 u + \beta_2 v)^{2i} + \ldots + \epsilon_n(\alpha_n u + \beta_n v)^{2i},$$

$$\frac{1}{2i} F'(u, v) = \beta_2\epsilon_2(\alpha_2 u + \beta_2 v)^{2i-1} + \ldots + \beta_n\epsilon_n(\alpha_n u + \beta_n v)^{2i-1},$$

for which the type-pencil is the trans-rotatory type $\rho_n, \rho_{n-1}, \ldots \rho_2$, of which by construction the variation-index is ν, so that its number of imaginary roots is $2i - 1 - \nu$, and consequently the number of real roots of $F(u, v)$ will be $\nu + 1$.

Thus, then, if the theorem be true for ν, whether ν be even or odd, it will be true for $\nu + 1$.

But when $\nu = 0$, the superlinear function becomes a sum of even powers of linear functions of x, y, all taken with the same sign, of which the number of roots is evidently 0. Hence, being true for this case, the proposition is true universally.

It will be noticed that the algebraical part (as distinguished from the purely polartactic part of the above demonstration) depends on the principle of which such abundant use has been made in the former part of this dissertation, namely that the number of imaginary roots in any ordinary algebraical equation in x cannot be increased when we operate any homographic substitution upon x, and take the derivative of the equation thus transformed in lieu of the original [24].

[24] For greater clearness I present in an inverted order of arrangement a summary of the foregoing argument.

By an ith derivative of $f(x, y)$ is meant any derived form

$$\left(\lambda_1 \frac{d}{dx} + \mu_1 \frac{d}{dy}\right)\left(\lambda_2 \frac{d}{dx} + \mu_2 \frac{d}{dy}\right) \ldots \left(\lambda_i \frac{d}{dx} + \mu_i \frac{d}{dy}\right) f(x, y),$$

the λ, μ quantities being any real quantities whatever. Then I say—

1. If T is the type-pencil (per-rotatory or trans-rotatory) of any superlinear form F, every derivative of T of the contrary name is the type-pencil of some first derivative of F, as shown in art. (18).

2. A derivative of T of contrary name may be found such that its variation-index shall be less by a unit than that of T itself, as shown in art. (16).

3. Hence if i is the variation-index of the type-pencil of F, an ith derivative of F may be found such that its variation-index shall be zero, and consequently having no real roots.

Hence, finally, since the number of real roots of any rational integral homogeneous function in x, y cannot exceed by more than i the number of the real roots in any of its ith derivatives, F cannot have more real roots than there are units in the variation-index of its type-pencil.

The subtle point of the argument, it will be noticed, lies in forming the conception of the variation-index to a trans-rotatory pencil, in which the singular phenomenon occurs of a reversal of *relative polarity* in passing from the last ray to the first, whereas in a per-rotatory pencil any ray indifferently may be regarded as the initial ray, no such reversal in that case taking place.

(19) The proposition above established leads immediately to the theorem and corollary following, namely:

THEOREM. If $c_1, c_2, \ldots c_n$ be a series of ascending or descending magnitudes, and m any positive integer, the equation

$$\lambda_1 (x + c_1)^m + \lambda_2 (x + c_2)^m + \ldots + \lambda_n (x + c_n)^m = 0$$

cannot have more real roots than there are changes of sign in the sequence $\lambda_1, \lambda_2, \ldots \lambda_n, (-)^m \lambda_1$.

For obviously $(1, c_1), (1, c_2), \ldots (1, c_n)$ will be points corresponding to rays within a semirevolution, and therefore forming a type-pencil.

Corollary. If the above equation be transformed by any real homographic substitution into the form

$$\mu_1 (y + \gamma_1)^m + \mu_2 (y + \gamma_2)^m + \ldots + \mu_n (y + \gamma_n)^m = 0,$$

where $\gamma_1, \gamma_2, \ldots \gamma_n$ are taken in ascending or descending order, the number of changes of sign in the series $\mu_1, \mu_2, \ldots \mu_n, (-)^m \mu_1$ is *invariable* [25]; for the effect of any such formation will be to leave the type-pencil unaltered except in its *phase*.

(20) If we look to the undeveloped form of the superlinear function

$$S = \epsilon_1 u_1{}^m + \epsilon_2 u_2{}^m + \ldots + \epsilon_n u_n{}^m,$$

and are supposed to possess no knowledge of the coefficients which enter into the linear elements u, we may still draw some general inferences as to the limit of the number of real roots in $S = 0$. Thus if the number of positive units ϵ is j, and of the negative units k, and j is not greater than k, it is obvious that, whatever may be the form of the type-pencil to S, its variation-index cannot be more than $2j$ when m is even, nor more than $2j + 1$ when m is odd; for the arrangement the most favourable to the largeness of the number of the real roots is that where every two rays with the signs belonging to the j group of ϵ are separated by one or more of the rays with a contrary sign to themselves. Thus it appears that when only the units $\epsilon_1, \epsilon_2, \ldots \epsilon_n$ are given, we may impose a maximum upon the number of real roots in the superlinear equation; this limit may be called the *absolute maximum*, being the double of the inferior number of like signs in the series $\epsilon_1, \epsilon_2, \ldots \epsilon_n$ when the degree is even, and one more than such double when the degree is odd [26].

[25] It may be noticed that, contrariwise, the limit to the number of real roots given by Newton's criteria is *not* an invariant; it fluctuates with the homographic transformations operated upon the equation; and a question suggests itself as to the maximum value the number of imaginaries indicated by the rule can attain. I presume this maximum is not in all cases necessarily the actual number of the imaginary roots possessed by the equation.

[26] (*a*) If a superlinear form of an odd degree contains an odd number of terms, say $2k + 1$,

The *specific maximum*, on the other hand, will depend on the form of the type-pencil, and cannot be ascertained until the coefficients of the linear elements are given. It can never exceed, but may be less than the absolute maximum. It may, indeed, be easily proved that *in general* the specific maximum will be less than the absolute maximum. Thus, by way of example, suppose the degree to be even, and the inferior number of like signs to be 2; the absolute maximum number of real roots will be four, but the specific maximum will more generally be only two. For let the number of linear terms in the superlinear function be $2+n$, n being 2 or any greater number; and first, to fix the ideas, suppose $n=2$. The type-pencil, which is to be read per-rotatorily, consists of four rays, say a, b, c, d, following each other in uninterrupted circular order, of which two are to bear positive and two negative signs. If the two negative signs fall on a, c or on b, d, the variation-index will be 4, but in the other four cases of incidence such index will be only 2. Consequently the chance is 2 to 1 [27] that the specific maximum, which may be 4, is not greater than 2; and consequently the chance that there will be four real roots in the equation will be only a chance (too difficult to be calculated, but which is a function of the degree of the equation) *of* the chance $\frac{1}{3}$ that there will be as many as four real roots in the equation $u_1{}^m+u_2{}^m-u_3{}^m-u_4{}^m=0$, where u_1, u_2, u_3, u_4 are unknown linear functions of x: thus we are entitled to say that *in general* the number of real roots in such an equation is *not* the maximum four, but a less number. This remark is of importance, as showing that on this subject it is possible to speak with scientific

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the greatest value of the *inferior* number of like signs is k, and the extreme limit to the number of real roots will be $2k+1$.

If it contain an even number of terms, say $2k$, the greatest value of the inferior index is k; but for this particular case it will readily be seen that a limit may be assigned to the variation-index closer than that given by the rule in the text; in fact the variation-index cannot in that case exceed $2k-1$, which will therefore be the extreme limit to the number of real roots. Now suppose the canonizant of an odd-degreed function of x, y to have all its roots real, then it may be expressed by a superlinear form of which the number of terms will be $2i+1$ or $2i$, according as the degree is $4i+1$ or $4i-1$. In the one case the number of real roots cannot exceed $2i+1$, in the other $2i-1$. Hence the following somewhat curious theorem:

(*b*) *If the canonizant of an odd-degreed quantic in x, y, of the degree $4i\pm1$, has* no *imaginary roots, the quantic itself must have at least i pairs of imaginary roots.* From the fact that when the roots of the canonizant of a quintic are all real there must be one pair at least of imaginary roots, we can infer that when the discriminant of a quintic is positive and that of its canonizant is negative, the equation has one real and four imaginary roots. This observation has led to a long train of reflections, which will be found embodied in the 3rd part of the memoir.

([27]) This, in fact, is identical in substance with the noted problem of determining the chance that two straight lines drawn on a black board will cross. Mr Cayley, of whom it may be so truly said, whether the matter he takes in hand be great or small, "nihil tetigit quod non ornavit," suggests the following independent proof of this. Taking unity as the length of the contour, fixing the extremity of one of the lines, and calling s the distance of its other end from it measured on the contour, the chance of the second line crossing this is easily seen to be $2s(1-s)$, which, integrated between $s=0$, $s=1$, gives $\frac{1}{3}$, as before obtained.

certainty, and on other than empirical grounds, of what may *in general* be expected to take place. Thus we find Newton declaring twice over in the chapter quoted, that *in general* his rule will give not merely the maximum, but the actual number of the imaginary roots in an equation. I am strongly inclined to doubt the truth of this assertion; but it is important to be satisfied by analogy that such an assertion may rest on a scientific and demonstrative basis, and not on the utterly fallacious foundation of arithmetical empiricism [28].

[28] A few additional words on this question of probability may not be unacceptable. In order to meet the case of the degree of the superlinear form or equation being odd as well as even, let it be supposed known under the form

$$\sum_{i=1}^{n} \lambda_i (x+c_i)^m,$$

the value of the quantities c being supposed to be left wholly indeterminate, and only the signs of the quantities λ to be given. Let ω be the inferior number of like signs in the λ series, meaning thereby that the number of signs of one sort is ω, and of the other sort ω, or more than ω.

Let the probability of the specific maximum of real roots being $2k$ when m is even, be represented by p_{2k}, and of its being $2k+1$ when m is odd by π_{2k+1}; also let s_{2k}, σ_{2k+1} represent the number of cases when ω and n are given which correspond to the specific maximum being $2k$, $2k+1$ respectively. Suppose $\omega=1$, then obviously, when m is even, we have $s_2=n$, $p_2=1$. But when n is odd $\sigma_1=2$ (for when either extreme element alone is negative the trans-rotatory cycle has the variation-index unity), and $\sigma_3=n-2$, so that

$$\pi_1=\frac{2}{n}, \quad \pi_3=\frac{n-2}{n}.$$

Again, suppose $\omega=2$, m being even; then obviously s_2 is the number of contiguous duads in a cycle of n elements, and s_4 is the remaining number of duads; hence

$$s_2=n, \quad s_4=n\frac{n-1}{2}-n=n\frac{n-3}{2};$$

so that

$$p_2=\frac{2}{n-1}, \quad p_4=\frac{n-3}{n-1}.$$

2nd. Suppose $\omega=2$, m being odd, so that σ_1, σ_3, σ_5 will have to be separately estimated. To fix the ideas, let the λ series be termed a, b, c, d, e, f, g, in which two of the elements are supposed of one sign, say negative, and the rest of the opposite sign, say positive; then the only dispositions of sign which correspond to the specific maximum being 1 are those in which a, b or else f, g are both negative. Hence $\sigma_1=2$. Again, the dispositions of sign which make the specific maximum equal to 3 are those in which a, g are both negative, those in which a and c, d, e, or f are negative, those in which g and e, d, c, or b are negative, and, finally, those in which any two contiguous elements except the a and g are negative. Hence $\sigma_3=1+2(n-3)+(n-3)=3n-8$; and it should be observed that this result cannot be prejudiced in its generality by the supposition of any of the components of σ_3 becoming negative, since $\omega=2$ implies that n is at least 4. Hence, finally,

$$\sigma_5=\frac{n^2-n}{2}-(3n-8)-2=\frac{n^2-7n+12}{2}=\frac{(n-3)(n-4)}{2};$$

so that

$$\pi_1=\frac{4}{n^2-n}, \quad \pi_3=\frac{6n-20}{n^2-n}, \quad \pi_5=\frac{n^2-7n+16}{n^2-n}.$$

This example serves to show how much more difficult is the computation of the respective probabilities when m is odd than when m is even, owing to the break of continuity in the cycle of readings on passing from the last to the first term.

It seems hardly worth while to pursue this subject in greater detail. I will only notice that

Notes to Part II.

On the probability of the specific superior limit to the number of real roots in a superlinear equation equalling any assigned integer.

(21) The question comes to that of determining the probability of a per-rotatory or trans-rotatory pencil with a definite number of rays of each kind possessing a given variation-index.

Since the footnote below was written, a method has occurred to me of obtaining the probability in question in general terms, as follows.

For a *per-rotatory* pencil of μ positive and ν negative rays. Let $[\mu, \nu, g]$ be the probability of the rays being so disposed as to give rise to $2g$ variations of sign in making a complete revolution. Then there will be g distinct groups of positive, and g of negative rays. The number of partitions with permutations of the parcels *inter se* of μ elements in g parcels is

$$\frac{(\mu-1)(\mu-2)..(\mu-g+1)}{1.2..(g-1)},$$

and of ν elements into g parcels is

$$\frac{(\nu-1)(\nu-2)..(\nu-g+1)}{1.2..(g-1)}.$$

If we combine each parcel with each in every possible way, and then imagine the combined parcels let into a circle containing $m+n$ places and shifted round in the circle through a complete revolution, we shall obtain

$$(\mu+\nu).\frac{(\mu-1)(\mu-2)..(\mu-g+1)}{1.2..(g-1)}.\frac{(\nu-1)(\nu-2)..(\nu-g+1)}{1.2..(g-1)}$$

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when m is even the chance of the specific maximum attaining the absolute maximum, that is, becoming 2ω, will depend on the proportion of the ways in which in a cycle of n elements ω of them may be marked with a distinctive sign in such a way that no two of such signs shall come together. Accordingly I find by a computation of no great difficulty,

$$s_{2\omega}=\frac{n(n-\omega-1)!}{\omega!(n-2\omega)!},$$

and hence, since the total number of combinations of n elements ω and ω together is $\frac{n!}{\omega!(n-\omega)!}$, I deduce

$$p_{2\omega}=\frac{(n-\omega)!(n-\omega-1)!}{(n-1)!(n-2\omega)!}.$$

Thus when n has its minimum value, namely, 2ω, $p_{2\omega}=\frac{\omega!(\omega-1)!}{(2\omega-1)!}$, and becomes very small as ω increases. When again n increases towards infinity $p_{2\omega}$ approaches indefinitely near to unity, and the chance approaches near to certainty of the specific not becoming less than the absolute maximum of real roots.

arrangements; but on examination it will be found that every arrangement so produced will be repeated g times; moreover it is obvious that no other arrangement giving rise to g groups of each sort can be found. Hence the true number of distinct groupings of the sort in question is

$$\frac{(\mu+\nu)}{g} \cdot \frac{(\mu-1)(\mu-2)..(\mu-g+1)}{1.2..(g-1)} \cdot \frac{(\nu-1)(\nu-2)..(\nu-g+1)}{1.2..(g-1)}.$$

And the total number of arrangements, which is the number of ways in which μ things can be distributed over $(\mu+\nu)$ places, is $\frac{(\mu+\nu)!}{\mu!\,\nu!}$. Hence we obtain

$$[\mu, \nu, g] = \frac{\mu!\,\nu!}{(\mu+\nu-1)!}\left\{\frac{(\mu-1)(\mu-2)..(\mu-g+1)\times(\nu-1)(\nu-2)..(\nu-g+1)}{1.2.(g-1)(1.2..g)}\right\}$$

$$= \frac{\mu!\,(\mu-1)!\,\nu!\,(\nu-1)!}{g!\,(g-1)!\,(\mu-g)!\,(\nu-g)!\,(\mu+\nu-1)!}.$$

If there should appear any obscurity in the statement of the method by which has been obtained the number of distinct distributions of the μ, ν elements into g groups of each, the reader is referred to the equation in differences obtained further on in this Note, by which all doubt of the correctness of the result will be removed.

(22) For a *trans-rotatory* pencil of rays, to ascertain the probability of the variation-index being $2g+1$.

Imagine a circular arrangement of μ positive elements and ν negative elements containing 2γ variations.

Let this circle be supposed opened out at any point and the variations of the open pencil so formed to be reckoned according to the trans-rotatory law, which is that in passing from one extremity to the other a change is to be seen as a variation, and a variation as a change. If the break is made between two negative or between two positive elements, the number of variations obviously becomes *increased* by one unit; but if between a positive and a negative element, that number becomes decreased by one unit. The number of these latter intervals is 2γ, and of the former $\mu+\nu-2\gamma$.

Hence the probability of the index becoming $2\gamma+1$ is $\frac{\mu+\nu-2\gamma}{\mu+\nu}$, and of its becoming $2\gamma-1$ is $\frac{2\gamma}{\mu+\nu}$.

If, then, we denote the probability to be calculated by $[\mu, \nu, g+\frac{1}{2}]$, it is obvious that we shall have

$$[\mu, \nu, g+\tfrac{1}{2}] = \frac{\mu+\nu-2g}{\mu+\nu}[\mu, \nu, g] + \frac{2(g+1)}{\mu+\nu}[\mu, \nu, g+1].$$

But by the formula previously obtained it will easily be seen that

$$[\mu, \nu, g+1] = \frac{(\mu-g)(\nu-g)}{g(g+1)}[\mu, \nu, g].$$

Hence

$$\begin{aligned}[\mu, \nu, g+\tfrac{1}{2}] &= \frac{[\mu, \nu, g]}{\mu+\nu}\left\{(\mu+\nu-2g) + \frac{2(\mu-g)(\nu-g)}{g}\right\} \\ &= \left(\frac{2\mu\nu}{g(\mu+\nu)} - 1\right)[\mu, \nu, g] \qquad (*) \\ &= \frac{2(\mu\,!)^2(\nu\,!)^2}{(g-1)!\,(g+1)!\,(\mu+\nu)!\,(\mu-g)!\,(\nu-g)!} \\ &\qquad - \frac{\mu\,!\,(\mu-1)!\,\nu\,!\,(\nu-1)!}{(g-1)!\,g\,!\,(\mu+\nu-1)!\,(\mu-g)!\,(\nu-g)!}.\end{aligned}$$

When $g=0$ the above expression fails; but reverting to the equation from which it is derived, we obtain

$$(\mu, \nu, \tfrac{1}{2}) = \frac{2}{\mu+\nu}[\mu, \nu, 1] = \frac{2\,.\,\mu\,!\,\nu\,!}{(\mu+\nu)!}.$$

(23) These combined results admit of an easy corroboration, for

$$\Sigma_\infty^0[\mu, \nu, g+\tfrac{1}{2}] = 1, \text{ and } \Sigma_\infty^0[\mu, \nu, g] = 1.$$

Hence the equation marked * gives

$$1 = [\mu, \nu, \tfrac{1}{2}] + \frac{2\mu\nu}{\mu+\nu}\Sigma\frac{[\mu, \nu, g]}{g} - 1.$$

Hence we ought to have

$$\frac{\mu\,!\,\nu\,!}{(\mu+\nu)!} + \frac{\mu\nu}{\mu+\nu}\Sigma\frac{[\mu, \nu, g]}{g} = 1,$$

that is

$$1 + \Sigma\frac{\mu\,!\,\nu\,!}{(\mu-g)!\,g\,!\,(\nu-g)!\,g\,!} = \frac{(\mu+\nu)!}{\mu\,!\,\nu\,!};$$

which is true, since the left-hand side of the equation is

$$1 + \mu\nu + \mu\frac{\mu-1}{2}\,.\,\nu\frac{\nu-1}{2} + \ldots,$$

which is obviously the coefficient of x^ν in $(1+x)^\mu(x+1)^\nu$, that is, in $(1+x)^{\mu+\nu}$.

(24) If we wish to find the chance of the specific superior limit becoming equal to the absolute superior limit, we must write g in the above formulæ

equal to ν, that one of the two quantities μ, ν which is not greater than the other, and we shall obtain

$$[\mu, \nu, \nu] \quad = \frac{\mu\,!\,(\mu-1)\,!}{(\mu+\nu-1)\,!\,(\mu-\nu)\,!},$$

$$[\mu, \nu, \nu+\tfrac{1}{2}] = \frac{\mu\,!\,(\mu-1)\,!}{(\mu+\nu)\,!\,(\mu-\nu-1)\,!};$$

so that, in fact, $[\mu, \nu, \nu+\frac{1}{2}] = [\mu, \nu+1, \nu+1]$, which relation may also be obtained by *à priori* considerations.

(25) With reference to the remark made concerning the mode of obtaining the value of $[\mu, \nu, g]$, I proceed to show how it may be obtained directly by the integration of an equation in differences, and by a method analogous in idea to that by which $[\mu, \nu, g+\frac{1}{2}]$ was made to depend on $[\mu, \nu, g]$. For as in that case we conceived an open pencil to be closed and then reopened, so we may imagine one of the rays to be withdrawn and then reinserted. In this way, observing that the effect of introducing a negative sign into a circle of μ positive and n negative signs consisting of ν distinct groups of each is to produce no change in the number of the groups if inserted between two negative signs, but to increase that number by unity if inserted between two positive signs, we may infer that the probability of ν becoming $\nu+1$, in consequence of such insertion, is $\dfrac{\mu-\nu}{\mu+\nu}$, and of ν remaining unaltered, is $\dfrac{n+\nu}{\mu+n}$.

Hence we obtain the equation in differences,

$$[\mu, \nu, g] = \frac{\nu-1+g}{\mu+\nu-1}[\mu, \nu-1, g] + \frac{\mu-g+1}{\mu+\nu-1}[\mu, \nu-1, g-1],$$

in which μ may be considered constant, and ν and g to vary.

The integral must satisfy the further condition that $[\mu, 1, g]$ shall be unity when g is 1, and zero for all values of g greater than 1.

Assume the value of $[\mu, 1, g]$ obtained by the method given in art. (21). This obviously satisfies the initial conditions corresponding to $g=1$. Moreover we may easily deduce from it the equalities

$$[\mu, \nu-1, g-1] = \frac{(g-1)\,g}{(\mu-g+1)(\nu-g)}[\mu, \nu-1, g],$$

and

$$[\mu, \nu, g] = \frac{(\nu-1)\,\nu}{(\mu+\nu-1)(\nu-g)}[\mu, \nu-1, g].$$

Hence the equation in differences will be satisfied if it be true that

$$\frac{(\nu-1)\nu}{\nu-g} = (\nu-1+g) + \frac{(g-1)\,g}{\nu-g},$$

which is obviously the case, since $\nu^2-\nu-g^2+g = (\nu-g)(\nu+g-1)$.

Since, then, the assumed value of $[\mu, \nu, g]$ is correctly determined when $\nu = 1$, it is obvious, from the form of the equation, that it holds good for all other values of ν, as was to be shown.

(26) From the equation

$$\frac{[\mu, \nu, g+1]}{[\mu, \nu, g]} = \frac{(\mu - g)(\nu - g)}{g(g+1)}$$

making $(\mu - g)(\nu - g) = g(g+1)$ or $g = \dfrac{\mu\nu}{\mu + \nu + 1}$, we may readily infer that the value of g for which the probability $[\mu, \nu, g]$ is greatest is the integer part of $\dfrac{\mu\nu}{\mu + \nu + 1}$, if that quantity is non-integer, or the quantity itself and the number next below it (indifferently) if it is an integer.

(27) If we apply a similar method to $[\mu, \nu, g + \frac{1}{2}]$, we obtain by aid of the formula above given,

$$\frac{[\mu, \nu, g+\frac{1}{2}]}{[\mu, \nu, g-\frac{1}{2}]} = \frac{2\mu\nu - (\mu + \nu)\gamma}{2\mu\nu + \mu + \nu - (\mu + \nu)\gamma} \cdot \frac{(\mu + 1) - \nu(\nu + 1 - \gamma)}{\gamma^2};$$

and equating this ratio to unity, we obtain

$$\frac{2\mu\nu - (\mu + \nu)\gamma}{2\mu\nu + \mu + \nu - (\mu + \nu)\gamma} = \frac{\gamma^2}{(\mu + 1)(\nu + 1) - (\mu + \nu + 2)\gamma};$$

or writing $\mu + \nu = p$, $\mu\nu = q$,

$$(p^2 + p)\gamma^2 - (3pq + 4q + p^2 + p)\gamma + 2q(q + p + 1) = 0.$$

The roots of this equation will be both of them real, for its *determinant* is

$$p^2q^2 + 16pq^2 + 16q^2 + (p^2 + p^3)(\mu^2 + \nu^2),$$

which is necessarily positive. Hence it follows that there are two positive roots of the equation. Whether there will exist values of g which give actual maxima or minima values, or one and the other to $[\mu, \nu, g + \frac{1}{2}]$, depends on the further condition being satisfied that the values of g in the above equation shall come out, one or both of them, not greater than either of the two numbers μ, ν. The inquiry connected with the satisfaction of this condition may be conducted by means of repeated applications of the processes of Sturm's theorem; but I shall not enter upon it, as it appears to lead to calculations of complexity disproportionate to the interest of the result.

(28) It may be noticed that the *average* value of $[\mu, \nu, g]$ can be calculated without any difficulty. This will be $\Sigma(g[\mu, \nu, g])$, or

$$\frac{\mu!\,\nu!}{(\mu + \nu - 1)!}\left[1 + \frac{(\mu - 1)(\nu - 1)}{1} + \frac{(\mu - 1)(\mu - 2)(\nu - 1)(\nu - 2)}{1 \,.\, 2^2} + \dots\right]$$

$$= \frac{\mu!\,\nu!}{(\mu + \nu - 1)!} \cdot \frac{(\mu + \nu - 2)!}{(\mu - 1)!\,(\nu - 1)!} = \frac{\mu\nu}{(\mu + \nu - 1)};$$

so that the average number of variations of sign in a per-rotatory pencil with μ positive and ν negative signs is $\frac{2\mu\nu}{\mu+\nu-1}$, or a little more than the harmonic mean between μ, ν.

In like manner, for a trans-rotatory pencil this number will be

$$\Sigma\,(2g+1)\,[\mu,\,\nu,\,g+\tfrac{1}{2}] = [\mu,\,\nu,\,\tfrac{1}{2}] + \Sigma\left\{(2g+1)\left(\frac{2\mu\nu}{g\,(\mu+\nu)}-1\right)[\mu,\,\nu,\,g]\right\},$$

which, observing that $\Sigma\,[\mu,\,\nu,\,g]=1$, and $[\mu,\,\nu,\,\tfrac{1}{2}]+\frac{2\mu\nu}{\mu+\nu}\,\Sigma\frac{[\mu,\,\nu,\,g]}{g}=2$, gives as the average number of variations of sign $\frac{4\mu\nu}{\mu+\nu}-\frac{2\mu\nu}{\mu+\nu-1}+1$.

(29) The simplest mode of calculating the value of $[\mu,\,\nu,\,g]$ is the following:

Let $[\mu,\,\nu,\,g)$, $[\mu,\,\nu,\,g-\tfrac{1}{2})$ denote the probabilities that an arrangement in open line (in which, as is the case in applying Des Cartes's rule of signs, no account is taken of the relation of the extreme signs to each other) shall contain respectively $2g$ and $2g-1$ variations. Conceive a circular arrangement of γ groups of positive and γ groups of negative signs. If this circle be opened out into a line at an interval between a positive and a negative sign (of which there are 2γ), one variation will be lost; but if at any of the remaining $\mu+\nu-\gamma$ intervals, the number of variations remains unaltered. Hence we derive immediately

$$[\mu,\,\nu,\,g) = \frac{\mu+\nu-2g}{\mu+\nu}\,[\mu,\,\nu,\,g] \quad\text{and}\quad [\mu,\,\nu,\,g-\tfrac{1}{2}) = \frac{2g}{\mu+\nu}\,[\mu,\,\nu,\,g].$$

But we may find $[\mu,\,\nu,\,g-\tfrac{1}{2})$ by counting the arrangements which give μ, ν, $2g-1$ variations of sign. These may be all obtained, and without repetition, by intercalating every distribution of μ into g groups with every distribution of ν into the same; and the intercalation may be performed in *two* ways, according as the parcels of the μ signs, or those of the ν signs, are taken first in order. Hence we have

$$[\mu,\,\nu,\,g-\tfrac{1}{2}) = \frac{2\,(\mu-1)\,(\mu-2)\,\ldots\,(\mu-g+1)}{1\,.\,2\quad\ldots\quad(g-1)}\cdot\frac{(\nu-1)\,(\nu-2)\,\ldots\,(\nu-g+1)}{1\,.\,2\quad\ldots\quad(g-1)}\cdot\frac{\mu\,!\,\nu\,!}{(\mu+\nu)\,!}$$

$$= \frac{2\,(\mu-1)\,!\,\mu\,!\,(\nu-1)\,!\,\nu\,!}{(\mu+\nu)\,!\,(g-1)\,!\,(g-1)\,!\,(\mu-g)\,!\,(\nu-g)\,!};$$

and thus

$$[\mu,\,\nu,\,g] = \frac{\mu+\nu}{2g}\,[\mu,\,\nu,\,g-\tfrac{1}{2}) = \frac{\mu\,!\,(\mu-1)\,!\,\nu\,!\,(\nu-1)\,!}{(\mu+\nu-1)\,!\,g\,!\,(g-1)\,!\,(\mu-g)\,!\,(\nu-g)\,!}.$$

as previously found; also

$$[\mu, \nu, g) = \frac{(\mu+\nu-2g)\,[\mu\,!\,(\mu-1)\,!\,\nu\,!\,(\nu-1)\,!]}{(\mu+\nu)\,!\,g\,!\,(g-1)!\,(\mu-g)\,!\,(\nu-g)\,!}.$$

(30) Moreover, we thus see that the average number of variations in an open line with μ positive and ν negative signs, which is

$$\Sigma\,(2g-1)\,[\mu, \nu, g-\tfrac{1}{2}) + \Sigma 2g\,[\mu, \nu, g),$$

or

$$\Sigma 2g\,\{[\mu, \nu, g-\tfrac{1}{2}) + [\mu, \nu, g)\} - \Sigma\,[\mu, \nu, g-\tfrac{1}{2})$$

will be equal to

$$\Sigma 2g\,[\mu, \nu, g] - \Sigma\,\frac{2g}{\mu+\nu}\,[\mu, \nu, g] = \frac{\mu+\nu-1}{\mu+\nu}\,\Sigma 2g\,[\mu, \nu, g]$$

$$= \frac{\mu+\nu-1}{\mu+\nu} \cdot \frac{2\mu\nu}{\mu+\nu-1} = \frac{2\mu\nu}{\mu+\nu}.$$

The total number of variations and continuations together is $\mu+\nu-1$. Hence the difference between the two is

$$\frac{4\mu\nu}{\mu+\nu} - (\mu+\nu-1),$$

or

$$\frac{(\mu+\nu)-(\mu-\nu)^2}{\mu+\nu};$$

so that the average number of variations is greater than, equal to, or less than that of the continuations, according as the difference between the numbers of the two sets is less than, equal to, or greater than the square root of the entire number of signs. Obviously the average should be the same for the variations as for the continuations if the number of signs, say $n+1$, is given, and each is supposed equally likely to be positive or negative. This is easily verified; for multiplying the probable value of each distribution of signs by the probable value of the number of variations corresponding thereto, we obtain the series

$$\frac{1}{(n+1)\,2^n}\left\{1\,.\,n\,.\,(n+1) + 2\,(n-1)\,(n+1)\,\frac{n}{2} + 3\,(n-2)\,\frac{(n+1)\,n\,.\,(n-1)}{1\,.\,2\,.\,3} + \ldots\right\}$$

$$= \frac{n\,(n+1)\,2^{n-1}}{(n+1)\,2^n} = \frac{n}{2}.$$

This is the final average of the number of variations of sign, and will be equal to that of the continuations, since the entire number of the two together is n.

Part III.—On the Nature of the Roots of the General Equation of the Fifth Degree.

(31) In a footnote, Part II. of this memoir, [p. 409 above] I have shown that when the discriminant of the canonizant (constituting an invariant of the twelfth order) of an equation of the fifth degree bears a particular sign, the character of the roots becomes completely determined by the sign of the discriminant of that equation.

This has naturally led me to investigate *de novo* the whole question of the character of the roots of an equation of that degree; and I have succeeded in obtaining under a form of striking and unexpected simplicity the invariantive criteria which serve to ascertain in all cases the nature of the equation as regards the number of real and imaginary roots which it contains; then passing to the expression for these criteria in terms of the roots themselves, I obtain expressions which exhibit the intimate connexion between this subject and a former theory of my own relative to the construction of the conditions for the existence of a given number and grouping of equal roots, which can hardly fail to lead eventually to the extension of the results herein obtained to equations of any odd degree whatever. It is the more needful that these results in a question of so high moment to the advancement of algebraical science should be made public, inasmuch as they do not seem to accord with those obtained by my eminent friend M. Hermite, who has preceded me in this inquiry in a classic memoir, published in the year 1854 in the ninth volume of the *Cambridge and Dublin Mathematical Journal*, since which time I am not aware that the subject has been resumed by any other writer. The discrepancy between our conclusions may be only apparent; but there can be no doubt of the superiority of the form in which they are herein presented, inasmuch as only three functions of the coefficients are required by my method, and five by M. Hermite's. The solution offered by M. Hermite is confessedly incomplete, but to this great analyst none the less will always belong the honour, not only of having initiated the inquiry, but of having emitted the fundamental conceptions through which it would seem best to admit of successful treatment. The arrow from my hand may have been the first to hit the mark, but it was his hand which had previously shaped, bent, and strung the bow.

Our methods of procedure, however, are widely dissimilar, and by employing my well-known canonical form for odd-degreed binary quantics, long since given to the world, I have succeeded in evading all necessity for the colossal labours of computation required in M. Hermite's method, and am able to impart to my conclusions the clearness and certainty of any

elementary proposition in geometry, not scrupling to avail myself for such purpose of that copious and inexhaustible well-spring of notions of continuity which is contained in our conception of space, and which renders it so valuable an auxiliary to Mathematic, whose sole proper business seems to me to be the development of the three germinal ideas—of which continuity is one and order and number the other two*.

SECTION I.—*Preparation of the General Binary Quantic of the Fifth Degree.*

(32) Let $$(a, b, c, d, e, i \between x, y)^5 = F(x, y);$$
a cubic covariant of F is the canonizant C, where C represents the determinant

$$C = \begin{vmatrix} a & b & c & d \\ b & c & d & e \\ c & d & e & i \\ y^3 & -y^2x & yx^2 & -x^3 \end{vmatrix}.$$

Let us first suppose that this form does not vanish identically, and has at least two distinct factors ξ, η, linear functions of x, y, where of course ξ, η are each of them determinate to a constant factor *près*; giving any value to the constant factor for either of them, we may write $F(x, y) = \Phi(\xi, \eta) = (\alpha, \beta, \gamma, \delta, \epsilon, \iota \between \xi, \eta)^5$, and the canonizant of Φ with respect to ξ, η becomes the determinant T, where T represents

$$T = \begin{vmatrix} \alpha & \beta & \gamma & \delta \\ \beta & \gamma & \delta & \epsilon \\ \gamma & \delta & \epsilon & \iota \\ \eta^3 & -\eta^2\xi & \eta\xi^2 & -\xi^3 \end{vmatrix}.$$

Hence since T to a constant factor *près* is identical with C, the coefficients of η^3 and ξ^3 in the above determinant must vanish in order that $\xi\eta$ may be contained in T.

Hence the two determinants

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \delta \\ \gamma & \delta & \epsilon \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} \beta & \gamma & \delta \\ \gamma & \delta & \epsilon \\ \delta & \epsilon & \iota \end{vmatrix}$$

both vanish.

* Herein I think one clearly discerns the internal grounds of the coincidence or parallelism, which observation has long made familiar, between the mathematical and musical ἔθος. May not Music be described as the Mathematic of sense, Mathematic as Music of the reason? the soul of each the same! Thus the musician *feels* Mathematic, the mathematician *thinks* Music,—Music the dream, Mathematic the working life—each to receive its consummation from the other when the human intelligence, elevated to its perfect type, shall shine forth glorified in some future Mozart-Dirichlet or Beethoven-Gauss—a union already not indistinctly foreshadowed in the genius and labours of a Helmholtz!

Hence either β, γ, δ, or otherwise γ, δ, ϵ, or else the first minors of

$$\left|\begin{matrix} \beta & \gamma \\ \gamma & \delta \\ \delta & \epsilon \end{matrix}\right|$$

are each zero.

The first two suppositions must be excluded, since either of them would lead to the conclusion of T, and therefore C, being a perfect cube, contrary to hypothesis. The last supposition implies either that β, γ, δ, or otherwise that γ, δ, ϵ, or else that $\beta\delta - \gamma^2$ and $\gamma\epsilon - \delta^2$ are each zero.

If β, γ, δ are each zero, T becomes a multiple of $\eta^2\xi$; if γ, δ, ϵ are each zero, T becomes a multiple of $\eta\xi^2$; that is to say, T, and consequently C, contains a square factor; and obviously the converse is true, so that when C contains a square factor F is reducible to the form $au^5 + 5euv^4 + fv^5$. When this is not the case $\delta = \dfrac{\gamma^2}{\beta}$, $\epsilon = \dfrac{\delta^2}{\gamma} = \dfrac{\gamma^3}{\beta^2}$. Hence

$$F = \left(\alpha - \frac{\beta^2}{\gamma}\right)\xi^5 + \frac{\beta^2}{\gamma}\left(\xi + \frac{\gamma}{\beta}\eta\right)^5 + \left(\iota - \frac{\epsilon^2}{\delta}\right)\eta^5,$$

which is of the form $\omega^5 + \phi^5 + \psi^5$, ω, ϕ, ψ being linear functions of x, y.

(33) We have supposed C not to be a perfect cube. When it is a perfect cube, say ξ^3, we may assume η any second linear function of x, y; and expressing F in the same manner as before in terms of ξ, η, it is clear that all the first minors of

$$\left|\begin{matrix} \alpha, & \beta, & \gamma, & \delta \\ \beta, & \gamma, & \delta, & \epsilon \\ \gamma, & \delta, & \epsilon, & \iota \end{matrix}\right|,$$

except the one obtained by cancelling the last column in the above matrix, must vanish; consequently δ, ϵ, ι must all vanish, so that Φ, and consequently F, must contain a cube factor identical with the canonizant itself.

Lastly, if the canonizant vanish entirely, every first minor in the above matrix, when we write again a, b, c, d, e, i in lieu of $\alpha, \beta, \gamma, \delta, \epsilon, \iota$, will be zero. Hence either a, b, c, d, or b, c, d, e, or c, d, e, i must each vanish or else that must be the case with the first minors of

$$\left|\begin{matrix} a, & b, & c, & d \\ b, & c, & d, & e \end{matrix}\right|,$$

or of

$$\left|\begin{matrix} b, & c, & d, & e \\ c, & d, & e, & i \end{matrix}\right|,$$

or of

$$\left|\begin{matrix} a, & b, & c, & d \\ c, & d, & e, & i \end{matrix}\right|.$$

Under the first or third supposition F must contain four equal factors; under the second Φ becomes $a\xi^5 + i\eta^5$; under the fourth or fifth it is readily seen that the form becomes

$$a\left(\xi + \frac{b}{a}\eta\right)^5 + \left(i - \frac{e^2}{d}\right)\eta^5, \text{ or } \left(a - \frac{b^2}{c}\right)\xi^5 + i\left(\eta + \frac{e}{i}\xi\right)^5$$

respectively, so that the second, fourth, and fifth suppositions conduct alike to the form $\omega^5 + \phi^5$, a particular case of the preceding one.

It remains only to consider the sixth supposition, namely that the first minors of

$$\begin{vmatrix} a, & b, & c, & d \\ c, & d, & e, & i \end{vmatrix}$$

are all zero.

In this case if we write

$$\sqrt{(a)}\,x + \sqrt{(c)}\,y = u,$$

$$\sqrt{(a)}\,x - \sqrt{(c)}\,y = v,$$

$$A + B = \frac{1}{a^{\frac{3}{2}}},$$

$$A - B = \frac{b}{a^2 c^{\frac{1}{2}}},$$

and if neither a nor c is zero, it will readily be seen that $F(x, y)$ becomes $Au^5 + Bv^5$ by virtue of the relations

$$d = \frac{c}{a}b, \quad e = \left(\frac{c}{a}\right)^2 a, \quad i = \left(\frac{c}{a}\right)^2 b \text{ (29)}.$$

If $a = 0$ or $c = 0$, the preceding transformation fails.

But unless also $i = 0$ or $e = 0$ at the same time as $a = 0$ or $c = 0$, a legitimate transformation similar to the above may be performed by interchanging a, c, x, y with i, a, y, x.

If now

$a = 0$, it will easily be seen that a, b, c, d or else a, c, e are each zero.

Similarly, if

$i = 0$, it will easily be seen that i, e, d, c or else i, d, b are each zero.

Again, if

$c = 0$, it will easily be seen that a, b, c, d or else c, e are each zero;

and if

$d = 0$, it will easily be seen that c, d, e, i or else d, b are each zero.

(29) Thus we see that the equation $ax^5 + 5bx^4 + 10acx^3 + 10bcx^2 + 5ac^2x + bc^2 = 0$ belongs to the class of soluble forms.

Thus, then, if $a=0$ and $i=0$, all the coefficients, or else all except one, namely b or e, are zero;

if $a=0$ and $d=0$, all the coefficients, or else only not e and i or only not b or only not i are zero;

so if $i=0$ and $c=0$, all must be zero except b and a or e or a;

if $c=0$ and $d=0$, only e and i or else a and b or else a and i will differ from zero.

Hence, then, in any case there will be at least four equal roots, or else F is of the form ax^5+iy^5.

Thus, then, for the first time has been here rigorously demonstrated, free from all doubt and subject to no exceptions, the following important proposition:

Every binary quintic function *not containing three or more equal roots* is reducible to one or the other of the two following forms,

$$u^5+v^5+w^5, \quad \text{or} \quad au^5+5euv^4+fv^5.$$

The former is the case when the discriminant of the canonizant is different from zero, the latter when it is equal to zero; for it will be observed that, whether the canonizant has equal roots or totally disappears, its discriminant in both cases alike is zero.

(34) It has been seen that when the quintic has three equal roots the canonizant becomes a perfect cube; and it may not be out of place here to point out what the conditions (necessary and sufficient) are to ensure the quintic having four equal roots. These are all comprised in that of the quadratic covariant vanishing. To prove this, let η be a factor of $F(x, y)$, so that

$$F(x, y)=\Phi(x, \eta)=(\alpha, \beta, \gamma, \delta, \epsilon, 0 \between x, \eta)^5.$$

Then, since the similar covariant *quoad* x, y must also vanish, we have

$$\alpha\epsilon-4\beta\delta+\gamma^2=0, \quad -3\beta\epsilon+2\gamma\delta=0, \quad -4\gamma\epsilon+3\delta^2=0.$$

If $\epsilon=0$, then $\delta=0$, $\gamma=0$ by virtue of the two extreme equations, and Φ, and therefore F, contains four equal factors. If ϵ is not zero,

$$\gamma=\frac{3\delta^2}{4\epsilon}, \quad \beta=\frac{\delta^3}{2\epsilon^2}, \quad \alpha=\frac{5\delta^4}{16\epsilon^3}, \text{ and } \Phi \text{ becomes } \frac{5\epsilon}{16}x\left(\frac{\delta}{\epsilon}x+2\eta\right)^4;$$

so that, as before, there are four equal factors. Conversely, it is obvious that if there are four equal factors u, so that $\Phi=au^5+5bu^4v$, the quadratic covariant of Φ disappears.

(35) The quadratic covariant also it was which led me to perceive the transformation applied in the antecedent article. For when the first minors of

$$\begin{vmatrix} a, & b, & c, & d \\ c, & d, & e, & f \end{vmatrix}$$

are all zeros, the quadratic covariant becomes

$$4(c^2 - bd)x^2 + 4(d^2 - ce)y^2.$$

Supposing neither of those coefficients to vanish, and calling its two factors u and v, and making

$$F(x, y) = \Phi(u, v) = (\alpha, \beta, \gamma, \delta, \epsilon, \iota \between u, v)^5,$$

it is clear that the minors of

$$\begin{vmatrix} \alpha, & \beta, & \gamma, & \delta \\ \gamma, & \delta, & \epsilon, & \iota \end{vmatrix}$$

can no longer all be zero, since in that case we should have

$$4(\gamma^2 - \beta\delta)u^2 + 4(\delta^2 - \gamma\epsilon)v^2$$

containing u, v as factors. Consequently the canonizant of Φ must vanish under one or the other of those remaining suppositions which had been previously shown to conduct to the form $au^5 + bv^5$, or else to the case of three or more equal roots. When the quadratic covariant vanishes, we know that there must be four equal roots; and when it becomes a perfect square but does not vanish, it will be found on examination that the equation has three equal roots.

(36) Returning to the general case, where $\Phi = u^5 + v^5 + w^5$, and making $\frac{u}{r^{\frac{1}{5}}} + \frac{v}{s^{\frac{1}{5}}} + \frac{w}{t^{\frac{1}{5}}}$ identically zero, and writing u', v', w' for $\frac{u}{r^{\frac{1}{5}}}, \frac{v}{s^{\frac{1}{5}}}, \frac{w}{t^{\frac{1}{5}}}$ respectively, Φ becomes $ru'^5 + sv'^5 + tw'^5$, or, if we please, $ru^5 + sv^5 + tw^5$, with the condition $u + v + w = 0$.

Moreover u, v, w will all three be factors of the canonizant of F. For taking the canonizant of F with respect to u, v, it becomes

$$\begin{vmatrix} r-t, & -t, & -t, & -t \\ -t, & -t, & -t, & -t \\ -t, & -t, & -t, & s-t \\ v^3, & -v^2u, & vu^2, & -u^3 \end{vmatrix}, \quad \text{or } rt \begin{vmatrix} 1, & 0, & 0, & 0 \\ -1, & -1, & -1, & -1 \\ 0, & 0, & 0, & s \\ v^3, & -v^2u, & vu^2, & -u^3 \end{vmatrix}$$

or $rst(uv^2 + vu^2)$, that is, $-rst\,uvw$.

Hence if $x + ey$, $x + fy$, $x + gy$ are three distinct factors of the canonizant of F with respect to x, y, if we choose the ratios $\lambda : \mu : \nu$ so that $\lambda + \mu + \nu = 0$, $e\lambda + f\mu + g\nu = 0$, we may make $u = \lambda(x + ey)$, $v = \mu(x + fy)$, $w = \nu(x + gy)$, and shall then have $F(x, y) = ru^5 + sv^5 + tw^5$, with the condition $u + v + w = 0$, where r, s, t may be found from three equations obtained by identifying any three of the six terms in F with the corresponding terms $ru^5 + sv^5 + tw^5$ expressed as a function of x, y. These equations being linear, it follows that ru^5, sv^5, tw^5 form a *single and unique* system of functions of x, y.

So when the canonizant has two equal roots and is of the form

$$C(x+py)(x+qy)^2,$$

in which case the reduced form is $au^5 + 5euv^4 + fv^5$, the canonizant in respect to u, v becomes

$$\begin{vmatrix} a, & 0, & 0, & 0 \\ 0, & 0, & 0, & e \\ 0, & 0, & e, & f \\ v^3, & -v^2u, & vu^2, & -u^3 \end{vmatrix},$$

that is, ae^2uv^2. Hence, writing

$$u = x + py, \quad v = x + qy, \quad F = au^5 + 5euv^4 + fv^5,$$

a, e, f may be obtained, as before, by means of three linear equations, and the terms au^5, $5euv^4$, fv^5 form a single and unique system.

Finally, when the canonizant vanishes entirely, so that the form becomes $au^5 + fv^5$, the quadratic covariant will take the form $C(x+ey)(x+fy)$; and making $u = x + py$, $v = x + qy$, a, f become determined by means of two linear equations, so that au^5, fv^5 form a single and unique system, as in the preceding cases.

(37) When the canonizant has three distinct roots, they may be all real, or one real and the other two imaginary. In the former case, in the expression $ru^5 + sv^5 + tw^5$, u, v, w may be considered as all real functions of x, y, and r, s, t will then also all of them be real. In the latter case w may be taken as a real function of x, y and u, v as conjugate imaginary functions; and consequently it is easy to see that, except when r, s are equal to each other, they will constitute a pair of conjugate imaginary quantities: in this case we may take for our canonizant form

$$r\left(\frac{-u+iv}{2}\right)^5 + s\left(\frac{-u-iv}{2}\right)^5 + tu^5,$$

or, if we please, $$ru_,^5 + sv_,^5 + tu^5,$$

understanding by $u_,$, $v_,$ respectively $\frac{-u+iv}{2}$ and $\frac{-u-iv}{2}$. And it should be noticed that the determinant of $u_,$, $v_,$ in respect to u, v will be

$$\begin{vmatrix} -\frac{1}{2} & \frac{i}{2} \\ -\frac{1}{2} & \frac{-i}{2} \end{vmatrix}$$

which is i.

(38) Let us proceed briefly to express the invariants of $ru^5 + sv^5 + tw^5$, which call Φ, with respect to u, v; the corresponding ones of $ru_,^5 + sv_,^5 + tw^5$,

which call $\Phi_{,}$, in respect to the same variables u, v, will be found by attaching to these suitable powers of i.

$$\Phi = (r-t,\ -t,\ -t,\ -t,\ -t,\ s-t \between u, v)^5.$$

Hence its quadratic covariant is the quadratic invariant of

$$((r-t)\,u - tv,\ -tu-tv,\ -tu-tv,\ -tu-tv,\ -tu+(s-t)\,v \between u',\ v')^4,$$

which is obviously

$$-rtu^2 - stv^2 + (rs - rt - st)\,uv.$$

Of this the quadratic invariant is

$$rt\,.\,st - \tfrac{1}{4}(rs - rt - st)^2;$$

or writing $\rho = st$, $\sigma = tr$, $\tau = rs$, and calling this invariant $-\frac{1}{4}(J)$,

$$(J) = \rho^2 + \sigma^2 + \tau^2 - 2\rho\sigma - 2\sigma\tau - 2\tau\rho.$$

Again, the cubic covariant or canonizant has been already shown to be $rst\,(u^2v + uv^2)$. Calling the discriminant of this $-\frac{1}{27}(L)$, we have

$$(L) = r^4s^4t^4\ ^{(30)} = \rho^2\sigma^2\tau^2.$$

Again, to find the discriminant (D) in respect to u, v.

When $ru^5 + sv^5 + tw^5 = 0$ has two equal roots, and $u + v + w = 0$, it is easy to see that we have $ru^4 + \lambda = 0$, $sv^4 + \lambda = 0$, $tw^4 + \lambda = 0$.

Hence to a constant factor *près* (D) will be the *Norm* of

$$(st)^{\frac{1}{4}} + (tr)^{\frac{1}{4}} + (rs)^{\frac{1}{4}}, \text{ that is of } \rho^{\frac{1}{4}} + \sigma^{\frac{1}{4}} + \tau^{\frac{1}{4}}\ ^{(31)}.$$

To find the value of this norm, suppose $\rho^{\frac{1}{4}} + \sigma^{\frac{1}{4}} + \tau^{\frac{1}{4}} = 0$, then

$$\rho + \sigma + \tau = 2\,(\rho^{\frac{1}{2}}\sigma^{\frac{1}{2}} + \sigma^{\frac{1}{2}}\tau^{\frac{1}{2}} + \tau^{\frac{1}{2}}\rho^{\frac{1}{2}}),$$

and $$\rho^2 + \sigma^2 + \tau^2 - 2\rho\sigma - 2\rho\tau - 2\sigma\tau = 8\rho^{\frac{1}{2}}\sigma^{\frac{1}{2}}\tau^{\frac{1}{2}}(\rho^{\frac{1}{2}} + \sigma^{\frac{1}{2}} + \tau^{\frac{1}{2}}).$$

Hence

$$\begin{aligned}(\rho^2 + \sigma^2 + \tau^2 - 2\rho\sigma - 2\rho\tau - 2\sigma\tau)^2 &= 64\rho\sigma\tau\,\{(\rho+\sigma+\tau) + 2\,(\rho^{\frac{1}{2}}\sigma^{\frac{1}{2}} + \sigma^{\frac{1}{2}}\tau^{\frac{1}{2}} + \tau^{\frac{1}{2}}\rho^{\frac{1}{2}})\} \\ &= 128\rho\sigma\tau\,(\rho+\sigma+\tau).\end{aligned}$$

Hence (D) must contain $(J)^2 - 128\rho\sigma\tau\,(\rho+\sigma+\tau)$ as a factor; and since when $t = 0$, $\rho = 0$, $\sigma = 0$, and $(D) = \tau^4 = (J)^2$, it is clear that $(D) = (J)^2 - 128(K)$, where $(K) = \rho\sigma\tau\,(\rho+\sigma+\tau)$.

(30) For this is $\left(0, \frac{rst}{3}, \frac{rst}{3}, 0 \between u, v\right)^3$, and the discriminant of $(a, b, c, d \between u, v)^3$ is

$$a^2d^2 + 4ac^3 + 4db^3 - 3b^2c^2 - 6abcd.$$

(31) It is worthy of observation that (J) is also a Norm, namely, of $\rho^{\frac{1}{2}} + \sigma^{\frac{1}{2}} + \tau^{\frac{1}{2}}$, so that (J) is the discriminant of $ru^3 + sv^3 + tw^3$. I have not been able to perceive the morphological significancy of this relation.

(39) Although in the investigation in view (K) will only figure as an abbreviation of $\frac{(J)^2-(D)}{128}$, it may not be amiss to indicate a direct process for finding it. Let us for this purpose act upon the Hessian of Φ, treated as a function of u, v, twice with the canonizant of Φ converted into an operator by substituting $\frac{d}{dv}$, $-\frac{d}{du}$ in place of u and v.

The Hessian of Φ may be obtained without difficulty under the form

$$rsu^3v^3 + stv^3w^3 + trw^3u^3 \text{ or } \tau u^3v^3 + \rho v^3w^3 + \sigma w^3u^3 \text{ [32]}.$$

Operating upon this with

$$r^2s^2t^2\left(\frac{d}{dv}\cdot\frac{d}{du}\left(\frac{d}{du}-\frac{d}{dv}\right)\right)^2,$$

we obtain $\rho\sigma\tau(A\tau + B\rho + C\sigma)$, where

$$A = -2\left(\frac{d}{du}\right)^3\left(\frac{d}{dv}\right)^3 u^3v^3 = -72;$$

and as we know that this quantity must be of the form $\lambda(K) + \mu(J)^2$, we have $\mu = 0$, $\lambda = -72$; so that, denoting the *operator* corresponding to the canonizant by T, and the Hessian by H, we have $(K) = -\frac{1}{72}T^2H\Phi$ [33]. This gives a ready practical method for finding the discriminant of a general quintic F by means of the identity $D = J^2 + \frac{16}{9}T^2H$, where D is the discriminant, H the Hessian, T the canonizantive operator, and J the quadratic invariant of F in respect to its own variables.

(40) If now we suppose the determinant of u, v in respect to x, y to be μ, where μ is by hypothesis a real quantity, and if we call the

Quadratic invariant in respect to x, y	. .	$-\frac{1}{4}J$,
Discriminant of primitive „ „	. .	D,
Discriminant of the canonizant „	. .	$-\frac{1}{27}L$,

we have obviously

$$\left.\begin{array}{ll} J = \mu^{10}(\rho^2+\sigma^2+\tau^2-2\rho\sigma-2\rho\tau-2\sigma\tau), & \\ K = \mu^{20}\rho\sigma\tau(\rho+\sigma+\tau), & D = J^2 - 128K, \\ L = \mu^{30}\rho^2\sigma^2\tau^2, & \end{array}\right\} \text{invariants of } \Phi.$$

This applies to the case where the reduced form is Φ, that is, where the roots of the canonizant are all real, and consequently where $-L$ is negative, that is, L positive.

[32] It will be the quadratic invariant of $ru^3\xi^2 + sv^3\eta^2 + tw^3\zeta^2$ with respect to ξ, η, $\xi+\eta+\zeta$ being zero; just as the quadratic covariant of Φ is the quadratic invariant of $ru\xi^4 + sv\eta^4 + tw\zeta^4$ with regard to the same variables. This latter is in fact $rsuv + stvw + trwu$.

[33] The intervening covariantic form of degree 3 in the variables and 5 in the coefficients, namely, $TH\Phi$, will easily be seen to be

$$rst^2(u^2v-uv^2) + str^2(v^2w-vw^2) + trs^2(w^2u-wu^2).$$

When L is negative and the reduced form is $\Phi_,$, then, since the determinant of $u_,$, $v_,$, in respect to u, v, is i, we have

$$\left.\begin{array}{l} J = -\mu^{10}(\rho^2+\sigma^2+\tau^2-2\rho\sigma-2\rho\tau-2\sigma\tau), \\ K = \quad \mu^{20}\rho\sigma\tau(\rho+\sigma+\tau), \qquad D = J^2 - 128\,K, \\ L = -\mu^{30}\rho^2\sigma^2\tau^2, \end{array}\right\} \text{invariants of } \Phi_,.$$

By means of the ratios $\frac{L}{J^3}$, $\frac{K}{J^2}$, it is obvious that in either case alike the ratios of ρ, σ, τ become determinable by means of the same cubic equation, namely

$$\theta^3 - K\theta^2 + \frac{K^2-JL}{4}\theta - L^2 = 0;$$

ρ, σ, τ will be to each other as the roots of this equation [34].

(41) Since $ru^5+sv^5+tw^5$ represents a function in x, y with real coefficients, it follows that when L is positive, u, v as well as w being real, $\alpha:\beta:\gamma$ are ratios of real quantities, and the roots of the preceding cubic will be real; when L is negative, u, v becoming conjugate imaginary functions of x, y, whilst w remains real, r, s, unless they are equal, must become conjugate imaginary constants. When r, s, t are all real, ρ, σ, τ will be so too; and when r, s are imaginary and t real, ρ, σ will be imaginary and τ real. Thus according as L is positive or negative the roots of θ are or are not all real. Hence understanding by Δ the discriminant of the preceding equation with respect to θ and 1, Δ/L must be always either zero or negative. We see *à priori* that Δ/L must be integer, because when $L=0$ the cubic has two equal roots, $\frac{1}{2}K$. To compute its value more conveniently, write $K=6k$, $J=12j$. Then the equation becomes

$$(1,\ 2k,\ 3k^2-jL,\ L^2 \between \theta,\ -1)^3,$$

[34] For since the absolute values of ρ, σ, τ are not in question, we may consider ρ, σ, τ as the roots of $\theta^3-K\theta^2+q\theta-r$, so that $\rho+\sigma+\tau=K$. We have then

$$\frac{\rho^4\sigma^4\tau^4}{(\rho\sigma\tau)^3(\rho+\sigma+\tau)^3}=\frac{L^2}{K^3},\ \text{or}\ \frac{r}{K^3}=\frac{L^2}{K^3},$$

which gives $r=L^2$. Again,

$$\frac{\rho\sigma\tau K^2}{(K^2-4q)^2}=\frac{K^2}{J^2},\ \text{or}\ \frac{(K^2-4q)^2}{r}=J^2,\ \text{or}\ (K^2-4q)^2=L^2J^2,\ \text{or}\ q=\frac{K^2\mp JL}{4}.$$

As regards the sign to be given to JL in q, since

$$\frac{J^3}{L}=\frac{(K^2-4q)^3}{r^2}=\frac{(K^2-4q)^3}{L^4},$$

we have $(K^2-4q)^3=J^3L^3$. Hence

$$q=\frac{K^2-1^{\frac{1}{3}}JL}{4}.$$

Consequently

$$q=\frac{K^2-JL}{4},\ \text{and } not\ \frac{K^2+JL}{4}.$$

of which the discriminant is

$$L^4 + 4(3k^2 - jL)^3 + 32k^3L^2 - 12k^2(3k^2 - jL)^2 - 12kL^2(3k^2 - jL).$$

Hence
$$\begin{aligned}\frac{\Delta}{L} &= L^3 - 108k^4j + 36k^2j^2L - 4j^3L^2 + 32k^3L \\ &\quad + 72k^4j - 12k^2j^2L \qquad - 36k^3L + 12jkL^2 \\ &= L^3 - 36k^4j + 24k^2j^2L - 4j^3L^2 - 4k^3L + 12jkL^2.\end{aligned}$$

Accordingly, multiplying the above equation by -3×12^2 in order to avoid fractions, replacing k, j by their values in terms of K, J, and naming G the quantity $-432\Delta/L$, positive, or to speak more strictly non-negative, we have

$$G = JK^4 + 8LK^3 - 2J^2LK^2 - 72JL^2K - 432L^3 + J^3L^2 \text{ (35)}.$$

It is evident that G must be identical to a positive numerical factor *près* with the function which M. Hermite denotes by I^2 (36).

(35) It will be observed that when $J=0$ and $L=0$, G vanishes. This is easily verifiable *à priori*; for when $J=0$ and $L=0$, the reduced form has been seen to be ax^5+5exy^4, of which the canonizant is

$$\begin{vmatrix} a, & 0, & 0, & 0 \\ 0, & 0, & 0, & e \\ 0, & 0, & e, & 0 \\ y^3, & -y^2x, & yx^2, & -x^3 \end{vmatrix}$$

which equals $aexy^2$.

Hence the form and its canonizant have a common factor x, and consequently their resultant vanishes; hence $I=0$ and $G=I^2=0$. G also vanishes when $K=0$ and $L=0$, which is also easily verifiable; for then the reduced form becomes u^5+v^5, of which the canonizant vanishes, and consequently the resultant of the form and its canonizant becomes intensely zero; which accounts for the high power of K in (JK^4), the sole term of G in which L does not appear.

(36) (*a*) Compare expression for $16I^2$, *Cambridge and Dublin Journal*, p. 203. This will be found to contain nine terms, and to rise as high as the fifth power in Δ (which to a constant factor *près* is identical with my J); whereas in $-\Delta/L$ there are only six terms, and no power of J beyond the third. This seems to indicate that the K and L are more fortunately chosen than M. Hermite's J_2, J_3, which are invariants of the like degrees 8 and 12. It is of course evident that the following relations exist between M. Hermite's Δ, J_2, J_3 and the J, K, L of this paper,

$$\Delta = lJ,$$
$$J_2 = mJ^2 + nK,$$
$$J_3 = pJ^3 + qJK + rL,$$

where l, m, n, p, q, r are certain numerical quantities. Until these are ascertained, it is impossible to confront M. Hermite's results with my own, to ascertain whether or not they are identical in substance, and, if not, wherein the difference consists. I therefore subjoin the necessary calculations for effecting this important object.

Let us first take the form $x^5+5exy^4+y^5$. The quadratic covariant of this is $x(ex+y)$.

Accordingly, to obtain M. Hermite's A, B, C, C', B', A' (*Cambridge and Dublin Journal*, vol. IX. p. 179), we must make

$$x = X; \quad ex+y = Y,$$

(42) In fact M. Hermite's octodecimal invariant is most simply obtained as the resultant of the primitive quintic and its canonizant. Using the

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which gives (vide *Cambridge and Dublin Journal*, p. 180)

$$F = X^5 + 5eX(Y - eX)^4 + (Y - eX)^5$$
$$= (A, B, C, C', B', A' \between X, Y)^5,$$

where $A = 1 + 4e^5, \quad B = -3e^4, \quad C = 2e^3, \quad C' = -e^2, \quad B' = 0, \quad A' = 1.$

Accordingly (vide *Cambridge and Dublin Journal*, p. 184),

$$AA' - 3BB' + 2CC' = 1 + 4e^5 - 4e^5 = 1 \qquad = \sqrt{\Delta},$$
$$AA' + BB' - 2CC' = 1 + 4e^5 + 4e^5 = 1 + 8e^5 = \frac{I_1}{2\sqrt{\Delta^3}},$$
$$AA' + 5BB' + 10CC' = 1 + 4e^5 - 20e^5 = 1 - 16e^5 = \frac{I_2}{2\sqrt{\Delta^5}}.$$

Hence $\Delta = 1, \quad I_1 = 2 + 16e^5, \quad I_2 = 2 - 32e^5.$

Again (vide *Cambridge and Dublin Journal*, p. 186, § vii.),

$$8J_1 = I_1 - \Delta^2 = 1 + 16e^5, \quad 24J_2 = I_2 - 2I_1\Delta + \Delta^3 = -1 - 64e^5;$$

but J_1, J_2 are subsequently *without warning* (compare expressions for AA', BB', CC', pp. 186, 192) renamed J_2, J_3; so that

$$8J_2 = 1 + 16e^5, \quad 24J_3 = -1 - 64e^5.$$

The corresponding values of J, K, L have been already calculated, and we have found

$$J = 1, \quad K = -2e^5, \quad L = 0.$$

Hence $A = 1, \quad \frac{1}{8} + 2e^5 = B - 2Ce^5, \quad \frac{-1}{24} - \frac{64}{24}e^5 = D - 2Ee^5.$

Thus $A = 1, \quad B = \frac{1}{8}, \quad C = -1, \quad D = -\frac{1}{24}, \quad E = \frac{4}{3}.$

To find F, take another form convenient for the purpose, as $x^5 + 10dx^2y^3 + y^5$.

Taking the emanant of this $(x, 0, dy, dx, y \between x', y')^4$, the quadratic covariant is obviously $xy + 3d^2y^2$, so that $J = 1$.

Also its discriminant is

$$\begin{vmatrix} 1, & 0, & 0, & d \\ 0, & 0, & d, & 0 \\ 0, & d, & 0, & 1 \\ y^3, & -y^2x, & yx^2, & -x^3 \end{vmatrix}$$

namely, $d^3y^3 - d(-dx^3 + y^2x) = d^3y^3 - dy^2x + d^2x^3,$

of which the discriminant is

$$d^{10} + 4d^2\left(\frac{-d}{3}\right)^3 = d^{10} - \tfrac{4}{27}d^5.$$

Hence by definition

$$L = e - \tfrac{27}{4}d^{10} + d^5.$$

Again, to find A, B, C, C', B', A', we must write

$$x + 3d^2y = X,$$
$$y = Y,$$

and we have then

$$(X - 3d^2Y)^5 + 10d(X - 3d^2Y)^2Y^3 + Y^5 = (A, B, C, C', B', A' \between X, Y)_5.$$

Since $J = 1$ and K is of the eighth order only in the coefficients, it is obvious that neither J^3 nor JK can contain a term involving d^{10}. In order therefore to find F, it will be sufficient to compare the coefficient of d^{10} in J_3 and in L.

Now $A = 1, \quad B = -3d^2, \quad C = 9d^4, \quad C' = 27d^5 + d, \quad B' = 81d^8 - 12d^3, \quad A' = 243d^{10} + 90d^5 + 1.$

Also $\Delta = J = 1$. Hence neglecting all but the terms which bring in d^{10}, $24J_3$ (p. 186, Memoir) is tantamount to I_2, and I_2 (p. 186) is tantamount to

$$2(243d^{10} - 5.3.81d^{10} + 10.9.27d^{10}),$$

which is $12 \times 243d^{10}.$

reduced forms for these two functions,

$$ru^5 + sv^5 - t(u+v)^5, \quad rstuv(u+v),$$

Hence in J_3 the term containing d^{10} is $\frac{243}{2}$.

Hence $$-\tfrac{27}{4}F = \tfrac{243}{2}, \text{ or } F = -18.$$

Hence we have, finally *,
$$\Delta = J,$$
$$J_2 = -K + \tfrac{1}{8}J^2,$$
$$J_3 = -18L + \tfrac{4}{3}JK - \tfrac{1}{24}J^3;$$

and conversely,
$$J = \Delta,$$
$$K = -\frac{J}{2} + \frac{1}{8}\Delta^2,$$
$$L = -\frac{J_3}{18} - \frac{2}{27}\Delta J_2 + \frac{1}{8}\Delta^3.$$

Unhappily a further step is wanting to bring M. Hermite's results to the final test of comparison; for the value of AA' (p. 192) does not agree with that given for AA' (p. 186) by simply changing J_1, J_2 into J_2, J_3 respectively; a further change of Δ into 2Δ becomes necessary to make the ratios of AA', BB', CC' (p. 192) accord with the ratios of the same quantities at p. 186. Finally, even after making this change the expression for $16I^2$ (p. 203) does not accord (even to a constant coefficient *près*) with that with which it is meant to be identical, namely, $16I_3{}^2$ (p. 187); so that after great labour I am still baffled in my attempt to ascertain the agreement or discrepancy of my conclusions with those of my precursor in the inquiry. As will appear hereafter, the two sets of conclusions are undoubtedly discrepant in form; but whether they are so in substance or not, or rather whether they are not in contradiction to each other, requires a close examination to discover, the more especially because, as will hereafter be shown, there is a certain necessary element of indeterminateness in the scheme of invariantive conditions which serve to fix the character of the roots. It is greatly to be lamented that so valuable a paper as M. Hermite's should be to some extent marred, in respect of the important end it would serve as a term of comparison, by the existence of these numerical and notational inaccuracies. I have spent hours upon hours in endeavouring to reconcile these several texts of the same memoir, and, after all my labour, the work is left unperformed without which the truth as between the two methods cannot be elicited. I feel, however, as confident of the correctness of my own conclusions as of the truth of any proposition in Euclid.

(*b*) It is worthy of notice that there is a failing case in M. Hermite's process for finding I^2 in terms of Δ, J_2, J_3, just as there is one in mine for finding G in terms of J, K, L,—the failure of the process, however, in neither case entailing any corresponding defect in the results obtained. The process employed in this memoir fails when $L=0$: for then the general form $ru^5+sv^5+tw^5$ is superseded by the supplementary one, $au^5+5euv^4+fv^5$. M. Hermite's fails when J (the J of *this* memoir) $=0$; for then the quadratic invariant becomes a perfect square, and the substitution of its factors in place of the original variables becomes inadmissible, since the two former coincide.

(*c*) It may be as well here to notice the form which M. Hermite's two linear covariants assume when referred to the canonical form above written. The quadratic covariant being $rsuv+stvw+trwu$, if we operate with the correlative of this, obtained by writing in it

$$\frac{d}{dv}, \quad -\frac{d}{du}, \quad \frac{d}{du} - \frac{d}{dv}$$

in lieu of u, v, w, namely with

$$-rs\frac{d}{du}\frac{d}{dv} - st\frac{d}{du}\left(\frac{d}{du} - \frac{d}{dv}\right) + tr\frac{d}{dv}\left(\frac{d}{du} - \frac{d}{dv}\right),$$

upon the primitive, we obtain to a factor *près* the canonizant $rstuvw$, which has been already

[* After Salmon, *Higher Algebra*, 1885, p. 250, $\Delta=J$, $J_2=-K$, $J_3=JK+9L$. Cayley's examination of Hermite's criteria is given, *Coll. Papers*, VI., p. 170.]

their resultant in respect to u, v is obviously

$$(rst)^5 (r-s)(s-t)(t-r)\ ^{(37)},$$

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obtained; repeating the process, it is easy to see that the first linear covariant of the fifth degree in the coefficient assumes the simple form $rst\,(stu+trv+rsw)$, or $rst\,(\rho u+\sigma v+\tau w)$. Taking again the correlative of this, namely,

$$rst\left\{\rho\frac{d}{dv}-\sigma\frac{d}{du}+\tau\left(\frac{d}{du}-\frac{d}{dv}\right)\right\},$$

and operating with it upon $rsuv+stvw+trwu$, it will be found without difficulty that the second linear covariant of the seventh degree in the coefficients becomes

$$rst\,\{(\sigma-\tau)(\sigma+\tau-\rho)\,u+(\tau-\rho)(\tau+\rho-\sigma)\,v+(\rho-\sigma)(\rho+\sigma-\tau)\,w\},$$

which is distinguishable in species from the former one by its symmetry being only of the hemihedral kind.

(d) It may not be out of place to notice here that the Hessian of the canonical form will be found to be

$$\rho v^3w^3+\sigma w^3u^3+\tau u^3v^3.$$

(e) Again, if we write

$$rst\,(\rho u+\sigma v+\tau w)=\xi,$$

$$rst\,\{(\sigma-\tau)(\sigma+\tau-\rho)\,u+(\tau-\rho)(\tau+\rho-\sigma)\,v+(\rho-\sigma)(\rho+\sigma-\tau)\,w\}=\eta,$$

$$u+v+w=0,$$

and from these equations deduce the values of u, v, w, and substitute them in $ru^5+sv^5+tw^5$, we shall obtain M. Hermite's "forme-type" expressed in terms of the parameters of the reduced form, and every coefficient therein will be invariantive.

The resultant of the equations above written (on making $\xi=0$, $\zeta=0$) will appear in the denominator of each such coefficient. Hence it appears, from M. Hermite's expressions (*Cambridge and Dublin Mathematical Journal*, vol. IX. p. 193), where J_3 will be seen to enter into the denominator of A, B, C, C', B', A', that this resultant to a factor *près* is his J_3. Its value may easily be calculated, and will be found to be

$$\rho\sigma\tau\,[(\rho+\sigma+\tau)^3-4\,(\rho+\sigma+\tau)\,(\rho\sigma+\rho\tau+\sigma\tau)+9\rho\sigma\tau]=JK+9L.$$

Accordingly as L (to use Dr Salmon's convenient elliptical expression) is the condition of the failure of my *general* reduced form, so is $9L+JK$ the condition of the failure of M. Hermite's "forme-type." As particular cases of this last failure, we may suppose $J=0$, $L=0$, or $K=0$, $L=0$. In the former case the reduced form is ax^5+5ex^4y, of which the simplest quadratic and cubic covariants are respectively aex^2; ae^2y^2x. Thus to find L, the first linear covariant, we have to operate upon ae^2y^2x with $ae\left(\dfrac{d}{dy}\right)^2$, which gives a^2e^3x; and to find L_2, we have to operate on $(aex^2)^2$ with $ae^2\left(\dfrac{d}{dx}\right)^2\dfrac{d}{dy}$, or, if we please (according to M. Hermite's method), with $\left(a^2e^3\dfrac{d}{dy}\right)$, on aex^2, showing that L_2 vanishes, but L_1 continues to subsist. When, secondly, $K=0$, $L=0$, the reduced form is ax^5+ey^5, and the canonizant disappears entirely, so that the first, and consequently also the second, linear covariants, each of them becomes a *null*.

(37) By aid of the reduced forms of the invariants J, K, L, I given in the text, it is easy to prove that every other invariant, say Ω of a quintic, is a rational integral function of these four. In what follows, let a parenthesis enclosing the symbol of any invariant signify its value when any two of the quantities u, v, w in the reduced form $ru^5+sv^5+tw^5$, where $u+v+w=0$, are taken as the independent variables. We have then

$$(J)=\rho^2+\sigma^2+\tau^2-2\rho\sigma-2\rho\tau-2\sigma\tau,\quad (K)=\rho\sigma\tau\,(\rho+\sigma+\tau),\quad (L)=\rho^2\sigma^2\tau^2,\quad (I)=\rho^2\sigma^2\tau^2\,(\rho-\sigma)(\sigma-\tau)(\tau-\rho),$$

ρ, σ, τ meaning st, tr, rs.

The degree of Ω must be of the form $4m$ or $4m+2$. (1) Let it be of the form $4m$. Then, since the interchange of any two of the variables, u, v, w must leave (Ω) unaltered, (Ω) will be

and consequently, if we call I the resultant in respect to x, y, we have

$$\pm I = \mu^{45}\rho^2\sigma^2\tau^2(\sigma-\rho)(\tau-\sigma)(\rho-\tau),$$

and

$$I^2 = \mu^{90}\rho^4\sigma^4\tau^4(\sigma-\rho)^2(\tau-\sigma)^2(\rho-\tau)^2$$
$$= \mu^{30}(\sigma-\rho)^2(\tau-\sigma)^2(\rho-\tau)^2L^2.$$

(43) Thus we see that the two quantities G, I^2, which are both rational integral functions of the degree 36 in the coefficients of $F(x, y)$, cannot one vanish without the other, at all events when L is not equal to zero. This is sufficient to show that they are identical to a numerical factor *près*, whatever L may be, zero or not zero [38], and consequently that the quantity called G, proved to be positive upon the supposition of L not being zero, must also remain positive when L is zero, because it is in fact the square of a rational function of the coefficients. But we may also prove this independently by virtue of the supplementary reduced form $au^5 + 5euv^4 + fv^5$ applicable to the case of L zero.

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unaltered by the interchange of any two of the letters r, s, t, and is consequently a symmetric function of ρ, σ, τ, the roots of the equation

$$\theta^3 - \frac{(K)}{(L)^{\frac{1}{2}}}\theta^2 + \frac{(K)^2-(J)(L)}{(L)} - (L^{\frac{1}{2}}) = 0.$$

Hence

$$(\Omega) = \frac{F\{(J),\ (K),\ (L)\}}{(L)^{2m}},$$

F denoting a rational integral function-form of the quantities it affects. Consequently

$$\Omega = \frac{F(J,\ K,\ L)}{L^{2m}}.$$

Hence since Ω cannot become infinite when $L=0$, which merely implies that the general form reduces to

$$(a,\ 0,\ 0,\ 0,\ e,\ i\c{)}(x,\ y)^5,$$

$\Omega = \Phi(J, K, L)$, a rational integral function of J, K, L.

(2) If the degree of Ω is of the form $4m+2$, (Ω) will be a function of r, s, t, which changes its sign when u and v or any two of the quantities u, v, w, are interchanged, such interchange having the effect of introducing as a multiplier the $5(2m+1)$th power of the determinant of substitution (-1). Hence (Ω) is of the form

$$(\rho-\sigma)(\sigma-\tau)(\tau-\rho)F(\rho,\ \sigma,\ \tau), \text{ that is } \frac{(I)\,.\,F(\rho,\ \sigma,\ \tau)}{(L)^{\frac{5}{2}}},$$

which again is of the form

$$\frac{(I)\,.\,F\{(J),\ (K),\ (L)\}}{(L)^{2m-8}},$$

so that Ω is of the form

$$\frac{I\,.\,F(J,\ K,\ L)}{L^{2m-8}}.$$

Hence since, as before, Ω cannot become infinite when $L=0$, and since, furthermore, I does not vanish (for if so then G, which is I^2, would vanish) when $L=0$, Ω must be of the form

$$I\Phi(J,\ K,\ L).\quad \text{Q. E. D.}$$

[38] For if $Q^2 = KI^2$ for an indefinite number of systems of values of a, b, c, d, e, f, of which Q, I are rational integral functions, Q^2 and KI^2 must be *absolutely* identical; this of course is the case when Q^2 and KI^2, as proved in the text, are known to be identical for all values of a, b, c, d, e, f which do not make L zero.

For when $L=0$, G becomes JK^4; so that the condition "G not negative" implies simply that J is positive unless K vanishes.

Now the canonizant, when it does not vanish, that is when e is not zero, contains v^2u as a factor, and, its coefficients being real, u, v are both of them necessarily real functions of x, y. Consequently J, which by definition is $-4\times$ discriminant of quadratic covariant, becomes $-4\mu^{10}\times$ discriminant of $au(eu+fv)$ in respect to u, v, which $=\mu^{10}a^2f^2$, μ being real. Consequently J is positive, since the reality of u, v implies that of a, e, f, when e is not zero. When e *is* zero u, v may be either real or imaginary; for u^5+v^5 may be real whether u, v be real or conjugate imaginary functions of x, y; but in that case K, which is found by operating twice upon the Hessian with a canonizant turned into an operator, vanishes, since then all the coefficients of the canonizant vanish [39]. Hence the rule that G cannot be negative is seen to be true, whatever L may be.

[39] (*a*) In the more general form $au^5+5euv^4+fv^5$, taking $\mu=1$, the canonizant is ae^2uv^2; this squared and turned into an operator becomes $a^2e^4\left(\frac{d}{dv}\right)^2\left(\frac{d}{du}\right)^4$, which, applied to the Hessian, namely $3aeu^4v^2+afu^3v^3-e^2v^6$, after multiplying by $-\frac{1}{72}$, gives $K=-2a^3e^5$, so that $D=J^2-128K=a^4f^4+256a^3e^5$, which is capable of easy verification. In fact D becomes the resultant of au^4+ev^4 and $v^3(4eu+fv)$; v^3 introduces the factor a^3 into D; and further, making $u:v::-f:4e$ and substituting in au^4+ev^4, we obtain the other factor af^4+256e^5.

If we adopt $u^5+5euv^4+v^5$ as the reduced form for the failing case (a form analogous to the well-known one, $u^4+6cu^2v^2+v^4$, for the general quartic), to find e we have $J=\mu^{10}$, $K=-2\mu^{20}e^5$. Hence $e^5=-\frac{K}{2J^2}$; thus when $K=0$, $e=0$.

(*b*) By a linear transformation we may always take away any two (except the two first or last) coefficients of a given quintic, but the vanishing of more than two coefficients always corresponds to some invariantive condition. Thus, for example, in the form

$ax^5+5exy^4+fy^5$	$L=0$		
ax^5+fy^5	$L=0$	$K=0$	
ax^5+5exy^4	$L=0$	$J=0$	
$ax^5+10dx^2y^3$	$J=0$	$K=0$*	
$ax^5+5bx^4y+10cx^3y^2$	$L=0$	$J=0$	$K=0$.

(*c*) The condition for the existence of four equal roots in a quintic is the vanishing of the quadratic covariant; that is to say, we must have

$$ae-4bd+3c^2=0,\quad af-3be+2cd=0,\quad bf-4ce+3d^2=0.$$

The three quantities equated to zero are not separately invariants, but constitute in their *ensemble* an invariantive plexus.

(*d*) [It may here be noticed incidentally that the conditions for equal roots in the biquadratic form are as follows. For two equal roots, of course, the discriminant is zero, for three equal roots the two lowest invariants are each zero, and for two pairs of equal roots the Hessian $(A, B, C, D, E\between x, y)^4$ becomes to a factor *près* identical with the primitive $(a, b, c, d, e\between x, y)^4$, so that all the first minors of the matrix

$$\begin{vmatrix} a, & b, & c, & d, & e, & f \\ A, & B, & C, & D, & E, & F \end{vmatrix}$$

vanish. *Quære*, whether the character of the five-rayed pencil (centre at origin), in which a, A; b, B; c, C; d, D; e, E mark points, may not serve to distinguish between the case of four real and four imaginary roots.]

[* Or, if $d=m^3$, $\omega^3=1$, $(x+my)^5+\omega(\omega x+m\omega^2y)^5+\omega^2(\omega^2x+m\omega y)^5$.]

It may be said that the case of three or more equal roots existing in $F(x, y)$ has been lost sight of; but we know, and it is capable of immediate verification by taking as the reduced form $au^5 + 5bu^4v + 10cu^3v^2$, that on such

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(*e*) When $J=0$ and $K=0$, but *not* $L=0$, it is obvious that $\rho:\sigma:\tau::1:\iota:\iota^2$, ι being any imaginary cube root of unity, and the reduced form is $u^5+\iota v^5+\iota^2 w^5$, with the relation $u+v+w=0$.

J and K being zero, D will be so too, and accordingly the equation $u^5+\iota v^5+\iota^2 w^5=0$ will have two equal roots. It will easily be found that these equal roots correspond to the system of ratios $u=1$, $v=\iota^2$, $w=\iota$. In fact, if we write $u=1+\rho$, $v=\iota^2+\iota\rho$, $w=\iota+\iota^2\rho$, the equation becomes $u^5+\iota v^5+\iota^2 w^5=\rho^2(30\rho+3\rho^3)=0$.

Hence, understanding by ϵ either of the two prime sixth roots of unity, the complete system of ratios of u, v, w may be expressed as follows :—

$$\begin{array}{lll}
u=1 & v=\iota^2 & w=\iota \\
u=1 & v=\iota^2 & w=\iota \\
u=1-\sqrt[3]{(10)} & v=\iota^2-\sqrt[3]{(10)} & w=\iota-\iota^2\sqrt[3]{(10)} \\
u=1+\sqrt[3]{(10)}\,\epsilon & v=\epsilon^4-\sqrt[3]{(10)} & w=\epsilon^2+\sqrt[3]{(10)}\,\epsilon^5 \\
u=1+\sqrt[3]{(10)}\,\epsilon^5 & v=\epsilon^4+\sqrt[3]{(10)}\,\epsilon & w=\epsilon^2-\sqrt[3]{(10)}.
\end{array}$$

Thus, when $J=0$ and $K=0$, u, v, w (with the relation $u+v+w=0$) may first be found, in terms of x, y, by solving the cubic equation, obtained by equating to zero the canonizant of $(a, b, c, d, e, f\between x, y)$, and then x, y will be known from the above system of values for any two of the quantities u, v, w.

(*f*) It is obvious that the form $ax^5+10dx^2y^3$ gives $J=0$ and $K=0$; but it seems desirable to prove the converse, namely that when $J=0$ and $K=0$, but not $L=0$, the form is always reducible to $au^5+10\delta u^2v^3$; which may be done as follows. Since $J=0$ and $K=0$ the discriminant is zero, and we may assume

$$F=ax^5+5bx^4y+10cx^3y^2+10dx^2y^3,$$

and we have $J=$ discriminant of

$$(-4bd+3c^2)\,\xi^2+2cd\xi\eta+3d^2\eta^2.$$

Hence
$$3d^2(3c^2-4bd)-c^2d^2=0\,;$$

d cannot be zero, for then we should have $J=0$, $K=0$, $L=0$, contrary to hypothesis. Hence $8c^2-12bd=0$.

If $b=0$ and $c=0$, F is already reduced to the desired form; but if not, $d=\dfrac{2c^2}{3b}$, and F becomes

$$ax^5+\frac{5b}{6}x^2\left(6x^2y+\frac{12c}{b}xy^2+\frac{8c^2}{b^2}y^3\right);$$

or, making
$$a-\frac{5b^2}{6c}=\alpha,\quad \frac{b^2}{6c}=2\delta,\quad x+\frac{2cy}{b}=v,$$

$F=\alpha x^5+10\delta x^2v^3$, as was to be shown.

The corresponding converses for the case of $J=0$, $L=0$, and of $K=0$, $L=0$ have been already established.

(*g*) It will be observed that under a certain point of view L for binary quintics is the analogue of Δ the *discriminant* for binary quartics, the condition of failure in the *general* reduced form in the two cases being $L=0$ and $\Delta=0$ respectively. The mere vanishing of the discriminant in the case of the quintic function, unattended by any other condition, does not affect the nature of the reduced form.

(*h*) It has been shown previously in the text that when $L=0$ the primitive is reducible to the form

$$(a, 0, 0, 0, e, f\between x, y)^5.$$

Hence if I_{12} is any duodecimal invariant which vanishes when $b=0$, $c=0$, $d=0$, I_{12} must vanish whenever L vanishes, and consequently, since L is of as high a degree as I_{12}, I_{12} must be a numerical multiple of L. In Mr Cayley's Third Memoir on Quintics, "No. 29" represents

hypothesis all the invariants J, K, L must vanish, so that JK^4 is still non-negative [40].

(44) It is most important to notice that G can only become zero by virtue of two of the quantities ρ, σ, τ, and therefore of r, s, t becoming equal. When u, v are imaginary, it is the coefficients r, s which must become equal, as otherwise the reduced form would not be a real function of x, y. By equating r to s, and using as an auxiliary variable the ratio $\frac{r}{t}$ or $\frac{s}{t}$, we shall be able to study the composition and inward nature of G with the utmost clearness and facility.

SECTION II.—*On the Criteria which decide the Number of Real and Imaginary Roots.*

(45) Since in the preceding section we have supposed that u, v are always real linear functions of x, y, it is obvious that the character of the roots of the given quintic in x, y is completely identical with that of the roots in the reduced form, and it has been shown that only one reduced form corresponds to a given system of values of J, D, L [41].

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a duodecimal invariant calculated by M. Faà de Bruno, and characterized* morphologically by Mr Cayley as being that duodecimal invariant in which "the leading coefficient a does not rise above the fourth degree." On examining No. 29 it will be found to contain no term in which b, c, d are all simultaneously absent. Hence it is, by virtue of the above observation, a multiple of my L: to determine the numerical factor, let all the coefficients in the primitive except a, d be supposed zero; then the canonizant becomes

$$\begin{vmatrix} a & 0 & 0 & d \\ 0 & 0 & d & 0 \\ 0 & d & 0 & 0 \\ y^3 & -y^2x & yx^2 & -x^3 \end{vmatrix} = d^3y^3 + ad^2x^3.$$

Hence L becomes $-27a^2d^{10}$, but "No. 29" becomes $27a^2d^{10}$. Hence we have the important relation "No. 29" $= -L$, so that No. 29 is a discriminant, an *intrinsic* property of the calculated invariant, which, I believe, was not suspected.

(i) It will at once be recognized that "No. 19" given in Mr Cayley's Second Memoir upon Quantics is identical with the J of this memoir, whence it follows from† Mr Cayley's equation (No. 26) = (No. 19)$^2 - 1152$ (No. 25), that $K = 9$ (No. 25). Thus abstraction made of a mere numerical factor, Mr Cayley and myself agree upon perfectly distinct grounds in recognizing K and L as the true simplest invariants of their respective degrees, an accordance as satisfactory as it was unexpected, and which must be considered as setting at rest the question of what should be deemed the, so to say, *staple* invariants of the Binary Quintic.

([40]) When the form is $au^5 + 5euv^4 + fv^5$ so that $L = 0$, the canonizant, as has been seen before, is ae^2v^2u; the resultant of these two is $a^5e^{10}a^2f = a^7e^{10}f$. Again, $J = a^2f^2$, $K = -2a^3e^5$; thus the square of the resultant $= \frac{1}{16}JK^4$; so that if we call this resultant, which we may take as the definition of the Octodecimal Invariant I, we have $G = 16I^2$.

([41]) It should be well noticed that the mere ratios $\frac{D}{J^2}, \frac{L}{J^3}$ do not suffice to determine the character of the roots. When these ratios are given, it is true that the ratios r, s, t in the

[* Cayley's *Coll. Papers*, II., pp. 294, 314.] [† *Ibid.*, II., p. 313 and VI., p. 148.]

Let us suppose J, D, L to be taken as coordinates of a point in space; when J, D, L are so related that the condition G non-negative is satisfied, the point will correspond to an equation with real coefficients, and may be termed a *facultative* point. But when G is negative it will correspond to an equation of the kind alluded to in the recent section of this paper, and there called conjugate: such a point may be termed non-facultative. Thus the whole of space will be divided into two parts, separated by the surface $G=0$, which may be termed respectively facultative and non-facultative (as being made up of facultative or non-facultative points [42]). It is clear that these two portions will be exactly equal, similar, and symmetrical with regard to the axis of D; by which I mean that, if two points be taken in any line perpendicular to the axis of D at equal distances from that axis, one will be facultative and the other non-facultative, as is evident from the fact that when J, L become $-J$, $-L$ (K, and therefore D or J^2-128K, remaining unaltered), G is converted into $-G$. Thus by a semirevolution round the axis of D the facultative and non-facultative portions may be made to exchange places.

(46) The axis of D itself lies on the surface of G, and like every other portion of this surface is facultative, for there is no reason for disallowing G to become zero. Conversely, if instead of a real equation, we take one of the conjugate class (described in the second section), the whole of the facultative portion of space (except the separating surface G) becomes non-facultative, and the non-facultative part becomes facultative, but G itself remains facultative. When the invariants, or any of them, become imaginary, we are put out of space altogether, and the system can belong neither to a real nor to a conjugate family, but to one with coefficients at the same time imaginary and non-conjugate. $G=0$ [43], it may be remarked, will in all cases be the condition of an equation capable of linear transformation into one of recurrent [44] form; for the reduced form then in general becomes

$$ru^5+rv^5-t(u+v)^5.$$

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reduced form are given, but according as L is positive or negative, the arguments u, v in $ru^5+sv^5+tw^5$ (supposing w to be the real linear function of x, y) will be real or imaginary. When J, L, D are all given *absolutely*, then the character of the roots is completely determined. The *indelible* marks of a quintic function are three in number, namely the ratios $\frac{K}{J^2}$, $\frac{L}{J^3}$, and the sign of L or J, as for a quartic function they are two in number, namely $\frac{s^3}{t^2}$ and the sign of s.

(42) It will also be convenient to call the coordinates J, D, L corresponding to any facultative point a facultative system of invariants, and $\frac{D}{J^2}$, $\frac{L}{J^3}$ corresponding to the same (for a *given sign* of J) a facultative system of invariantive ratios.

(43) I shall hereafter allude to the surface denoted by $G=0$ under the name of the Amphigenous Surface, as being the locus of the points which give birth to real and conjugate forms indifferently.

(44) The roots of recurring equations, geometrically represented, in general go in quadruplets,

The case when G becomes zero by virtue of $J = 0$ and $L = 0$, that is to say when the function is reducible by real or imaginary linear substitutions (see footnote (39) (f)) to the form $u(u^4 \pm v^4)$, is the one which might for a moment be supposed to offer an exception to the rule; but the exception is only apparent, since $u(u^4 - v^4)$, on writing $u = p + q$, $v = p - q$, becomes

$$8(p+q)pq(p^2+q^2).$$

(47) To every point in space, it has been remarked, will correspond one particular family of equations all of the same character as regards the number they contain of real or imaginary roots, because capable of being derived from one another by real linear substitutions, such family consisting of an infinite number of ordinary or conjugate equations according as the point is facultative or non-facultative; but it may be well to notice that, conversely, every point does not correspond to a distinct family. In fact the equations $D = pJ^2$, $L = qJ^3$ (p, q being constants) will denote a curve divided into two branches by the origin of coordinates, one of which will be facultative and the other non-facultative; but in each separate branch every point will represent the same family. Any such separate branch may be termed an isomorphic line; and we see that the whole of space may be conceived as permeated by and made up of such lines radiating out from the origin in all directions.

(48) The origin at which $J = 0$, $D = 0$, $L = 0$, as already noticed, corresponds to the case of three equal roots. The theorem that, when more than half as many roots are equal to each other as there are units in the degree of any binary form, all the invariants vanish, was remarked by myself originally in the very infancy of the subject, before Mr Cayley's paper, alluded to by M. Hermite, appeared in Crelle. The method of proof which then occurred to me is the simplest that can be given. For instance, in the case before us, if the quintic have three equal roots, we may reduce it to the form

$$ax^5 + 5bx^4y + 10cx^3y^2.$$

Suppose now, if possible, an invariant of the degree m; the *weight* of each term therein, say $a^r b^s c^t$, in respect to x or y would be the same (namely $5m/2$), so that we should have

$$5r + 4s + 3t = \frac{5m}{2} = s + 2t, \text{ or } 5r + 3s + t = 0,$$

and therefore $r = 0$, $s = 0$, $t = 0$, $m = 0$. So for a sextic with four equal

A, A'; B, B', where A and B, as also A', B', are mutual optical images of each other in respect to a fixed line, and A, A', as also B, B', are electrical images of each other in respect to a circle of which the fixed line is a diameter—with liberty, of course, for the images taken in either mode of combination to coalesce so as to reduce the quadruplet to a simple pair.

roots reduced to the form $(a, b, c, 0, 0, 0, 0 \between x, y)^6$. Supposing any term in one of its invariants to be $a^r b^s c^t$, we should have

$$6r + 5s + 4t = \frac{6m}{2} = s + 2t, \text{ or } 6r + 4s + 2t = 0,$$

which is absurd, unless $r = 0$, $s = 0$, $t = 0$, $m = 0$, and so in general for a binary form of any degree. If in the above example for the degree m only three roots were equal *inter se*, the form assumed being $(a, b, c, d, 0, 0, 0 \between x, y)^6$, any term in a supposed invariant being $a^r b^s c^t d^u$, where $r + s + t + u = m$, we should have

$$6r + 5s + 4t + 3u = 3m = s + 2t + 3u,$$

and, as before,

$$6r + 4s + 2t = 0, \quad r = 0, \quad s = 0, \quad t = 0;$$

no longer, however, $m = 0$, but $m = u$, which is left undetermined.

(49) Before proceeding further it will be proper to consider under what circumstances a variation (in the coefficients of any equation) arbitrary, except that the coefficients are to remain real, can affect the character of the roots.

Let $F(x) = 0$ be any algebraical equation with real coefficients, and let $\delta F(x)$ be the variation of F due to the variation of the coefficients, $dF(x)$ the variation due to the change of x into $x + dx$. If, now, r be a root of $F(x) = 0$, and $r + dr$ the corresponding root of $F(x) + \delta F(x) = 0$, we have

$$F(r) = 0, \quad F(r + dr) + \delta F(r) = 0,$$

or

$$\delta F(r) + \frac{d}{dr} F(r)\, dr + \frac{1}{1\,.\,2} \left(\frac{d}{dr}\right)^2 F(r)\, dr^2 + \&\text{c.} = 0.$$

Hence, unless $\frac{\delta F}{dr} = 0$, that is, unless there are two equal roots r, we shall have $dr = -\dfrac{\delta F(r)}{\dfrac{d}{dr} F(r)}$ = a real quantity; so that the character of the root $r + dr$ will be the same as that of r.

But if

$$\frac{dF}{dr} = 0, \quad \frac{d^2 F}{dr^2} = 0, \ldots \left(\frac{d}{dr}\right)^{i-1} F = 0,$$

so that there are i roots r, i being any integer greater than zero, then to find dr we have the equation

$$(dr)^i + \frac{\Pi(i)\, \delta F r}{\left(\dfrac{d}{dr}\right)^i F(r)} = 0.$$

Thus dr will have i distinct values; of these, if i is odd, all but one will be imaginary, but if i is even they will be all imaginary, or only all but two

imaginary and the remaining two real, according as the sign of $\delta F(r)$ is the same as or the contrary to that of $\left(\frac{d}{dr}\right)^i F(r)$. Accordingly, if r is real [45] and i even, the nature of the *ensemble* of the i roots $r + dr$ will not be the same when $\delta F(r)$ is positive as when $\delta F(r)$ is negative.

(50) So, further, if $F(x) = 0$ have $2m$ equal roots r, $2n$ equal roots s, and so on, the deduced corresponding groups of roots in $F(x) + \delta F(x) = 0$ will, or may at least each of them, undergo a change of character to the extent of one pair of the r group changing their nature with the sign of $\delta F(r)$, one pair of the s group changing their nature with the sign of $\delta F(s)$, and so on; but in no case, except $F(x)$ possess some equal roots (that is unless its discriminant be zero), can an infinitesimal variation in the constants affect the character of the roots [46].

(51) To every facultative point corresponds a certain set of values of J, D, L; and when these are given, it has been shown that the equation $(a, b, c, d, e, f \between x, y)^5$ is reducible to the form $ru^5 + sv^5 + tw^5$, where

$$u + v + w = 0,$$

or to the form $ru_1{}^5 + sv_1{}^5 + tw^5$, where

$$u_1 + v_1 + w = 0, \text{ and } u_1 = \frac{-w + iv}{2}, \quad v_1 = \frac{-w - iv}{2},$$

or to the form $au^5 + 5euv^4 + fv^5$, u, v, w being always real linear functions of x, y, with the sole exception that when $J = 0$, $K = 0$, $L = 0$, the reduced form is

$$au^5 + 5bu^4v + 10cu^3v^2.$$

When these three invariants are not all zero, the coefficients in the reduced form r, s, t or a, e, f are known functions of J, D, L, and the character of the roots is perfectly determinate; so that to every facultative point corresponds an infinite family of equations with real linear coefficients all deducible from each other by real linear substitutions. Thus then, with the sole exception of the origin, every facultative point corresponds to a determinate character of equation, namely to an equation with four, or two, or no imaginary roots; so that by a bold figure of speech we may be permitted to speak of

[45] r, although supposed to be one of a group of equal roots, is not *necessarily* real, for it may belong to a factor $(x^2 + 2xe\cos\theta + e^2)^2$.

[46] Compare this statement with the corresponding one given by M. Hermite, *Camb. and Dub. Journal*, vol. IX. p. 204, where only one parameter is supposed to undergo a change. I think that greater breadth and at the same time greater precision and clearness are gained by the mode of exposition employed in the text above. It will be observed that for a change of character to be possible when the function passes through a phase of equal roots, it is not enough that there shall exist a group of equal roots r, but there must be an even number of such roots in the group, and, furthermore, the equal roots must be *real*; when this last supposition is not satisfied, no change in the character of dr will affect the character of $r + dr$: an instructive exemplification of this remark will occur in the sequel.

every point but one in facultative space having a determinate quality, as masculine, feminine, or neuter. The origin alone is exempt from this law, and may be considered to be of epicene gender, since the factor

$$au^2 + 5buv + 10v^2$$

may have its roots real or imaginary. As we travel continuously from point to point in the facultative portion of space we pass from family to family, or, if we please, from an individual of one family to an individual of another family, differing from the former individual by an infinitesimal variation of the constants.

(52) If, then, we insulate any portion of facultative space, and in the block so insulated it is possible to pass from one point to any other—that is to say, if we can draw a *continuous* curve of any sort from one point to another without passing out of the block, and without cutting or touching the plane $D = 0$, then by virtue of the principle just laid down, we see that all the points in such block have the same character, and the nature of the roots will be the same in the infinite number of families, each containing an infinite number of individuals which the points in that block severally represent. Now imagine a block taken so extensive as to admit of no further augmentation, except accompanied with a violation of the condition of the capability of free communication between point and point without cutting or touching the surface D; such a block may be termed a *region*, and the whole of facultative space will be capable of subdivision into a certain number of these regions. This being supposed effected, the character of each region will be known when we know the character of a single point in it; that is to say, every region will have a determinate character of positive, negative, or neuter. It will presently be shown that the number of such regions is only three [47] (the least number it could be to meet the three cases of four, two, or no imaginary roots), one masculine, one feminine, one neuter; and consequently there will be but three cases to consider when the invariantive coordinates J, D, L are given; according as J, D, L belong to one or the other of these three regions, the equation to which they belong will have all its roots real, or only one real, or three real and two imaginary. The origin, it need hardly be added, constitutes a region *per se*, in which, so to say, the characters of masculine and feminine are blended.

(53) Let it be observed that we can see *à priori* that, were it not for the distinction between facultative and non-facultative portions of space, it would be impossible for each point corresponding to a given system of invariants to possess an unequivocal character; for in such case there would necessarily

(47) It is clear from the definition, that a *region* can only be bounded by G the amphigenous surface*, and D the plane of the discriminant: and granted (as will be shown hereafter) that G and D *touch* each other in only one continuous line, it becomes obvious *à priori* that there can be but two regions on one side of D and a single region on the other. [* p. 436, Footnote (43).]

be free continuous communication possible between all the points on each side of D *inter se*, and consequently we should be landed in the absurdity of conceiving the general equation of the fifth degree not to admit of division into cases of four, two, or no imaginary roots; D being negative, we know, would imply two roots, and not more than two, being imaginary; and accordingly D positive would imply either that four roots are imaginary or none—not sometimes one and sometimes the other, but in all cases alike four imaginary, to the exclusion of the supposition of the roots being all real—or else that all the roots are real and never four imaginary. Thus we see that the mere fact of a given system of invariants communicating a definite character to the roots, implies the necessity of the invariants exercising a restraining action over each other's limits, and that where this restraint does not exist it is impossible that the character of the roots can be determined by the values of the invariants.

(54) This is precisely what happens in biquadratic equations. In such we know the fundamental invariants t, s, or, if we please,

$$t, \ \Delta \ (\text{where } \ \Delta = s^3 + 27t^2),$$

are perfectly independent and subject to no equation of condition; so that if we consider t, Δ as the coordinates of points in a plane, the whole of the plane will be made up of facultative points. When Δ is negative, that is for representative points lying on one side of the line Δ, it is true we know that there is just one pair of imaginary roots constituting what may be termed the neuter case; but when the representative points lie on the other side of this plane, they cannot be said to be either masculine or feminine, but will every one of them possess that epicene character which is peculiar to the origin alone in the case of quintic forms. A single example will make this clear.

Take the two reduced forms

$$u^4 + 6(1+\epsilon)\,u^2v^2 + v^4,$$
$$\omega^4 + 6(1-\epsilon)\,\omega^2\theta^2 + \theta^4,$$

where u, v are real linear functions of x, y, and ω, θ conjugate imaginary ones of the same; and suppose s, the quadrinvariant in respect to x, y, to be the same for both forms. For greater convenience of computation consider ϵ to be infinitesimal.

Then in the one case the t is of the same sign as

$$(1+\epsilon)(1-(1+\epsilon)^2), \text{ that is, } -2\epsilon,$$

and in the other the t is of the contrary sign to

$$(1-\epsilon)(1-(1-\epsilon)^2), \text{ that is, } \quad 2\epsilon,$$

so that t is of the same *sign* (namely negative) in each case.

Again, in the two cases respectively

$$\frac{t^2}{s^3} = \frac{4\epsilon^2}{1 + 3(1 \pm \epsilon)^2} = 4\epsilon^2.$$

Hence t as well as s, and consequently t and Δ are alike for both forms.

But in the one first written the roots are of the same nature as those of $u^4 + 6u^2v^2 + v^4$, that is, are all impossible, and in the other of the same nature as in

$$\left(\frac{u+iv}{2}\right)^4 + 6\left(\frac{u+iv}{2}\right)^2\left(\frac{u-iv}{2}\right)^2 + \left(\frac{u-iv}{2}\right)^4 = 0,$$

where u, v are real linear functions of x, y and $i = \sqrt{-1}$, in which case the roots are all possible. Thus we see that the very same values of t, Δ may correspond either to the case of four real or four imaginary roots, showing that the point t, Δ is what we have termed *epicene*. If we choose to take s, t as the coordinates, the same remarks would apply, except that Δ instead of a straight line would become a semicubical parabola. All the points on one side of this curve would have a definite neuter character, but those on the opposite side would be neither masculine nor feminine, but epicene.

(55) With a view to its subsequent distribution into regions, I now proceed to ascertain the form of that moiety of space which I have termed facultative.

Let $J^2 = qK$, $J^3 = \nu L$. Then

$$\frac{G}{J^9} = \frac{1}{q^4} + \frac{8}{\nu q^3} - \frac{2}{\nu q^2} - \frac{72}{\nu^2 q} - \frac{432}{\nu^3} + \frac{1}{\nu^2}, \text{ and } \frac{D}{J^2} = 1 - \frac{128}{q}.$$

We may for the moment make abstraction of the section of G made by the plane of L; that being done, and J, K, L being referred to the form

$$ru^5 + sv^5 + tw^5 \text{ or } ru_1^5 + sv_1^5 + tw^5,$$

calling μ^{10}, M, and, as before, using ρ, σ, τ to denote st, tr, rs, we have [pp. 426–7]

$$\pm J = M(\rho^2 + \sigma^2 + \tau^2 - 2\rho\sigma - 2\rho\tau - 2\sigma\tau),$$
$$K = M^2\rho\sigma\tau(\rho + \sigma + \tau),$$
$$\pm L = M^3\rho^2\sigma^2\tau^2.$$

Now when $G = 0$, we may suppose $\rho = \sigma$, $\frac{\tau}{\rho} = \frac{\tau}{\sigma} = \theta + 4$, θ being a new auxiliary variable [real. Cf. § 44]. We have then

$$\pm J = M(\tau^2 - 4\rho\tau) = M\rho\tau\theta,$$
$$K = M^2\rho^2\tau(2\rho + \tau) = M^2\rho^2\tau^2\left(1 + \frac{2}{\theta + 4}\right),$$
$$\pm L = M^3\rho^4\tau^2 \qquad = M^3\rho^3\tau^3\frac{1}{\theta + 4},$$

and consequently

$$\nu = \frac{J^3}{L} = \theta^4 + 4\theta^3,$$
$$q = \frac{J^2}{K} = \frac{\theta^2(\theta + 4)}{\theta + 6}.$$

(56) In general we have $\theta^4+4\theta^3-\nu=0$.

By a well-known corollary to Descartes' rule this equation can never have more than two real roots; when ν is positive there will always be two real roots of opposite signs; but when ν is negative and inferior to a certain negative limit, *all the roots become imaginary.* When ν lies between zero and that limit, two roots of θ will be real and both negative. To find that limit we may make $4\theta^3+12\theta^2=0$, or $\theta=-3$, which gives $\nu=81-108=-27$.

(57) When
$$D=0,\ q=\frac{J^2}{K}=128,$$
that is,
$$\theta^3+4\theta^2-128\theta-768=0, \text{ or } (\theta+8)^2(\theta-12)=0;$$
thus the roots of θ, when $D=0$, are -8, -8, $+12$, and the corresponding values of ν are 2^{11}, 2^{11}, $27\cdot 2^{10}$.

If now we make $\theta^4+4\theta^3=2^{11}$, one of the real values of θ we know is -8, and the other will be the real root of the cubic equation
$$\theta^3-4\theta^2+32\theta-256=0.$$
When $\theta=5$, the left-hand side of the equation
$$=125+160-100-256=-71.$$
When $\theta=6$, the left-hand side of the equation
$$=216+192-144-256=\quad 8.$$
Hence the real root lies between 5 and 6, and q lies between $\frac{225}{11}$ and $\frac{360}{12}$. Thus $q<30$ and $\frac{D}{J^2}=1-\frac{128}{q}$ is *negative.*

Again, if we take $\theta^4+4\theta^3=27\cdot 2^{10}$, and take out the root $\theta=12$, the resulting cubic becomes
$$\theta^3+16\theta^2+192\theta+2304=0,$$
where it will easily be seen the real root lies between -12 and -16.

When $\theta=-12$,
$$q=\theta^2\frac{\theta+4}{\theta+6}=144\times\frac{8}{6}=192;$$
and when $\theta=-16$,
$$q=256\times\frac{12}{10}=307\tfrac{1}{5}.$$
Moreover, when q is a maximum or minimum, it will readily be found that $\theta^3+11\theta+24=0$; so that $\theta=-3$, or $\theta=-8$. Hence for the value of θ found from the above cubic $q<192$ and $\frac{D}{J^2}=1-\frac{128}{q}$ is *positive.*

(58) When $J=0$, $\nu=0$; and when $L=0$, $\nu=\infty$.

For these two cases it will be more simple to dispense with the auxiliary variable θ, and to revert to the original equation between J, K, L.

Accordingly, when $J=0$, we find $8LK^3-432L^3=0$. Hence

$$J=0,\ \text{or}\ K^3=54L^2,\ \text{that is}\ \left(\frac{-D}{128}\right)^3=54L^2;$$

thus the complete section of G made by the coordinate plane J becomes a straight line, namely the axis of D, and a semicubical parabola whose axis is the negative part of D.

When J is very nearly zero, ν becomes a positive or negative infinitesimal in the equation $\theta^4+4\theta^3=\nu$.

One real root of this equation is $\theta=\left(\frac{\nu}{4}\right)^{\frac{1}{3}}$.

The other is $-4+\delta$, where $[4(-4)^3+12(-4)^2]\,\delta=\nu$,

or $$\delta=-\frac{\nu}{64}.$$

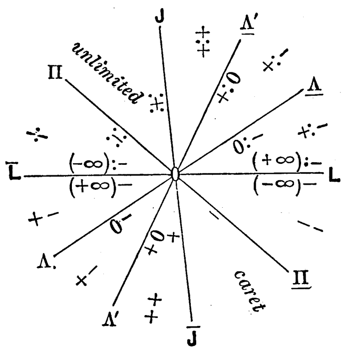
Now $$\frac{K^3}{L^2}=\left(\frac{\theta+6}{\theta+4}\right)^3(\theta+4)^2=\frac{(\theta+6)^3}{(\theta+4)}.$$

The first value of θ gives $K^3=54L^2$ to an infinitesimal *près*; the other value gives

$$K^3=-\frac{512}{\nu}L^2,$$

or, to an infinitesimal *près*,

$$\left(\frac{D}{128}\right)^3=\frac{512}{\nu}L^2;$$



so that D passes from $+\infty$ to $-\infty$, that is, $\frac{J^3}{L}$ passes through zero.

(59) In the annexed figure [48], the plane of the paper represents the plane of D, that is, the plane for which $D=0$; $JO\bar{J}$ is the axis of J, OJ being the positive and $O\bar{J}$ the negative direction; $LO\bar{L}$ is the axis of L, OL being the positive and $O\bar{L}$ the negative direction. In order to avoid any appearance of an attempt at a practicably impossible accuracy of drawing, I use straight lines to denote cubical parabolas, and pay no attention whatever to relative magnitudes, but only to the order or progression of magnitudes, using the lines which are drawn in the figure

([48]) I shall refer, when I have occasion to do so, to this figure, which contains a synopsis of the whole theory, under the name of the Dial figure.

not as *copies* but as *symbols* of the actual curves which are to be mentally imagined.

Thus the line $JO\bar{J}$ is used to represent the straight line $L=0$; $\Lambda'O'\underline{\Lambda}'$ the cubical parabola $J^3 = 27\cdot 2^{10}L$; $\Lambda O\underline{\Lambda}$ the cubical parabola $J^3 = 2^{11}L$; $\Pi O\underline{\Pi}$ the cubical parabola $J^3 = -27L$ [49].

It will be observed that certain combinations of *plus, zero, minus,* positive and negative *infinity* are placed along the lines and inside the sectorial spaces. The meaning of these will be sufficiently obvious from what has preceded. They refer to the signs of the two values of D in the surface G for each point in the line or sector along or within which they are placed. At every point along the line $O\bar{J}$, $\dfrac{D}{J^2}$ has only one value, and that positive; along $\Lambda'O\underline{\Lambda}'$, $\dfrac{D}{J^2}$ has two values, one positive and the other zero. Along $\Lambda O\underline{\Lambda}$, $\dfrac{D}{J^2}$ has two values, one zero, the other negative. Immediately below $\bar{L}OL$ two values, one $+\infty$, the other finite and negative. Immediately above $\bar{L}OL$ two values, one $-\infty$, the other finite and negative. Along $\Pi O\underline{\Pi}$ one value, finite and negative.

Moreover D has been shown to be never zero, except along $\Lambda'O\underline{\Lambda}'$, $\Lambda O\underline{\Lambda}$. Hence it is obvious that *inside* $\Lambda'O\bar{J}$ and the opposite sector D has two values, both *plus*; inside the next pair of opposite sectors two values, one *plus*, the

(49) It has been shown in the preceding articles that corresponding to the line $\bar{J}OJ$ and to the line $\Pi O\underline{\Pi}$, the vertical ordinate D of the amphigenous surface ($G=0$) has only one value, positive for the former, negative for the latter; along the line $\Lambda'O\underline{\Lambda}'$ two values, one positive the other zero; for the space between $\Lambda O\underline{\Lambda}$, $\bar{L}OL$ indefinitely near to the latter two values, one positively infinite, the other negative; and for the space indefinitely near to the same on the opposite of it, two values, one negatively infinite, the other negative. These results are collected and represented symbolically in the Table annexed.

$\bar{J}$	Λ'	Λ	$\bar{L}$	Π
	$+$	0	$\overbrace{(+\infty)\ -}$	
$+$				$-$
	0	$-$	$-\ (-\infty)$	

Thus, corresponding to the upper sheet of G, we have the succession

$+\quad +\quad 0\quad (+\infty)\quad -\quad -$

and to the lower sheet

$+\quad 0\quad -\quad -\quad (-\infty)\quad -$

the two sheets coming together at a cuspidal edge above $\bar{J}OJ$ and below $\Pi O\underline{\Pi}$.

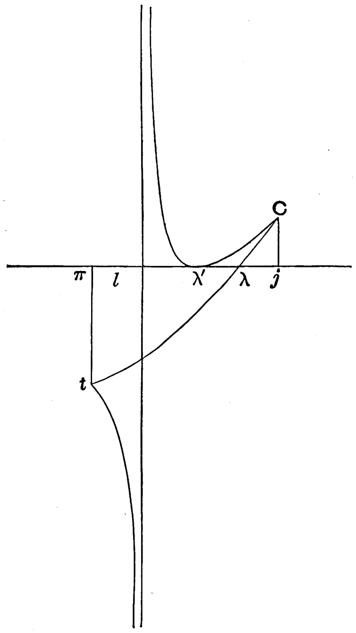
Moreover these are the only positions of the line revolving in the plane of D corresponding to which a change in the nature of D can take place, and thus we can without further examination fill up the Table, giving the nature of D for the intervening spaces, and may thus obtain the Table embodied in the *dial-figure* above, that is,

$\bar{J}$		Λ'		Λ		$\bar{L}$		Π
	$+$	$+$	$+$	0	$+$	$\overbrace{(+\infty)\ -}$	$-$	
$+$								$-$
	$+$	0	$-$	$-$	$-$	$-\ (-\infty)$	$-$	

other *minus*; inside the next pair of sectors also two values, one *plus*, the other *minus*; inside the next pair of sectors two values both *minus*, and in the pair of sectors left vacant, for which $\nu < -27$, it has been shown that D becomes impossible.

(60) Thus it will be seen that the surface G consists of two opposite portions precisely similar and symmetrical in respect to the axis of D.

Let us trace that one of these whose ground-plan is comprised within the sector $\Pi O\bar{J}$. It will consist of two sheets coming to a cuspidal edge (a common parabola) in the superior part of the plane of L. The upper sheet will touch the plane of D in $O\Lambda$ (50), and, remaining above the plane of D, approach continually to the plane of J as an asymptotic plane. The lower sheet will cut the plane of D in $O\Lambda'$, pass under the plane of D, cut the plane of J, progress to a maximum distance from it, and then approach indefinitely to J as its asymptotic plane. This will become apparent by taking a vertical section of this portion, cutting the lines $O\bar{L}$, $O\bar{J}$; for the nature of the flow of the two branches of the section will evidently be as figured here, where j, λ, λ', l, π represent the points in which the lines $O\bar{J}$, $O\Lambda'$, $O\Lambda$, $O\bar{L}$, $O\Pi$ are cut by the secant plane. [It should be particularly noticed that this figure is only intended to exhibit, under its most general aspect, the nature of the flow of the two branches of the curve; it is drawn in other respects almost at random, and makes no pretension whatever to giving a representation of the actual form of the curve.]



No part of the surface G lies under or above the sector ΠOJ, except the axis of D. The cusp C, where the two branches meet, is the intersection of the cutting plane with the parabola $J = D^2$ lying in the plane of L, and there will be another cusp at t, the point of maximum recession from the plane of J.

(61) I now proceed to discriminate, by aid of this surface, the facultative from the non-facultative portion of space.

If in the expression for G as a function of J, K, L we substitute for K its value $-\dfrac{D}{128} + \dfrac{J^2}{128}$, we obtain $G = \dfrac{J}{(128)^4} D^4 +$ terms involving only lower powers

(50) For the value of D for this sheet is zero all along $O\Lambda$, and positive on either side of it.

of D; so that, calling D_1, D_2 the two real values of D in the upper and lower sheets of G respectively corresponding to any point J, L,

$$G = J(D - D_1)(D - D_2)Q,$$

Q being a quantity essentially positive.

Hence when J is negative the *facultative* points in any line parallel to D will be those for which D lies between D_1, D_2, but when J is positive, the facultative points must be exterior to the segment D_1D_2; I denote this difference in the figure by placing a colon between the signs in each sector for which J is positive, indicating thereby that the facultative points lie between $+\infty$ and D_1, and between D_2 and $-\infty$; but where no colon is interposed, then it is to be understood that the facultative points lie between D_1 and D_2. Thus, if we turn back for a moment to the section of G last drawn, the whole of the space included between the two branches and the asymptote is facultative, because up to the asymptote J is negative, and beyond the asymptote the whole of the space not included between the asymptote and the lower branch is facultative, because beyond the asymptote J becomes positive. Thus, then, we see that the whole of that portion of the plane which lies on the left-hand side of the entire curve is facultative, and the portion on the right-hand side of the same non-facultative; the curve separating facultative from non-facultative space as a coast-line, indefinitely extended, separates land from water; so that there is, as of course we might have anticipated, no break of continuity in passing through the plane J.

If we take a corresponding section of the opposite portion of space corresponding to the ground-plan $JL\underline{\Pi}$, it is obvious that precisely the contrary takes place, because the sign of J is opposite in the opposite sectors; so that what was facultative becomes non-facultative, and *vice versâ*.

(62) It is now clear that the whole of the facultative part of space is divided into three, and only three of the *regions* previously defined. One region will consist of that portion of it which is entirely under the plane of D: the second region will be so much of the upper portion as stands upon the acute sector $\bar{J}O\Lambda$; and the third of so much of the remainder of this portion as stands on the sector $\Lambda OJJO\underline{\Pi}$ (51). Again, as regards the second region, the line $O\Lambda'$ is quite inoperative against its unity, because we have vertical ordinates above $O\Lambda'$ through which free communication can take place between the blocks over $JO\Lambda'$ and $\Lambda'O\Lambda$; but when we come

(51) It will be borne in mind that the whole of the infinite prism, both above and below, standing on ΠOJ belongs to *facultative* space: the prism standing on the opposite sector $\bar{J}O\bar{\Pi}$, or, to speak more strictly, on the *inside* of this last-named sector, is wholly unfacultative. The facultative line D which passes through O is completely isolated from the facultative portion which stands over $\Lambda O\bar{J}$, except at the point O (which we are forbidden to pass through if we would remain in the same region), and is of course a rectilinear edge to the facultative prism above referred to.

to $O\Lambda$, where G touches the plane of D, there we have an effective line of demarcation between the adjoining blocks *above* the plane of D; for it is impossible to pass from one into the other without going under D and coming up again through that plane, or else descending to the line $O\Lambda$ and so meeting the plane of D [52].

(63) It remains only to fix the characters of the several regions; but this requires no calculation to effect, for we know that when D is negative there is one and only one pair of imaginary roots. This disposes of the first of the regions above enumerated. Again, we know that when L is positive so that the reduced form is the superlinear equation $ru^5 + sv^5 + tw^5 = 0$, u, v, w being *real* functions, D being also positive, there must be four imaginary roots, as follows from the theory of the second section. Hence the third region has for its character two pairs of imaginary roots; and consequently the only remaining region, the second described, must correspond to the case of no imaginary roots, since otherwise we should be absurdly assuming the impossibility in any case of a quintic equation having all its roots real.

(64) It may, however, be an additional satisfaction to see how the change of character comes to pass at the critical line $O\Lambda$ from one to five real roots.

Along the line $O\Lambda$ [with $G=0$] we have found [p. 443] that, calling the reduced form

$$ru_1^5 + sv_1^5 + tw^5,$$

$$r = s, \ \frac{\tau}{\rho} = \frac{rs}{st} = \frac{r}{t} = \theta + 4 = -4.$$

Hence the equation becomes

$$4u_1^5 + 4v_1^5 + (u_1 + v_1)^5 = 0,$$

u_1, v_1 being of the form $\dfrac{-u+iv}{2}$, $\dfrac{-u-iv}{2}$, because L is negative.

Hence, beside $u_1 + v_1 = 0$,

$$4(u_1^4 - u_1^3v_1 + u_1^2v_1^2 - u_1v_1^3 + v_1^4) + (u_1 + v_1)^4 = 0,$$

that is $$5u_1^4 + 10u_1^2v_1^2 + 5v_1^4 = 0,$$

that is $$(u_1^2 + v_1^2)^2 = 0;$$

[52] Two superior regions we know *à priori* must exist to correspond respectively to the two cases of five and of one real root. Moreover we know *à priori* that two regions can only meet on the plane of D, and an inspection of the *dial-figure* shows that only $O\Lambda$ can be such line. Thus without completely making out the geometry of the question as regards the remarkable line ($J=0$, $L=0$) (the axis of D) which lies on the surface G, we may feel assured that the upper part of this line (which is easily found to belong to the 1-real-root region) cannot have any point except the origin in common with the 5-real-roots region, since otherwise these two regions would communicate along this line and merge into one. When it is considered that G is a surface of the ninth order in J, D, L, it will not appear surprising that some difficulty arises in forming a mental conception of certain of its local properties; on the contrary, the subject of wonder rather is that enough can be ascertained about it in a very brief compass to shed all the needful light upon the analytical problem which it illustrates.

so that there are two pairs of equal roots of $\frac{u_1}{v_1}$, namely $\pm i$; to these values of $\frac{u_1}{v_1}$ correspond

$$\frac{u-iv}{u+iv}=i, \quad \frac{u-iv}{u+iv}=-i.$$

Hence $\qquad (1-i)u=(i-1)v, \text{ or } (1+i)u=(i+1)v;$

so that the two pairs of equal roots of u/v are ± 1, the outstanding root corresponding to $u_1+v_1=0$ being $u/v=0$.

Now, *still keeping upon the surface* G, which we know is facultative, let θ become $-8+4\epsilon$, where ϵ is an infinitesimal, then

$$\delta\left(\frac{J^3}{L}\right)=\delta v=(4\theta^3+12\theta^2)\,\delta\theta=-5120\epsilon;$$

also the supposed equation becomes

$$(4-4\epsilon)(u_1^5+v_1^5)+(u_1+v_1)^5=0,$$

or
$$(iv-u)^5-(iv+u)^5-8(1+\epsilon)u^5=0;$$

or, calling $v/u=\rho$,

$$(i\rho-1)^5-(i\rho+1)^5-8(1+\epsilon)=0.$$

Let $\rho=\pm 1+\sigma$, where σ is an infinitesimal. Hence

$$[-10(\pm i-1)^3+10(\pm i+1)^3]\sigma^2-8\epsilon=0,$$

or
$$20(-3+1)\sigma^2-8\epsilon=0,$$

or
$$\sigma^2=\frac{-\epsilon}{5}=+\frac{1}{25600}\,\delta\left(\frac{J^3}{L}\right).$$

Hence calling σ_1, σ_2 the two values of σ, the four roots that at $O\Lambda$ were 1, 1, -1, -1 become $1+\sigma_1$, $1+\sigma_2$, $-1+\sigma_1$, $-1+\sigma_2$, when J^3/L becomes varied by $\delta(J^3/L)$, and consequently become all real if J^3/L is increased, and all imaginary if J^3/L is decreased, that is, become real or imaginary according as the line $O\Lambda$ sways towards or away from $O\bar{J}$, conformably with what has been shown on other grounds.

It will be noticed that in the line $O\Lambda$ produced in the opposite direction, that is, along the line $O\underline{\Lambda}$, L being positive, the reduced form is

$$4(u^5+v^5)+(u+v)^5=0,$$

and the roots of $\frac{u}{v}$ become $\frac{u}{v}=-1$, $\frac{u}{v}=\pm i$, $\frac{u}{v}=\pm i$; so that, according to the canon laid down at the commencement of this discussion (see foot-note [46]), no change in the character of the roots can possibly take place along $O\Lambda$, and accordingly we have seen that this curved line does not correspond to any demarcation of regions.

(65) It is easy to express the conditions to be satisfied by the coordinates

of a point according as it lies in one or another of the three regions which have now been mapped out, and it is clear that we have the following rule:

When D is negative the equation has two imaginary roots.

When D is positive the equation has *no* imaginary roots, provided the two criteria J and $2^{11}L - J^3$ are both* negative [53]; but if either of these is zero or positive, there are two pairs of imaginary roots [54].

The duodecimal criterion-invariant, $2^{11}L - J^3$, and the invariants of the like order, $27 \cdot 2^{10}L - J^3$, $-27L - J^3$, I shall henceforth call Λ, Λ', Π respectively. It has been shown just above that the three invariants J, D, Λ of the 4th, 8th, and 12th orders respectively are sufficient for ascertaining the character of the roots of the quintic to which they appertain.

(66) The assertion that the *whole* of facultative space is divisible into three regions, in strictness requires a slight modification. It is obvious that the plane of D itself cannot be said to belong to any of the regions; and in order to make our theory quite complete, so as to furnish criteria applicable to equations having equal roots, and to enable us to distinguish between the case of the unequal roots being all three real, or two imaginary and one real, we must examine what takes place in this plane, and under what circumstances a passage from one point of it to another will or may be accompanied with a change of character in the roots.

If the roots of $f(x) = 0$ are supposed to be a, a, c, d, e, where c, d, e are unequal, on varying the constants of $f(x)$ in such a manner that the variation of the discriminant D is zero, the two equal roots a, a will remain equal. Now *in general* we have $\delta f(a) + \frac{1}{2} f''(a)\, da^2 = 0$; if this, under the particular supposition made, continued to obtain, da would have two distinct values, and the two equal roots would cease to continue to be equal, contrary to hypothesis. Hence we see that $D = 0$, $\delta D = 0$ necessarily implies $\delta f(a) = 0$ [55],

(53) Observe that this implies L also being negative; so that $2^{11} - \frac{J^3}{L}$ is positive and $\frac{J^3}{L} < 2^{11}$.

(54) (*a*) Observe that in general when $2^{11}L - J^3$ is zero there are no facultative points above the plane of D, but when J and $2^{11}L - J^3$, and consequently L and J are both simultaneously zero, a facultative right line springs into existence, namely, the axis of D extending both above and below the plane of D. The reduced form of equation (as previously demonstrated) corresponding to this singular line is $u^5 \pm uv^4 = 0$.

(*b*) It may further be noticed that on each side of the line $O\underline{\Lambda}$ the limits of D are between positive infinity and a positive quantity, and between negative infinity and a negative quantity; so that as we pass from $O\underline{\Lambda}$ to either side of it no facultative point can be found lying in the plane of D, showing that we cannot pass by a real infinitesimal variation of coefficients from an equation with two pairs of equal imaginary roots to an equation with a single pair of equal roots, as is apparent also on purely analytical grounds.

(55) (*a*) This is a somewhat curious theorem (whether new or otherwise I know not) thus incidentally established in the text, namely, that if $D(f)$ represent the discriminant of f, and if

[* $2^{11}L - J^3$ should be positive, and L negative. Cf. p. 372 above.]

and consequently $\delta f(a+da)$ is no longer $\delta f(a)$, but $\delta f'(a)\,da$; so that we obtain $da=0$, or $da=-\dfrac{2\delta f'(a)}{f''(a)}$, and no change of character in the five roots results. If, however, the original roots are a, a, c, c, e, then, as shown in the general case, δc will have two distinct values, which will be both real or both imaginary. Accordingly we see that in the plane of D no change can possibly take place except in crossing the line which corresponds to a family of *two pairs* of equal roots.

(67) It has already been pointed out, in a foot-note, that we cannot pass facultatively from $O\Lambda$ to either side of this curve line. Hence the separation of the plane of D into subregions can only take place along the line $O\Lambda$, and it remains but to ascertain the character of the points on either side of this line, which we know, therefore, *à priori,* must possess opposite characters, since otherwise we should be admitting the absurd proposition of its being impossible to construct an equation of the fifth degree having two equal roots without the remaining three being always of *one character*, either all real or all not real. Let us, then, ascertain the character of the points in OJ for which $D=0$, $L=0$, and J is positive [56].

$D(f)=0$ and $\delta D(f)=0$, then when $f=0$ we must have $\delta(f)=0$. The very simplest example that can be chosen will serve to illustrate this proposition. Let

$$f=ax^2+2bxy+cy^2.$$

Suppose
$$D(f)=ac-b^2=0,$$
and also
$$\delta D(f)=a\delta c+c\delta a-2b\delta b=0,$$
we have
$$\delta(f)=x^2\delta a+2xy\delta b+y^2\delta c.$$

Now if $f=0$ we may write $x=b$, $y=-a$, and δf becomes

$$\begin{aligned}&b^2\delta a-2ab\delta b+a^2\delta c\\ &=b^2\delta a-2ab\delta b+2ab\delta b-ac\delta a\\ &=(b^2-ac)\,\delta a=0,\end{aligned}$$

according to the theorem.

If we make $f=(x,\,1)^n$, D we know becomes a syzygetic function of f and f' or df/dx.

Hence since δD vanishes when $f(x)=0$, $D=0$, and $\delta f(x)=0$, we learn that $\delta(D)$ is a syzygetic function of $(f, f', \delta f)$.

The theorem thus stated easily admits of extension to the higher variations of D, and so extended takes, I believe, the following form:

$$\delta^i(D)=\text{a syzygetic function of } (f, f', f'', \ldots f^i, \delta f).$$

(*b*) Professor Cayley has since informed me that the theorem in ([55]) (*a*), about whose originality I was in doubt, will be found in Schläfli's *De Eliminatione.* This is not the first unconscious plagiarism I have been guilty of towards this eminent man, whose friendship I am proud to claim. A much more glaring case occurs in a note by me [? p. 242 above] in the *Comptes Rendus*, on the twenty-seven straight lines of cubic surfaces, where I believe I have followed (like one walking in his sleep), down to the very nomenclature and notation, the substance of a portion of a paper inserted by Schläfli in the *Mathematical Journal*, which bears my name as one of the editors upon its face!

([56]) We could not take J negative, for the facultative points of D in $\bar{J}$ are two positive quantities. See dial figure.

Since $L=0$, the reduced form is $u^5+5euv^4+v^5$.

This equation, by Descartes's rule, must contain imaginary roots. Hence in the sector $\Lambda O\bar{J}$ the roots are all real, and in the remainder of the facultative portion of the plane (from which it may be noticed the sector $\underline{\Lambda}O\bar{J}$ is excluded) two of the roots are imaginary.

Along $O\Lambda$ itself there are, as already observed, two pairs of real equal roots, and along $O\underline{\Lambda}$ two pairs of imaginary equal roots. Thus, finally, we have the *complete rule**.

If D is negative,	2 roots imaginary.	
If D is positive.		
When J, Λ are both negative,	0 roots imaginary.	
„ J, Λ are *not* both negative,	4 roots imaginary.	
If D is zero.		
When J, Λ are both negative,	0 roots imaginary	1 pair of equal roots.
„ J, Λ are *not* both negative,	2 roots imaginary	
„ J is negative, Λ zero,	0 roots imaginary	2 pairs of equal roots.
„ J is positive, Λ zero,	4 roots imaginary	
„ J is zero, Λ zero,		3 equal roots [56 bis].

Thus we see our space referred to an arbitrary origin, and with the invariants J, D, Λ for the coordinates, has been first divided into facultative and non-facultative space. The former has then been resolved prismatically into two regions above and one below the plane of D. The plane of D itself, or the facultative part of it, into two planar regions on opposite sides of the line $\Lambda O\underline{\Lambda}$; and again this line into two linear regions on either side of the origin O, which last corresponds to the case of three equal roots, and constitutes a region or microcosm in itself.

(68) It may as well be noticed here that the ambiguity of character in the points representing the different families of biquadratic forms when t and D are taken as the coordinates (and the same would be true if s and D were

[56 bis] When $D=0$, $\Lambda=0$, there are two pairs of equal roots. If J is negative these pairs are both real. If J is positive they are both imaginary. When J is zero there are no longer two pairs, but a single triad of equal roots. This perfectly explains what at first sight has the air of a paradox, namely, that the discrimination between the two kinds of double equality of an apparently equal order of generality that may subsist between the roots of an equation, depends on the fulfilment or failure of an algebraical equality. The fact is, as shown above, that there are not, as commonly supposed, two, but three kinds of double equality, according as there are two pairs of real, two pairs of imaginary, or one triad of equal roots; and the last is a sort of transition case between the other two.

[* For $D=0$, the sector $\bar{J}O\Lambda'$ of the dial figure is non-facultative, as follows from the diagram of p. 446. Thus the rule is: $D<0$, 2 roots imaginary; $D>0$, 5 roots real when $L<0$, $\Lambda>0$, 1 root real when one or both of these is reversed; $D=0$, 5 roots real when $\Lambda>0$, $\Lambda'<0$, 3 roots real when $\Lambda<0$, $\Lambda'<0$; $D=0$, $\Lambda=0$, 5 roots or 1 root real according as $L<0$ or $L>0$.]

employed), which prevails when these points lie above the line $D=0$, equally obtains along this line itself. For the reduced form, when $D=0$, is

$$ax^4+4bx^3y+6cx^2y^2.$$

In that case, calling the determinant of transformation μ, we have

$$s=3\mu^{12}c^2, \quad D=-\mu^{24}c^3;$$

and thus, whatever s and D may be, the character of the unequal roots is left undecided.

It may also be noticed that the blending of characters at the *origin* for the quintic form is not precisely of the same nature as that for the points above the line D in the biquadratic form; for at these points it is the cases of 4 and 0 imaginaries which become undistinguishable invariantively; whereas at the origin for quintics the reduced form becomes $ax^5+5bx^4y+10x^3y^2$, and the characters left undistinguished are those of 4 and of 2 imaginary roots—unless, indeed, we consider equal real roots as belonging indifferently to the class of real and imaginary; on which supposition all the three genders (so to say), masculine, feminine and neuter, become blended together at that point. But if we consider equal real roots as exclusively of the real class, then the *origin* for quartics ceases to be epicene; for when there are three equal roots all of them must be real. Thus the origin in quintics is the only epicene point, and in quartics the only non-epicene point—understanding by epicene the blending of the masculine (4 *imaginary roots*) and feminine (*no imaginary roots*) characters.

(69) We may draw some further important inferences from an inspection of the "dial figure," or the section of facultative space which follows it.

Within the prism $\bar{J}O\Lambda'$ [57] it will be observed D is always positive [58]. Hence, when J is negative and Λ' is negative, all the roots *must* be real, and the necessity for using the criterion D is done away with.

Again, when J and L are both negative, D is always negative, so that just two of the roots must be imaginary; and in this case also it becomes unnecessary to apply the criterion D.

Again, since there is no facultative prism corresponding to $\underline{\Pi}OJ$, the combination of L and D, both negative, can never occur unless $\underline{\Pi}$ is negative.

When L is negative, but J not negative, there may be two or four imaginary roots, according to the sign of D; but all the roots cannot be real.

(57) By which I mean within the facultative prism of which $\bar{J}O\Lambda'$ is the section made by the plane of D.

(58) The vertical section of facultative space in this supposition (see figure) is the area $\lambda C\lambda'$, which lies wholly *above* the plane of D.

(70) M. Hermite's rule is as follows. For remarks on the relation between his Δ, J_2, J_3 and the J, K, L of this paper, see [p. 430, above] footnote [36]. D is still the discriminant.

If D is negative (of course) two roots are imaginary.

If D is positive.

When Δ is negative, $25\Delta^3 - 3 \cdot 2^{10}J_3$ negative and J_3 positive, no roots are imaginary.

Δ is negative, $25\Delta^3 - 3 \cdot 2^{10}J_3$ positive, $25\Delta^3 - 2^{11}J_3$ negative, no roots are imaginary.

Δ is positive, $25\Delta^3 - 3 \cdot 2^{10}J_3$ positive, $25\Delta^3 - 2^{11}J_3$ negative, four roots are imaginary.

Δ is negative, $25\Delta^3 - 3 \cdot 2^{10}J_3$ positive, $25\Delta^3 - 2^{11}J_3$ positive, four roots are imaginary [59].

(71) What is the effect of the condition "Λ *positive* or *negative*," as the case may be? or rather, how does this condition arise? The ground of it is simply this, that $\Lambda = 0$ represents a cylindrical surface passing through the curve $O\Lambda$ (see dial figure), which curve is the *edge* of separation between two regions of opposite characters above the plane of D; the cylinder in question cuts the facultative portion of space below the plane of D, but above this plane (except along the vertical line $J = 0$, $L = 0$, that is, the axis of D)

[59] (*a*) The last four conditions ought to tally (and be in effect coextensive) with the two given by me for the case of D positive. The third of them, namely the case of D positive Δ positive, I have already noticed, as inferences from the dial figure; for M. Hermite's Δ, if not identical with my J, is at all events a positive multiple of it. I do not see how the case of Δ negative, $25\Delta^3 - 3 \cdot 2^{10}J_3$ negative with D positive, is met by this system of criteria, since J_3, as well as Δ, may be negative consistently with the second condition. I have not been able to ascertain whether in the memoir such a combination is shown to be impossible. M. Hermite admits, and indeed has been always aware of, the existence of a *lacuna* in the conditions above stated, which, I understand from him, it is his intention at some future time to fill up, and thus to complete his original solution. In the meanwhile he has been led to study the question from a different point of view, and has succeeded in obtaining a new set of criteria adequate to a complete solution of the question without calling in the aid of the principle of continuity. In this new system my Λ criterion is replaced by an invariant of the twenty-fourth degree, which is of course an objection as far as it goes, but in no wise diminishes the extraordinary interest that attaches to this altered mode of approaching the question, which bears to his original method and my own the same relation as the proof of Sturm's theorem by the law of inertia for quadratic forms bears to that given by Sturm himself.

(*b*) It is apparent from the fact that when $D=0$, G (M. Hermite's I^2) becomes

$$(25\Delta^3 - 3 \cdot 2^{10}J_3)\,(25\Delta^3 - 2^{11}J_3)^2$$

(*Camb. and Dub. Journal*, vol. IX. p. 206), that the factors of this product are respectively of the form $a\Lambda' + bJD$, $c\Lambda + eJD$, a, b, c, e being certain numerical quantities. This gives rise to a singular reflection, *to wit*, that my own criteria for the case of D positive may be varied by the addition of a term λDJ to Λ (λ being a numerical coefficient), provided λ lies within certain limits, the form of the criteria in all other respects remaining unchanged. This proposition, fraught with the most important consequences, and not unlikely to lead to an entire revolution in the mode of attacking the general problem of criteria, I proceed to establish in the text.

it passes exclusively through non-facultative space, never again cutting or meeting the surface G (the facultative boundary). Now it is clear that any surface whatever which passes through $O\Lambda$ and never meets the surface G above the plane $D=0$, except along the axis of D (that is, the line $J=0$, $L=0$), may be substituted for Λ [60] and will serve equally well with Λ to distinguish between the masculine and feminine regions of space. $\Lambda-\rho JD$ will fulfil the condition of passing through the line $O\Lambda$, whose equation is $\Lambda=0$, $D=0$, and obviously is the only invariant not exceeding the twelfth order capable of so doing; it only remains to ascertain within what limits the numerical coefficient ρ must be taken so as to fulfil the condition that the combined equations $\Lambda-\rho JD=0$, $G=0$ shall be incapable of being satisfied by any positive value of D.

(72) Substituting for Λ and D their values, the equation to be combined with $G=0$ becomes

$$J^3-2^{11}L+\rho J(J^2-128K)=0.$$

Returning to the notation of Art. (55) [p. 442], and dividing by JK, this equation, when $G=0$, becomes

$$q-2^{11}\frac{q}{\nu}+\rho(q-128)=0,$$

or $$(1+\rho)\,q\nu-2^{11}q=128\rho\nu,$$

which, substituting for q, ν in terms of θ, gives

$$\frac{(1+\rho)\,\theta^5(\theta+4)^2}{\theta+6}=2^{11}\frac{\theta^3+4\theta^2}{\theta+6}-128\rho\theta^3(\theta+4),$$

or $$(\theta+4)\,\theta^2(\theta+8)\left[(\theta^3-4\theta^2+32\theta-256)+(\theta^3-4\theta^2-96\theta)\rho\right]=0.$$

When $\theta+8=0$, $D=0$, see Art. (57); neglecting, then, this factor, the condition to be satisfied is that when from the equation

$$(\theta+4)\,\theta^2\left[(\theta^3-4\theta^2+32\theta-256)+(\theta^3-4\theta^2-96\theta)\,\rho\right]=0$$

a value of θ has been deduced, the value of D corresponding thereto shall not be a positive finite quantity.

(73) Now

$$\frac{D}{J^2}=1-\frac{128(\theta+6)}{\theta^2(\theta+4)}=\frac{\theta^3+4\theta^2-128(\theta+6)}{\theta^2(\theta+4)}=\frac{(\theta+8)^2(\theta-12)}{\theta^2(\theta+4)}.$$

If $\theta=0$, or $\theta+4=0$, since D cannot be infinite, we have $J=0$, so that $\Lambda-\rho JD$ becomes identical with the original criterion Λ. Hence the factor

[60] The surface to be employed will be $\Lambda-\rho JD$, which call M. Λ and M (or at least their upper portions above the plane of D) may then be regarded as the two sides of a sack, of infinite dimensions, open at the top, and seamed together at the bottom, along the curved line $D=0$, $\Lambda=0$, and in the vertical direction along the straight line $J=0$, $L=0$. The surface Λ serving as a screen of separation between the two upper regions, it is clear that M will serve equally well as such screen, provided no superior facultative points lie in the interior of the sack.

$(\theta+4)\theta^2$ in the quantity just above equated to zero may be neglected, and the condition to be fulfilled by ρ is that if θ be any root of the equation

$$\frac{-\theta^3+4\theta^2-32\theta+256}{\theta^3-4\theta^2-96\theta}=\rho,$$

θ shall be between -4 and 12; this equation on making $\theta=-4\phi$, so that $1>\phi>-3$, becomes

$$-\rho=\frac{\phi^3+\phi^2+2\phi+4}{\phi^3+\phi^2-6\phi},$$

or, writing $\sigma=\dfrac{-1-\rho}{4}$,

$$\sigma=\frac{2\phi+1}{\phi^3+\phi^2-6\phi}=\frac{2\phi+1}{(\phi-2)\,\phi\,(\phi+3)}.$$

(74) We wish to ascertain what values of σ will be incompatible with the violation of the limits just assigned to ϕ, and accordingly we must inquire what is the range of values assumed by σ when $\phi>1$ or $\phi<-3$; any values of σ *not* included within this range will be admissible for the purpose in view.

When $\phi<-3$, σ is always positive, and proceeds continuously from ∞ to 0 as ϕ passes from $-3-\epsilon$ (ϵ being infinitesimal) to $-\infty$. Consequently σ must not be allowed to have any positive value. When $\phi=\infty$, $\sigma=0$, and when $\phi=1$, $\sigma=-\frac{3}{4}$.

Hence, if no minimum value of σ (that is, no maximum value of $-\sigma$) occurs between $\phi=1$, $\phi=\infty$, σ may have any value between 0 and $-\frac{3}{4}$; but if such a minimum value, $-M$, where $M>\frac{3}{4}$, should exist, the admissible values of σ would become more enlarged, and might be taken between 0 and $-M$.

Making then $\delta\sigma=0$, we have

$$\frac{2}{2\phi+1}=\frac{3\phi^2+2\phi-6}{\phi^3+\phi^2-6\phi},$$

or

$$4\phi^3+5\phi^2+2\phi-6=0;$$

which, substituting $1+\psi$ for ϕ, becomes

$$4\psi^3+17\psi^2+24\psi+5=0;$$

so that there can be no real root of the equation in ϕ greater than unity.

Hence the admissible values of σ are defined by the inequalities

$$0>\sigma>-\tfrac{3}{4},$$

that is, $\quad 0>-\dfrac{1+\rho}{4}>-\frac{3}{4}, \quad$ or $0>-(1+\rho)>-3, \quad$ or $2>\rho>-1$.

(75) We have thus obtained the complete solution of the problem of assigning invariantive criteria, such that their signs (positive, negative, or zero) shall serve to fix the nature of the roots. These criteria we now see are

$$J,\ D,\ \Lambda+\mu JD,$$

where μ (the negative, it must be noticed, of ρ) is any numerical quantity intermediate between 1 and -2 [61].

(76) This important modification of the original criteria J, D, Λ I proceed to apply to the problem of obtaining the *simplest* and *most symmetrical* expression for the criteria in terms of the roots of the equation. Let a, b, c, d, e be the roots, and write

$$Z = \Sigma \{(a-b)^2 (a-c)^2 (b-c)^2 (a-d)^4 (a-e)^4 (b-d)^4 (b-e)^4 (c-d)^4 (c-e)^4\},$$

or say
$$Z = \Sigma \left\{\zeta(a, b, c) \begin{pmatrix} a & b & c \\ & d & e \end{pmatrix}\right\} \text{ [62]}.$$

Then, since each letter occurs the same number of times (12) in each term, Z will be an invariant.

(77) Again, suppose any two roots to become equal, say that e becomes d, then Z reduces to the single term $\zeta(a, b, c) \begin{pmatrix} a & b & c \\ & d & d \end{pmatrix}$; for any such factor as $\zeta(a, b, d)$ will be accompanied with the factor $\begin{pmatrix} a & b & d \\ & c & d \end{pmatrix}$ which vanishes.

If, further, we suppose any two of the letters a, b, c to become equal, then Z disappears entirely, since on that supposition $\zeta(a, b, c)$ vanishes. Hence Z is an invariant of the twelfth order, possessing the property of vanishing when the equation to which it belongs has two pairs of equal roots. Hence Z is of the form $p\Lambda + qJD$, and it becomes of importance to ascertain the value of the ratio $\frac{q}{p}$.

To do this let us suppose $e = 0$, $a = -b$, $c = -d$.

The ten terms in Z correspond to the following ten partitions:—

(1)	(2)	(3)	(4)
abc	*abd*	*acd*	*bcd*
de	*ce*	*be*	*ae*
	(5)	(6)	
	abe	*cde*	
	cd	*ab*	
(7)	(8)	(9)	(10)
ace	*bde*	*ade*	*bce*
bd	*ac*	*bc*	*ad*

[61] Strictly it has only been proved that the surface $\Lambda + \mu JD$, which passes through the line Λ, D, contains no superior facultative points except those comprised in the line $L=0$, $J=0$. It is, I think, not difficult to see from this, that, if in the "*sack*" formed between Λ and $\Lambda + \mu JD$ any such points were contained, $L=0$, $J=0$, that is the axis of D would be a double or multiple line on the surface G, which is easily disproved by examining the algebraical form of G in Art. 41, where K represents $\frac{-D+J^2}{128}$; any obscurity, however, which may be supposed to cling to this view is immaterial, as a demonstration capable of being followed *in plano* and leaving nothing to be desired in point of perspicuity, will be found in the Note appended to this Part.

[62] Agreeable to the meaning assigned to ζ and to a couple of rows of letters in my memoir on Syzygetic Relations in the *Philosophical Transactions*. [Vol. I. of this Reprint, p. 429.]

(78) The corresponding values of the terms will be

$$4a^2(a^2-c^2)^2\,16a^8c^8(a^2-c^2)^4;\ 4a^2(a^2-c^2)^2\,16a^8c^8(a^2-c^2)^4;$$
$$4c^2(a^2-c^2)^2\,16a^8c^8(a^2-c^2)^4;\ 4c^2(a^2-c^2)^2 16a^8c^8(a^2-c^2)^4;$$
$$4a^6c^8(a^2-c^2)^8;\ 4c^6a^8(a^2-c^2)^8;\ (a-c)^2\,256a^{10}c^{10}(a+c)^8;\ (a-c)^2\,256a^{10}c^{10}(a+c)^8;$$
$$(a+c)^2\,256a^{10}c^{10}(a-c)^8;\ (a+c)^2\,256a^{10}c^{10}(a-c)^8.$$

Collecting and simplifying these terms, and observing that

$$(a-c)^2(a+c)^8+(a+c)^2(a-c)^8=(a^2-c^2)^2[(a+c)^6+(a-c)^6]$$
$$=4(a^2-c^2)^2(a^2+c^2)(a^4+14a^2c^2+c^4),$$

we find $$Z=128(a^2+c^2)a^8c^8(a^2-c^2)^6+4(a^2+c^2)a^6c^6(a^2-c^2)^8$$
$$+1024(a^2+c^2)(a^4+14a^2c^2+c^4)(a^2-c^2)^2a^{10}c^{10}.$$

Let $(a^2-c^2)^2=p$, $a^2c^2=q$, and let $Z_1=\dfrac{Z}{(a^2+c^2)q^3}$. Then

$$Z_1=16384pq^3+1024p^2q^2+128p^3q+4p^4$$
$$=2^{14}pq^3+2^{10}p^2q^2+2^7p^3q+2^2p^4.$$

(79) We must now calculate J, D, L:

$$D=\frac{1}{5^5}\zeta(a,-a,c,-c,0)$$
$$=\frac{1}{5^5}4a^6c^6(a^2-c^2)^4;$$

or writing $$D_1=\frac{D}{q^3},$$
$$D_1=\frac{4}{5^5}p^2.$$

Again, for J. The form to which it belongs is

$$x^5-(a^2+c^2)x^3y^2+a^2c^2xy^4,$$

or $$\left(1,\ 0,\ -\frac{a^2+c^2}{10},\ 0,\ \frac{a^2c^2}{5},\ 0\between x,\ y\right)^5;$$

so that the coefficients of the biquadratic Emanant are

$$x;\quad -\frac{a^2+c^2}{10}y;\quad -\frac{a^2+c^2}{10}x;\quad \frac{a^2c^2}{5}y;\quad \frac{a^2c^2}{5}x.$$

Hence the quadratic covariant becomes

$$\frac{a^2c^2}{5}x^2+\frac{2}{25}(a^2+c^2)a^2c^2y^2+\frac{3}{100}(a^2+c^2)^2x^2$$
$$=\frac{20a^2c^2+3(a^2+c^2)^2}{100}x^2+\frac{2}{25}(a^2+c^2)(a^2c^2)y^2.$$

Hence, by definition, J (which $= -4 \times$ Discriminant of the Quadratic Covariant)

$$= -\frac{4}{1250}(a^2c^2)(a^2+c^2)[3(a^2-c^2)^2+32a^2c^2];$$

and making
$$J_1 = \frac{J}{(a^2+c^2)\,q},$$

$$J_1 = -\frac{6}{625}p - \frac{64}{625}q = -\frac{6}{5^4}p - \frac{2^8}{5^5}q.$$

Finally, to calculate L. The canonizant of the form,

$$\begin{vmatrix} 1 & 0 & A & 0 \\ 0 & A & 0 & B \\ A & 0 & B & 0 \\ y^3; & -xy^2; & x^2y; & -x^3 \end{vmatrix},$$

is
$$(A^3-AB)\,x^3 + (B^2-A^2B)\,xy^2,$$

of which the discriminant is

$$-4\,\frac{AB^3}{27}(A^2-B)^4,$$

where
$$A = -\frac{a^2+c^2}{10}, \quad B = \frac{a^2c^2}{5}.$$

Hence, by definition,

$$L = AB^3(A^2-B)^4 = -\frac{1}{125\,.\,10^9}(a^2+b^2)(a^6b^6)[(a^2-b^2)^2-16a^2b^2];$$

and making

$$L_1 = -\frac{L}{(a^2+c^2)\,q^3},$$

$$L_1 = \frac{1}{125\,.\,10^9}(p-16q)^4 = -\frac{1}{5^{12}\,.\,2^7}(p^2-16q)^4.$$

(80) Now let us write

$$\frac{1}{5^{12}}Z = \eta L + eJD^{(63)} + \epsilon J^3.$$

This gives
$$\frac{1}{5^{12}}Z_1 = eqJ_1D_1 + \epsilon(p+4q)\,J_1{}^3 + \eta L,$$

or
$$4p^4 + 128qp^3 + 1024q^2p^2 + 16384pq^3$$
$$= 125\,(256p^2q^2 + 24p^3q)\,e + (p+4q)(6p+64q)^3\epsilon + \frac{1}{2^7}(p-16q)^4\eta,$$

by means of which identity we can obtain linear equations for finding the values of e, ϵ, η.

(63) Since Z has been proved to be of the form $p\Lambda + qJD$, we know *à priori* the value of $\frac{\epsilon}{\eta}$; but I have thought it safer to determine ϵ, η independently, as an additional check upon the accuracy of the computations.

Thus, equating the coefficients of p^4, q^4, p^3q respectively, we obtain

$$4 = 216\epsilon + \frac{1}{2^7}\eta,$$

$$4 \cdot 64^3\epsilon + \frac{16^4}{2^7}\eta = 0,$$

which gives $\quad \eta = -2^{11}\epsilon$ (as it ought to do),

$$128 = (24 \times 125)e + (4 \times 216 + 108 \times 64)\epsilon + 64 \cdot 2^{11}\epsilon$$
$$= 3000e + 8800\epsilon.$$

Hence $\quad 200\epsilon = 4, \quad \epsilon = \frac{1}{50}, \quad \eta = -\frac{2^{10}}{25},$

$$3000\epsilon = 128 - 176 = -48, \quad e = -\frac{2}{125} \text{ and } \frac{e}{\epsilon} = -\frac{4}{5}.$$

In order to verify the value of e, let $p = -4$, $q = 1$; then, assuming the correctness of the above determinations, we ought to find

$$4^5 - 128 \cdot 4^3 + 1024 \cdot 16 + 16384$$
$$= 125(256 \cdot 16 - 24 \cdot 64) \cdot \frac{-2}{125} + \frac{1}{128} \cdot 160000 \cdot -2^{11} \cdot \frac{1}{50},$$

or $\quad 2^{10}(1 - 8 + 16 - 64) = (-32 \cdot 256 + 48 \cdot 64) - \frac{8}{25} \times 160000,$

or $\quad 2^{10}(-55) = -5120 - 25 \cdot 2048 = 2^{10}(-5 - 50),$

which is right.

(81) Thus
$$-Z = \frac{5^{10}}{2}\left(2^{11}L - J^3 + \frac{4}{5}JD\right)$$
$$= \frac{5^{10}}{2}\left(\Lambda + \frac{4}{5}JD\right);$$

and accordingly we have proved that $-Z$ is of the form $(\Lambda + \frac{4}{5}JD)$; and consequently, since $\frac{4}{5}$ lies within the allowed limits 1 and -2, $-Z$ may be used to replace Λ in the system of criteria.

(82) On examining the composition of Z, it will be found to have a remarkable relation to the lower criterion J.

J we know is, to a numerical factor *près*, of the form

$$\Sigma\{(d-e)^4\zeta(a, b, c)\},$$

ζ denoting, as usual, the squared product of the differences of the quantities which it affects; and Z, it will readily be seen, is of the form

$$\{\zeta(a, b, c, d, e)\}^2 \Sigma \frac{1}{\zeta(a, b, c)(d-e)^4};$$

and the squared factor is always positive whatever the roots may be, for ζ is always real.

Hence the essential part of our rule thus transformed comes to this, that if

$$\Sigma\{\zeta(a, b, c)\times(d-e)^4\} \text{ and } \Sigma\{(\zeta(a, b, c))^{-1}(d-e)^{-4}\}$$

are both of them positive, then when the discriminant is positive, so that the case of two of the five quantities a, b, c, d, e being conjugate and the other three real is excluded, and the choice lies between supposing all or only one of them real, we are able to affirm that they will all be real. Nothing could be easier than to multiply tests expressed by simple symmetric functions of differences of the roots, any infringement of which would contradict the hypothesis of all the five letters denoting real quantities; the difficulty consists in discovering a system of the least number that will suffice of decisive tests, such that not only their infringement shall contradict the hypothesis of imaginary roots, but whose fulfilment shall ensure the roots being all real. This is what has been proved to be effected by means of the invariants J, D, $\Lambda+\frac{4}{5}JD$.

In the case before us it is clear that when the roots are all real, each of the sums above written must be positive and greater than zero. That their being both positive and greater than zero is inconsistent with four of the letters a, b, c, d, e being imaginary would probably not admit of an easy direct demonstration.

Z we have seen is only a particular value of the general invariant $\Lambda+\mu JD$, which may be called M, where μ is an arbitrary constant limited to lie between 1 and -2.

(83) It may be well to notice the effect of using *as a criterion*, in conjunction with J and D, the value of M corresponding to either extreme value of μ. In such case, supposing M to become zero, it might for a moment appear doubtful to which region that point representing the family of forms is to be referred. But since the doubt can only arise when J is negative and D positive, and since by hypothesis we have $\Lambda=-\mu JD$, we see that Λ takes the sign of μ; and consequently the sign of M, when it becomes zero, is to be understood as following the sign of μ, that is, as positive when μ is 1 and negative when μ is -2.

(84) The method above given for ascertaining the nature of the roots of a quintic involves the use of only three criteria. It may be inquired how many would become needful in applying Sturm's method. In the case of a cubic equation only the last of the two Sturmian criteria comes into use; and it seems therefore desirable to ascertain whether all four of the Sturmian criteria applicable to that case are required, or whether a smaller number are sufficient. I speak of four criteria, inasmuch as the leading terms fx and $f'x$ cannot be considered as such, their signs being fixed; so that we are at liberty to consider them both positive. Suppose all six Sturmian functions to be written down, including fx (a function of x of the fifth degree) and $f'x$,

and let us characterize by the index (r, s) any succession of signs of the leading coefficients which contain r continuations and s variations, and which therefore will correspond to the case of $(r-s)$ roots.

The total number of cases to be considered are the sixteen following:

$$\begin{array}{lcccccc}
(5, 0) & + & + & + & + & + & + \\
\multirow{4}{*}{(4, 1)} \left\{ \right. & + & + & + & + & + & - \\
 & + & + & + & + & - & - \\
 & + & + & + & - & - & - \\
 & + & + & - & - & - & - \\
(3, 2) & + & + & + & + & - & + \\
 & + & + & + & - & + & + \\
 & + & + & + & - & - & + \\
 & + & + & - & + & + & + \\
 & + & + & - & - & + & + \\
 & + & + & - & - & - & + \\
(2, 3) & + & + & + & - & + & - \\
 & + & + & - & + & + & - \\
 & + & + & - & + & - & - \\
 & + & + & - & - & + & - \\
(1, 4) & + & + & - & + & - & +
\end{array}$$

the successions corresponding to the indices (2, 3), (1, 4) will become impossible, as corresponding to a *negative* number of real roots. An inspection of the eleven cases corresponding to the indices (5, 0), (4, 1), (3, 2) will show that no *ternary* combination of signs in the third, fourth, and sixth columns belongs to any of the three characters (5, 0), (4, 1), (3, 2) exclusively, and consequently all four signs must be used; and therefore, if the method of Sturm is employed, four criteria are indispensable for determining effectually the character of the roots in an equation of the fifth degree [64]; whereas in the symmetrical and invariantive method which I have employed three have been seen to suffice.

(64) (*a*) For an equation of the nth degree there are $n-1$ variable criteria, each capable of being $+$ or $-$, and thus giving rise to 2^{n-1} conceivable diversities of combination. The actual number possible, however, is considerably less than this; and I find by an easy method that this number, when n is odd, is $2^{n-2}+\dfrac{\Pi(n-1)}{2\left(\Pi\frac{n-1}{2}\right)^2}$, and when n is even, is $2^{n-2}+\dfrac{\Pi(n-1)}{\Pi\frac{n}{2}\,\Pi\left(\frac{n}{2}-1\right)}$.

(*b*) Not quite foreign to this subject is the inquiry as to the comparative probability of each different succession or each different family of successions possessing equivalent characters; and, as connected therewith, the comparative probability of a certain specified number of the roots of an equation of a given degree being real and the remainder imaginary. In the simplest case of a quadratic equation of which the coefficients are real but otherwise arbitrary, I find that upon the particular hypothesis of the squares of the three coefficients being limited by one and the same quantity, the probability of the roots being imaginary is $\frac{31}{72}-\frac{\log 2}{12}$, or ·3727932, a little

In an equation of the seventh degree the case of 0 or 4 will be distinguishable from that of 2 or 6 imaginary roots by the sign of the discriminant, and then again the case of 0 from that of 4, and of 2 from that of 6, by other invariantive criterion-systems. So for an equation of the ninth degree, the first separation will be that of the 0, 4, or 8 case from that of 2 or 6; then it may be conjectured the 2 case will be invariantively separated from the 6, and the 0 or 8 from that of 4, and, finally, 0 and 8 from each other—the reduction of cases apparently depending upon the relation of the index of the equation to the powers of the number 2. This much we know (from Art. 49) as matter of certainty, that no single criterion other than the discriminant can ever serve to distinguish one form of roots from another so that all other criteria must accompany each other in groups; and accordingly the scheme of criteria established in the foregoing investigation is in kind the very simplest *à priori* conceivable.

Note on the arbitrary constant which appears in one of the criteria for distinguishing the case of four from that of no imaginary roots, and on the curve whose coordinates express the limiting relations of all the octodecimal invariants of a binary quintic, &c.

(85) The appearance of an arbitrary constant in a criterion is a circumstance so unexampled and remarkable that I have thought it desirable to give a more complete, or at least a more palpable proof of the validity of the substitution of $\Lambda + \mu JD$ for Λ than that furnished in the foregoing text, where some indistinctness arises from the difficulty of raising up in the mind a clear conception of the form of the amphigenous surface, and the two portions of space which it separates. That difficulty is entirely obviated, and

less than $\frac{5}{8}$, this being the value of the integral $\int_0^{\frac{1}{2}} db \int_{b^2}^{1} da \left(1 - \frac{b^2}{a}\right)$; but we are not at liberty to infer from this the value of the probability in question when the coefficients are left absolutely unlimited. A case in point, as illustrating the effect of imposing a limit in questions of this kind, occurs in the problem (which I raised in my lectures on Partitions) of finding the probability that four points placed at hazard in a plane will form the angles of a reentrant quadrilateral, which Professor Cayley has shown is exactly $\frac{1}{4}$ in the absence of any limit. For if $ABCD$ be the four points, and ABC the greatest of the four triangles of which they may be regarded as the angular points, and if through A, B, C be drawn lines parallel to BC, CA, AB respectively, the triangle $\alpha\beta\gamma$ so formed will be four times as great as ABC, and the point D must be somewhere within $\alpha\beta\gamma$, otherwise ABC would not be less than each of the three other triangles ABD, BCD, CAD; and consequently, since D must lie within ABC when the quadrilateral is reentrant, the probability in question is $\frac{ABC}{\alpha\beta\gamma}$, or $\frac{1}{4}$. Now it is easy to see, by using the very same construction, that if any contour whatever be imposed as a limit upon the positions of the four points, the probability referred to will exceed $\frac{1}{4}$ by a finite quantity—a result somewhat paradoxical, since *à priori* one would have supposed that the value of it for the case of *no limit* would be the *mean* of the values corresponding to the respective suppositions of every possible form of limit.

the theory rendered palpable to the senses by the following investigation, where the problem is so handled as to involve the contemplation of two dimensions only of space. We have in general

$$D = J^2 - 128K, \quad \Lambda = 2048L - J^3,$$

and at the amphigenous surface (see Art. 57)

$$\frac{K}{J^2} = \frac{\theta + 6}{(\theta + 4)\theta^2}, \quad \frac{L}{J^3} = \frac{1}{(\theta + 4)\theta^3}.$$

Let $$\theta = 4\phi, \quad y = \frac{D}{J^2}, \quad x = \frac{\Lambda}{J^3}.$$

Then

$$y = 1 - 128\frac{\theta + 6}{(\theta + 4)\theta^2} = 1 - \frac{8\phi + 12}{\phi^2(\phi + 1)} = \frac{(\phi + 2)^2(\phi - 3)}{\phi^2(\phi + 1)},$$

$$x = -1 + \frac{2048}{(\theta + 4)\theta^3} = -1 + \frac{8}{\phi^4 + \phi^3} = \frac{-(\phi + 2)(\phi^3 - \phi^2 + 2\phi - 4)}{\phi^3(\phi + 1)};$$

and consequently

$$\delta y = \frac{4(\phi + 2)(4\phi + 3)}{\phi^3(\phi + 1)^2}\delta\phi, \quad \delta x = -\frac{8(4\phi + 3)}{\phi^4(\phi + 1)^2}\delta\phi, \quad \frac{\delta y}{\delta x} = -\frac{\phi^2 + 2\phi}{2}:$$

x, y may be considered as the coordinates (inclined to each other at any angle) of a curve of the fourth order, whose form, so far as is essential to the object in view, I proceed to determine. It is obvious, furthermore, that this curve will be a section of the amphigenous surface made by the plane $J = 1$.

(86) This curve will be seen to consist of four branches, coming together in pairs or two cusps, so as to form two distinct horns[65]. For when $\phi = \infty$, or $\phi = -\frac{3}{4}$, δy, δx will each of them be zero. Hence there is a cusp at the point where $x = -1$, $y = 1$[66], and again at the point where

$$x = -1 + \frac{8 \times 256}{81 - 108} = -76\tfrac{23}{27}, \quad y = \frac{(\frac{5}{4})^2(-\frac{9}{4})}{(\frac{3}{4})^2\frac{1}{4}} = -25.$$

(87) When $\phi = 0$, and also when $\phi = -1$, x and y each become infinite; when $\phi = \pm\infty$, x and y each become unity.

[65] (*a*) Since $$\phi^4 + \phi^3 - \frac{J^3}{256L} = 0,$$
we see at once, from Descartes's rule, that ϕ can never have more than two real values to one of $\frac{L}{J^3}$, or consequently of x, and consequently there can only be two values of y to each of x.

(*b*) When $J = 0$, the cusp of the left-hand horn and the two points of intersection of the dexter horn with the axis of L coincide at the origin; the upper branch of the latter and the lower of the former become the lower and upper parts of the axis of D, whilst the lower and upper branches of the same respectively become the left and right-hand branches of the semi-cubical parabola $27 . 2^{22}L^2 = -D^3$.

[66] Where this branch cuts the axis of y we have $\phi^3 - \phi^2 + 2\phi - 4 = 0$, of which the real root will be a trifle less than $\frac{3}{2}$.

As ϕ passes from $+\infty$ to 0, δy is always negative, and x always positive; so that there will be one branch of the curve (CMP in Figure [p. 479]) extending from $x=-1$ to $x=+\infty$, for which y commences at $y=1$, which cuts the axis of x when $\phi=3$, that is $x=-\frac{25}{27}$ (67), and which, for the remaining part of its course, lies completely under the axis of x, becoming infinite when x becomes indefinitely great.

Again, as ϕ passes from $-\infty$ to -1, δx remains always positive, but δy is negative so long as $\phi<-2$ vanishes when $\phi=2$, and ever afterwards continues positive. Thus there is a second branch, COQ, which starts from the cusp C, touches the axis of x at the origin, ever afterwards remaining positive, and increasing up to positive infinity.

Since when $\phi=\infty$, $\dfrac{\delta y}{\delta x}=\infty$, the tangent at C is parallel to the axis of y, and consequently the two branches which start from C lie on the same side of the tangent, so that the cusp at this point is of the second or ramphoidal kind; in Professor Cayley's nomenclature a cusp-node, and equivalent to the union of a double point and a cusp of the first kind.

There remains to account for the values of ϕ in the interval between 0 and -1. Throughout this interval y and x remain both of them negative, and $\dfrac{\delta y}{\delta x}=-\dfrac{\phi(\phi+2)}{2}$ (68, 69) is always positive.

(67) From this it is easily seen that, whatever may be supposed to be the inclination of the axes x, y, the curve in question is rectifiable by means of elliptic functions; for $\dfrac{ds}{d\phi}$ will be expressible as a rational function of ϕ and the square root of a quartic function of ϕ. The same conclusion will hold for the curve obtained by making J constant when J, together with any invariant of the eighth and any of the twelfth order, are taken as the coordinates of the amphigenous surface.

(68) To ascertain which range of ϕ gives the superior and which the inferior outline of the sinister horn, let $\phi=\epsilon$, an infinitesimal; then $\phi^4+\phi^3=\epsilon^3$, and the other value of ϕ is $-1-\eta$, where $\eta=\epsilon^3$. Hence the two values of y corresponding to ϕ nearly zero and ϕ nearly -1 respectively will be

$$y_1=-\frac{12\epsilon}{\epsilon^3}=-\frac{12}{\epsilon^2}\text{ and }y_2=\frac{-4(-1-\eta)}{\epsilon^3}=\frac{4}{\epsilon^3}.$$

Thus y_1 is negative for ϵ positive or negative, but y_2 is positive in the one case and negative in the other, as already seen for the dexter horn. We see also that y_2 becomes indefinitely greater than y_1, so that it is the value of ϕ near to -1 which gives the inferior branch; and consequently the superior branch of the sinister horn belongs to the range from $-\frac{3}{4}$ to 0, and the inferior to the range from $-\frac{3}{4}$ to -1.

(69) It may further be noticed that each horn so called is a true horn, being destitute of any point of contrary flexure, except at infinity; for otherwise we should have

$$\frac{d^2y}{dx^2}=\frac{d\phi}{dx}\cdot\frac{d\dfrac{dy}{dx}}{d\phi}=-\frac{d\phi}{dx}(\phi+1)=\frac{(\phi+1)^3\phi^4}{8(4\phi+3)}=0,$$

which implies $\phi=0$ or $\phi=-1$, for each of which values of ϕ, x and y become infinite. It will be seen hereafter that it is only for the value corresponding to $\phi=0$ that there does exist at infinity a point of inflexion.

There will thus be two branches, in each of which x and y increase simultaneously in the negative direction, coming to a cusp necessarily of the first kind at the point $x = -76\frac{23}{27}$, $y = -25$, one branch corresponding to the values of ϕ from $-\frac{3}{4}$ to 0, the other to the values of ϕ from $-\frac{3}{4}$ to -1, both of them lying completely under the axis of x, and becoming respectively infinite at the extreme values of ϕ (0 and -1).

Again,

$$2y - x + 5 = \frac{\phi + 2}{\phi^4 + \phi^3}\{(2\phi^3 - 2\phi^2 + 2\phi) + (\phi^3 - \phi^2 + 2\phi - 4)\} + 5$$

$$= \frac{\phi + 2}{\phi^3}(3\phi^2 - 6\phi - 4) + 5 = \frac{8\phi^3 - 16\phi - 8}{\phi^3}.$$

Hence when $\phi = -1$, for which value of ϕ x and y both become infinite, $2y - x + 5 = 0$; hence the straight line $2y - x + 5 = 0$, represented by AN in the diagram, will be an asymptote to the curve (70).

If now we draw the straight line $2y - x = 0$, represented by OB in the figure and join OC, the curvilinear triangle OCM will be completely under OC, and the curvilinear infinite sector XOP completely under OB.

(88) What we have to prove is, that so long as μ lies between -2 and 1, so long may $\Lambda + \mu JD$ be substituted as a criterion in lieu of Λ, it being remembered that Λ only plays the part of a criterion when D is positive and J is not positive. Hence, since when $J = 0$, $\Lambda + \mu JD$ and Λ coincide, we have only to show that, so long as D is positive and J is negative, $\Lambda + \mu JD$ and Λ will bear the same sign for all such values of J, D, L as constitute a facultative system, that is coordinates to a facultative point in space.

Now at any facultative point G (the function of the amphigenous surface), or say rather $G(J, K, L) > 0$, or $\frac{1}{J^3}G\left(1, \frac{D}{J^2}, \frac{L}{J^3}\right) > 0$, and consequently considering $\frac{D}{J^2}, \frac{L}{J^3}$ as the coordinates of a plane curve, the line $G\left(1, \frac{D}{J^2}, \frac{L}{J^3}\right) = 0$ (the sign of J being fixed) will separate those points for which J, K, L constitute a facultative system from those in which

(70) The two points where the asymptote cuts the curve will be found by writing

$$\frac{\phi^3 - 2\phi - 1}{\phi + 1} = \phi^2 - \phi - 1 = 0,$$

which gives

$$\phi = \frac{1 \pm \sqrt{5}}{2}.$$

The superior sign corresponds to a point x, y in the inferior branch of the dexter horn, and the lower sign, for which $\phi > -\frac{3}{4}$, to the superior branch of the sinister horn. It is easy to see that there can be no other asymptote; for x, y only become infinite when $\phi = -1$, or $\phi = 0$; so that if $\lambda x + \mu y + \nu$ is an asymptote, it must contain $(\phi + 1)^2$, or ϕ^2 as a factor. The first condition is only satisfied when $\lambda : \mu : \nu :: -1 : 2 : 5$; and the latter cannot be satisfied at all.

J, K, L constitute a non-facultative one. But the curve above traced is obviously a homographic derivative of that line

$$\left(\text{for } G \text{ is the resultant of } \frac{K}{J^2} = \frac{\theta + 6}{(\theta + 4)\,\theta^2},\ \frac{L}{J^3} = \frac{I}{(\theta + 4)\,\theta^3}\right).$$

Hence this latter curve will also separate systems of values of J, D, Λ corresponding to facultative from those corresponding to non-facultative points. Moreover when J is negative and D positive, it has been shown (see dial figure) that the values of D (in facultative systems) corresponding to finite values of J are *limited* in magnitude; hence, upon the same suppositions, facultative systems of J, D, Λ will correspond to the interior and contour of the curve we have been considering.

(89) Accordingly, since D is supposed positive, our sole concern will be with the curvilinear triangle CMO and the infinite sector QOX, and we have to show that for all points not exterior to those areas Λ and $\Lambda + \mu JD$ have the same sign; that is to say, $1 + \mu \dfrac{JD}{\Lambda}$, or $1 + \mu \dfrac{y}{x}$ is *positive*.

When y and x have opposite signs (as is the case in the triangle CMO), all negative values of μ, and when y and x have the same signs (as is the case in the sector XOQ), all positive values of μ obviously make $1 + \mu \dfrac{y}{x}$ positive. But furthermore $\dfrac{y}{x}$, which is -1 for the line OC, is greater than -1 for all points in the triangle just named; and again, $\dfrac{y}{x}$, which is $\frac{1}{2}$ for OB (the parallel to the asymptote through O), will be less than $\frac{1}{2}$ for all points in the sector QOX. Thus, then, as regards points either in the triangle or in the sector, $\dfrac{y}{x}$ is always intermediate between -1 and $\frac{1}{2}$; so that when μ lies between 1 and -2, $1 + \mu \dfrac{y}{x}$ will be always positive, and Λ and $\Lambda + \mu JD$ will bear the same sign, so that $\Lambda + \mu JD$ may be used to replace Λ as a criterion. Q.E.D.

(90) It is apparent from the nature of the preceding demonstration that Λ may be replaced by an invariant containing not one merely, but an infinite number of arbitrary constants (limited), provided we are indifferent to the degree which the substitute for Λ may assume. To this end we have only to draw any algebraical curve $F(x, y) = 0$ passing through the origin, and with its parameter subject to such conditions of inequality as will ensure the mixtilinear triangle and sector COM, XOQ lying on opposite sides of the curve. If its degree be n, the number of parameters in F left arbitrary within limits will be $\dfrac{n^2 + 3n - 2}{2}$, and $\epsilon F(\Lambda, JD)$, where ϵ means one of the

two quantities $+1$ or -1, may be used as a criterion in lieu of Λ. For instance, a common parabola with its axis coincident with that of x and passing through O will obviously serve as a screen between these figures; its equation will be $y^2 - x = 0$, and the invariant $D^2 - J\Lambda$, which is of the sixteenth degree in the coefficients, will serve together with J and D to fix the nature of the roots; so in general we may obtain invariants of any degree of the form $4i$ from twelve upwards. Thus M. Hermite, by a method not introducing the notion of continuity, has found one of the twenty-fourth degree, which he has been so obliging as to communicate to me, namely $(D_1 - 5\Delta J_2)^2 + (9D - 25\Delta^2) J_2^2$, where $D_1 = 16J_3 + 25\Delta J_2$; and D is his discriminant, which I cannot safely attempt to express in terms of x, y for want of a certain knowledge of the arithmetical relations between his Δ, J_2, J_3, D, and my own J, K, L; but were this transformation effected, the curve so represented must, *ex necessitate*, pass clear between the triangle and sector above referred to, or else the invariant in question could not be a criterion. I have ascertained without difficulty that it passes through the origin and represents one of the principal species of Newton's diverging parabolas.

(91) The curve which we have been discussing will, on reference to Plücker's *Algebraischen Curven*, p. 193, be seen to belong to his sixteenth species of curves of the fourth order having two double points; but as in reality one of these is tantamount to the union of two, it may be considered as having three, the maximum possible number of such points, and consequently comes under the operation of Clebsch's rule, given in the last Number of Crelle's *Journal*, and accordingly its coordinates have been seen to be rational functions of a single variable. The equation connecting x, y may of course be obtained by means of a simple and obvious substitution operated upon the G of Art. 41, or it may be found directly by writing

$$\frac{x+1}{8} = \xi = \frac{1}{\phi^4 + \phi^3}, \quad \frac{y-1}{4} = \eta = -\frac{2\phi + 3}{\phi^3 + \phi^2},$$

whence we obtain

$$\phi^4 + \phi^3 - \frac{1}{\xi} = 0, \tag{1}$$

$$2\phi^2 + 3\phi + \frac{\eta}{\xi} = 0. \tag{2}$$

Calling ϕ_1, ϕ_2 the two roots of equation (2), making

$$\left(\phi_1^4 + \phi_1^3 - \frac{1}{\xi}\right)\left(\phi_2^4 + \phi_2^3 - \frac{1}{\xi}\right) = 0,$$

and substituting the values of the symmetric functions of ϕ_1, ϕ_2 found from the same equation, we obtain without difficulty

$$\eta^4 - \xi\eta^3 - 8\xi\eta^2 + 36\xi^2\eta + 16\xi^2 - 27\xi^3 = 0$$

for the equation in question. The curve thus denoted I propose to call the Bicorn. Its figure is given below [p. 479], in which ξ, η are taken at right angles, but they may of course be supposed to be inclined at any angle whatever. If we now assume at pleasure any two new axes U, V in the plane of the Bicorn, the coordinates u, v will be always respectively proportional to two invariants of the twelfth order of the given quintic, whose particular forms will depend upon the positions of the two new axes so taken. If one of these axes, say that of u, be made coincident with the axis of ξ, v will be proportional to JD, and u to some other invariant of the twelfth degree. When this is the case, then in general v, as u travels from one end of infinity to the other, will sometimes have four, and sometimes two, or else sometimes two and sometimes no real values, as will be obvious by inspection of the figure. There is, however, one direction of the axis of v which will cause v in all cases to have two, and only two real values. This direction is that of the line joining the two cusps. At the node-cusp, for which $\phi = \infty$, $\xi = 0$, $\eta = 0$; at the other cusp, for which $\phi = -\frac{3}{4}$, $\xi = -\frac{256}{27}$, $\eta = -\frac{32}{3}$. Hence the equation of the joining line is $9\xi - 8\eta = 0$. Now

$$\frac{K}{J^2} = -\frac{\eta}{32}, \quad \frac{L}{J^3} = \frac{\xi}{256}.$$

Hence along this line $9L + JK = 0$; and consequently, if the axis of v be taken parallel to this line and passing through the origin, whilst u is proportional to $9L + JK$, v will be proportional to JD; and thus we see that for every value of $9L + JK$, which is Hermite's J_3 (see footnote [36] (e)) [p. 431], D at the amphigenous surface (that is when $G = 0$, and therefore when Hermite's $I = 0$) will always have two, and only two real values. This perfectly agrees with M. Hermite's conclusion[71], and in an unexpected manner confirms the correctness of the concordance established, in the footnote cited, between his J_3 and my J, K, L. Had M. Hermite employed any duodecimal invariant whatever other than J_3, a mere inspection of the Bicorn shows that a similar conclusion could not have obtained.

(92) The intersections of the curve whose equation is written in the preceding article with infinity evidently lie in the lines $\eta^3 = 0$, $\eta - \xi = 0$. This latter is the equation to a line parallel to the asymptote which touches the highest and lowest of the four branches of the curve, and corresponds to the value -1 of ϕ. Thus we see that there is a point of inflexion corresponding to the point at infinity at which the second and third branches of the Bicorn may be conceived to unite. It is easy to show that the Bicorn has no double tangent; for we have seen that

$$\frac{dy}{dx} = -\frac{\phi^2 + 2\phi}{2},$$

(71) Lemma 3, p. 202, *Cambridge and Dublin Journal*, vol. IX.

and consequently the values of ϕ corresponding to the two supposed points of contact may be regarded as the two roots ϕ_1, ϕ_2 of the equation $\phi^2+2\phi+2\lambda=0$, and we shall have

$$-\frac{2\phi_1+3}{\phi_1^3+\phi_1^2}+\frac{2\phi_2+3}{\phi_2^3+\phi_2^2}=\lambda\left(\frac{2}{\phi_1^4+\phi_1^3}-\frac{2}{\phi_2^4+\phi_2^3}\right),$$

that is

$$-(2\phi_1+3)(\phi_2^3+\phi_2^2)+(2\phi_2+3)(\phi_1^3+\phi_1^2)=(\phi_2^4+\phi_2^3)-(\phi_1^4+\phi_1^3),$$

or $$4\lambda.(-2)+4\lambda+3(4-2\lambda)+6\{-2(4-4\lambda)+(4-2\lambda)\}=0,$$

or $$(-8+4-6+8-2)\lambda+12-6-8+4=0,$$

that is $$-4\lambda+2=0,\quad \lambda=\tfrac{1}{2},\quad \phi^2+2\phi+1=0,$$

and the two values of ϕ coincide, contrary to hypothesis.

It is also easy to find its class; for when $\frac{d\eta}{d\xi}$ corresponds to any point in which the curve is met by a tangent drawn from the point whose ξ, η coordinates are a, b, we have

$$\left(\frac{2\phi+3}{\phi^3+\phi^2}+b\right)+\frac{d\eta}{d\xi}\left(\frac{1}{\phi^4+\phi^3}-a\right)=0;$$

but $$\frac{d\eta}{d\xi}=2\frac{dy}{dx}=-(\phi^2+2\phi);$$

hence $$\frac{(2\phi+3)-(\phi+2)}{\phi^3+\phi^2}+(\phi^2+2\phi)a+b=0;$$

hence $$a\phi^4+2a\phi^3+b\phi^2+1=0;$$

and ϕ having four values, four tangents (real or imaginary) can be drawn to the Bicorn from every point in its plane. It is thus of the fourth order, fourth class, possesses a common cusp and a cusp-node, no double tangent, and one point of inflexion at infinity. These results accord with those given by Plücker (*Algebraischen Curven*, p. 222).

(93) The canonical form of the equation to the Bicorn is

$$(pr+q^2)^2+pq^3=0,$$

as seen in Plücker, p. 193, where $p=0$, $r=0$, $q=0$ will obviously be the equations to the tangent at the node-cusp, to the tangent at the common cusp, and to the line of junction of the two cusps. This leads to a remarkable transformation of the invariant G of Art. 41. Thus we may write $p=\xi$, $q=\mu(9\xi-8\eta)$; and to find r, we must draw the tangent to the lower cusp, for which $\phi=-\frac{3}{4}$, which gives

$$\xi=-\frac{256}{27},\quad \eta=-\frac{32}{3},\quad \frac{d\eta}{d\xi}=-\frac{15}{16}\,{}^{(72)};$$

(72) I find, by a calculation which offers no difficulty, that the value of ϕ at the point where this tangent cuts the curve will be given by the equation

$$-256\phi^4-256\phi^3+288\phi^2+432\phi+135=0;$$

and taking away the factor $(4\phi+3)^3$ which belongs to the cusp, there remains $\phi=\frac{5}{4}$, which corresponds to a point in the lower branch of the superior horn.

consequently we may write $r=\lambda(144\eta-135\xi+256)$, and then proceed to satisfy, by assigning suitable values to λ, μ, ν, the identity

$$\{\lambda(144\eta\xi-135\xi^2+256\xi)+\mu^2(8\eta-9\xi)^2\}^2+\mu^3\xi(8\eta-9\xi)^3$$
$$=\nu(\eta^4-\eta^3\xi-8\eta^2\xi+36\eta\xi^2+16\xi^2-27\xi^3)=\nu\,.\,2^{20}G.$$

On performing the necessary calculations it will be found that

$$\lambda=-\frac{1}{2^{12}},\quad \mu=\frac{1}{2^6},\quad \nu=\frac{1}{2^{12}}.$$

Hence we see that J^3G may be expressed under the form $(LL_1+cJ_3^2)^2+eLJ_3^3$, where L_1 is a new duodecimal invariant, and c, e are two known numbers; in fact

$$J^3G=\{L(18JK+135L^2-J^3L)+(JK+9L)^2\}^2+64L(JK+9L)^3.$$

I am indebted to my friend Dr Hirst for these references to the immortal work of Plücker.

(94) The existence has been demonstrated of a linear asymptote which is a tangent at infinity to the first and fourth branch. A cubic asymptote touches the intermediate branches in the point at infinity corresponding to $\phi=0$. For we have

$$\xi=\frac{1}{\phi^3(1+\phi)}=\frac{1}{\phi^2}(1-\phi+\phi^2-\phi^3\ldots);$$

and writing v for $-\eta$,

$$v=\frac{3+2\phi}{\phi^2(1+\phi)}=\frac{1}{\phi^2}(3-\phi+\phi^2-\phi^3\ldots),$$

$$v^{\frac{1}{2}}=\frac{3^{\frac{1}{2}}}{\phi^3}\left(\phi^2-\frac{1}{6}\phi^3+\ldots\right),\quad v^{\frac{3}{2}}=\frac{3^{\frac{1}{2}}}{\phi^3}\left(3-\frac{3}{2}\phi+\frac{13}{8}\phi^2-\frac{27}{16}\phi^3\ldots\right).$$

Hence we may determine A, B, C, D so that

$$Av^{\frac{3}{2}}+Bv+Cv^{\frac{1}{2}}+D-\xi \text{ shall }=\lambda\omega^n+\mu\omega^x+\ldots,$$

and I find $$A=\frac{1}{3^{\frac{1}{2}}},\quad B=-\frac{1}{6},\quad C=\frac{7}{72},\quad D=-\frac{2}{9}.$$

Thus the cubic asymptote will have for its equation

$$\left(\xi+\frac{1}{6}v+\frac{2}{9}\right)^2=3v\left(\frac{v}{9}+\frac{7}{72}\right)^2,$$

which is a divergent cubic parabola with a conjugate point, namely the point for which

$$v=-\frac{7}{8},\quad \xi+\frac{1}{6}v+\frac{2}{9}=0,\quad \text{or } \eta=\frac{7}{8},\quad \xi=-\frac{9}{128}.$$

(95) It is obvious from the preceding article, that we may expand ξ in terms of v by the descending series

$$\xi = Av^{\frac{3}{2}} + Bv + Cv^{\frac{1}{2}} + D + \frac{E}{v^{\frac{1}{2}}} + \dots.$$

But we may also obtain an ascending series for ξ in terms of v which will exhibit the nature of the curve of the cusp-node at which point $\phi = \infty$. Let $\phi = \frac{1}{\omega}$, then

$$\xi = \frac{1}{\phi^3(\phi+1)} = \frac{\omega^4}{1+\omega} = \omega^4(1 - \omega + \omega^2 - \omega^3 \dots \qquad),$$

$$v = \frac{2\phi+3}{\phi^2(\phi+1)} = \omega^2\left(\frac{2+3\omega}{1+\omega}\right) = \omega^2(2 + \omega - \omega^2 + \omega^3 \dots).$$

Hence

$$\begin{aligned}
v^2 &= \omega^4(4 + 4\omega - 3\omega^2 + 2\omega^3 \dots \qquad),\\
v^{\frac{5}{2}} &= \omega^4(\quad 4\sqrt{(2)}\,\omega + 5\sqrt{(2)}\,\omega^2 - \tfrac{25}{8}\sqrt{(2)}\,\omega^3 \dots \qquad),\\
v^3 &= \omega^4(\quad 8\omega^2 + 12\omega^3 \dots \qquad),\\
v^{\frac{7}{2}} &= \omega^4(\quad \sqrt{(2)}\,\omega^3 \dots \qquad),\\
\&c. &= \&c.,
\end{aligned}$$

from which we may easily deduce

$$\xi = 2\left(\frac{v}{2}\right)^2 - \left(\frac{v}{2}\right)^{\frac{5}{2}} + \frac{7}{4}\left(\frac{v}{2}\right)^3 - \frac{109}{32}\left(\frac{v}{2}\right)^{\frac{7}{2}}, \ \&c.,$$

in which it will be observed that the indices of the powers of v are precisely complementary to those in the preceding expansion, the two series of indices together comprising all multiples of $\frac{1}{2}$ from positive to negative infinity.

(96) We now see how, supposing the curve to be given with ξ and η at any angle, the axes corresponding to $\frac{K}{J^2}$, $\frac{L}{J^3}$ may be defined: namely, the origin of coordinates will be at the cusp-node; η, along which $\frac{K}{J^2}$ is reckoned, will be in the direction of the tangent at that point; and ξ, along which $\frac{L}{J^3}$ is reckoned, will be the axis of that common parabola which at the same point has the closest contact with the given curve.

It seems desirable, with a view to a more complete comprehension of the form of the amphigenous surface, that is the *limiting surface* of invariantive parameters, to ascertain the nature of the systems of plane sections of it, parallel to each of the three coordinate planes. The sections parallel to J, which are curves of the fourth order, have been already satisfactorily elucidated. It remains to consider briefly the sections parallel to J and D, which will be curves of the ninth order.

(97) When L is constant, writing $J=z$, $D=y$, where for facility of reference we may conceive y horizontal and z vertical, and making $L=\frac{k^3}{256}$, we have

$$z^3=k^3\phi^3(\phi+1),\quad y=z^2\frac{(\phi+2)^2(\phi-3)}{\phi^2(1+\phi)}=k^2\frac{(\phi-3)(\phi+2)^2}{(1+\phi)^{\frac{1}{3}}},$$

$$\frac{\delta y}{y}=\frac{2}{3}\frac{(\phi-1)(4\phi+3)}{(\phi+2)(\phi-3)(\phi+1)}\delta\phi,\quad \frac{\delta z}{z}=\frac{1}{3}\frac{4\phi+3}{\phi(\phi+1)}\delta\phi,$$

$$\frac{\delta z}{\delta y}=\frac{1}{2k}\frac{(\phi+1)^{\frac{2}{3}}}{(\phi-1)(\phi+2)}\delta\phi,$$

when	$\phi=-1$,	$z=0$,	$y=\infty$,
„	$\phi=-\frac{3}{4}$,	$\delta y=0$,	$\delta z=0$,
„	$\phi=0$,	$z=0$,	$y=-12k^2$,
„	$\phi=1$,	$\frac{\delta y}{\delta z}=0$,	
„	$\phi=+\infty$,	$z=+\infty$,	$y=+\infty$,
„	$\phi=-2$,	$y=0$,	$\frac{\delta z}{\delta y}=\infty$,
„	$\phi=-\infty$,	$z=+\infty$,	$y=+\infty$.

Hence it appears that the curve consists of three branches, two coming together at an ordinary cusp at the point corresponding to $\phi=-\frac{3}{4}$, and the third completely separate. The nature of the sign of k does not affect the nature of the curve. If, for greater clearness, k be supposed positive, the first branch, having the negative part of the axis of y for its asymptote, lies entirely in the $-y$, $-z$ quadrant, and is always convex to the axis of y; the second branch, joining the first at a cusp corresponding to $\phi=-\frac{3}{4}$, is concave to the origin, cuts the axis of y negatively and of z positively, and goes off to infinity; the third branch, having the positive part of the axis of y for its asymptote, lies in the $+y$, $+z$ quadrant, is always convex to the axis of z, which it touches at a point below that where it is cut by the second branch, and also goes off to infinity, lying entirely under the second branch. A straight line, according to the direction in which it is drawn, may cut the curve in one, three, or five real points.

(98) When D is constant, writing $J=z$, $L=x$, we have

$$z^2=D\frac{\phi^2(\phi+1)}{(\phi+2)^2(\phi-3)},\quad x=\frac{Dz}{(\phi-3)\phi(\phi+2)^2}.$$

The form of the curve changes with the sign of D. For sections parallel to and above the plane of D, we may make

$$D=c^2,\quad \tau^2=\frac{\phi+1}{\phi-3},\quad \text{or}\quad \phi=\frac{3\tau^2+1}{\tau^2-1};$$

then the complete equation-system to the curve will be

$$z = c\tau \frac{3\tau^2+1}{5\tau^2-1}, \quad x = c^3\tau \frac{(\tau^2-1)^4}{4(5\tau^2-1)^3},$$

it being unnecessary to affect c with a double sign, since z and x change their signs with that of τ.

Also

$$\frac{\delta x}{x} = \frac{(\tau^2+1)(15\tau^2+1)\,\delta\tau}{\tau(\tau^2-1)(5\tau^2-1)}, \qquad \frac{\delta z}{z} = \frac{(\tau^2-1)(15\tau^2+1)\,\delta\tau}{\tau(3\tau^2+1)(5\tau^2-1)},$$

$$\delta x = \frac{c^3}{4}\frac{(\tau^2+1)(15\tau^2+1)(\tau^2-1)^3}{(5\tau^2-1)^4}\delta\tau, \quad \delta z = c\frac{(15\tau^2+1)(\tau^2-1)}{(5\tau^2-1)^2}\delta\tau,$$

$$\frac{\delta x}{\delta z} = \frac{c^2}{4}\frac{(\tau^2+1)(\tau^2-1)^2}{(5\tau^2-1)^2}.$$

To the values of τ included between $+\sqrt{(\frac{1}{5})}$ and $-\sqrt{(\frac{1}{5})}$ will correspond one branch of the curve passing through the origin, where it has a point of contrary flexure, and extending to infinity in both directions.

When $(5\tau^2-1)$ is positive $\frac{x}{\tau}$ is always positive; and when $\tau^2=1$,

$$\delta x = 0, \quad \delta z = 0, \quad \frac{\delta x}{\delta z} = 0.$$

Hence there will be a cusp of the second kind when $x=0$, $z=\pm c$, the axis of z being a tangent to the curve at each cusp. One pair of branches has its cusp at the point $x=0$, $z=c$, and the values of x, z increase indefinitely in the respective branches as τ passes from 1 to $+\infty$ and from 1 to $\sqrt{(\frac{1}{5})}$. This pair lies in the $+x$, $+z$ quadrant, and there will be a precisely similar and similarly situated pair in the $-x$, $-z$ quadrant. Thus there will be in all one infinite $\int$-formed branch passing through the origin, and two detached pairs of infinite branches lying in opposite quadrants[73]. The value $\frac{1}{5}$ for τ^2, it will of course be seen, corresponds to -2 for ϕ, and gives, as it ought to do, the position of the cusp.

[73] Let ϵ be an infinitesimal, and $\theta^2 = 1+\epsilon$; then

$$\delta z = \frac{4}{c^2}\frac{(4+5\epsilon)^2}{(2+\epsilon)\epsilon^2}\delta x = \frac{32}{c^2}(1+2\epsilon)\frac{\delta x}{\epsilon^2}.$$

Hence at either cusp the branch the further removed from the axis of x corresponds to the values of θ^2 between 1 and ∞, and the inferior branch to its values between 1 and $\frac{1}{5}$; so that the order of continuity of the five branches of the curve may be read as follows:—from the infinite point in the higher branch of the upper pair to its cusp; thence to the infinite point in the connected branch, which is contiguous to the infinite point in the opposite extremity of the middle branch; thence along this branch to its contrary infinite extremity; thence to the infinite point in the upper branch of the inferior pair; along that branch to its cusp; and thence, finally, along the lower branch to infinity.

(99) Finally, for sections parallel to the plane of the discriminant and lying below it, making $D=-k^2$, $t^2=\dfrac{1+\phi}{3-\phi}$, we obtain in like manner

$$z=kt\frac{3t^2-1}{5t^2+1},\quad x=k^3t\frac{(t^2+1)^4}{4(5t^2+1)^3},\quad \frac{\delta x}{x}=\frac{(t^2-1)(15t^2-1)}{t(t^2+1)(5t^2+1)}\delta t,$$

$$\frac{\delta z}{z}=\frac{(t^2+1)(15t^2-1)}{t(3t^2-1)(5t^2+1)},$$

$$\delta x=\frac{k^3}{4}\frac{(t^2-1)(15t^2-1)(t^2+1)^3}{(5t^2+1)^4},\quad \delta z=k\frac{(15t^2-1)(t^2+1)}{(5t^2+1)^2},$$

$$\frac{\delta x}{\delta z}=\frac{k^2}{4}\frac{(t^2-1)(t^2+1)^2}{(5t^2+1)^2}.$$

When $t^2=\frac{1}{15}$ there will be an ordinary cusp, and when $t^2=1$, $\dfrac{\delta x}{\delta z}=0$.

There will therefore be three branches,—one corresponding to the values of t between $-\sqrt{(\frac{1}{15})}$ and $+\sqrt{(\frac{1}{15})}$, the other two to values of t between these limits and $-$ and $+$ infinity respectively. The middle branch passes through the origin, where it undergoes an inflexion, and comes to a cusp at a finite distance from the origin in two opposite quadrants. The connected branch at each cusp crosses the axis of x, sweeps convexly towards the axis of z, arrives at a minimum distance from it, and then goes off to infinity.

The value $\frac{1}{15}$ for t^2 corresponds to $-\frac{3}{4}$ for ϕ, and gives, as it ought to do, the cusp-node. In fact the values $\phi=-\frac{3}{4}$, $\phi=-2$ correspond respectively to a cuspidal and to a cusp-nodal line in the limiting surface whose sections we have been considering.

When the cutting plane is that of D itself, the section becomes a double cubic parabola and a single cubical parabola crossing each other at the origin.

DESCRIPTION OF THE FIGURES [pp. 478, 479].

FIGURE I. [see p. 395 above].

The (ϵ, η) equation is $(1, \epsilon, \epsilon^2, \eta^2, \eta, 1 \between x, y)^5 = 0$, of which two roots are always imaginary; its extreme criteria are 0, 0; its middle criteria $\epsilon^4 - \epsilon\eta^2$, $\eta^4 - \eta\epsilon^2$,

$$p = \epsilon\eta - 1, \quad \sigma = (\epsilon^3 - \eta^2)(\epsilon^2 - \eta^3).$$

Points (p, σ) above the discriminatrix indicate 2 pairs of associated roots in the (ϵ, η) equation.

Points (p, σ) on the discriminatrix indicate 2 equal roots in the (ϵ, η) equation.

Points (p, σ) under the discriminatrix indicate 3 solitary roots in the (ϵ, η) equation.

Points (p, σ) above the equatrix indicate ϵ, η real and unequal.

Points (p, σ) on the equatrix indicate ϵ, η equal.

Points (p, σ) under the equatrix indicate ϵ, η imaginary and conjugate.

Points (p, σ) above the loop of the indicatrix indicate middle criteria not *both* positive.

Points (p, σ) on the loop of the indicatrix indicate middle criteria of opposite signs.

Points (p, σ) under the loop of the indicatrix indicate middle criteria not *both* negative.

The discriminatrix is a closed curve, the *whole* of which is figured, and is shaped somewhat like a harp: it has a cusp of the fourth order at the origin.

The equatrix consists of two branches coming together at a cusp at the distance 1 from the origin; the upper branch touches the horizontal axis at the origin; the lower branch, after touching the discriminant at a single point, sweeps out from and round it, cutting the vertical axis at the distance 4 below the origin. Both branches go off to infinity to the right, and lie completely under the horizontal axis. Where the lower branch touches the discriminatrix, the discriminant of the (ϵ, η) equation passes through zero without changing its sign.

The indicatrix consists of a single branch extending indefinitely in both directions. It passes from infinity below and to the left until, at the distance 1 from the origin, it touches the axis, which at the origin it crosses at an angle of 45°, after which it goes off to infinity in the positive direction. Its *loop* extends from $p = 0$ to $p = -1$. The two portions of it figured join on together, coming to a maximum at a great distance below the horizontal axis. The narrow tract marked "Region of Real parameters" is that portion of the harp-shaped space for which alone, ϵ, η being *real*, the (ϵ, η) equation can have more than one real root. The areas of each of the three regions into which the discriminatrix is divided by the equatrix and indicatrix may readily be expressed numerically in terms of algebraic and inverse circular functions only.

I am indebted to Gentleman Cadet S. L. Jacob, of the Royal Military Academy, for the tracing of the curves of which the Figure is a somewhat imperfect reproduction.

FIGURE II.

Described in text, p. 465, etc.

CONTENTS.

SUPPLEMENTAL REFERENCES.

Proposed new reduced forms for binary quartics and ternary cubics (note [11]).

Theorem on the imaginary roots of odd-degreed equations (note [26]).

Concordance between Hermite's invariants and those of the memoir (note [34]).

Identification of the latter with the corresponding numbered Tables of Professor Cayley (note [39] (*h*) (*i*)).

Proof that every invariant of a quintic is a rational integral function of the four basic invariants (note [35]).

Invariantive conditions for certain special forms of quintics (note [37]).

Conditions necessary in order that an infinitesimal variation of the coefficients of an equation may be accompanied with a change of character in the roots (note [43]).

Schläfli's theorem (proof and extension of) (note [52]).

On a number of cases capable of arising under Sturm's theorem, and on certain questions of probability (note [61]).

All the invariants of a binary form vanish when more than half the roots are equal to one another, art. 48.

Identification of section of limiting surface of invariants as a variety of the sixteenth species in Plücker's enumeration of quartic curves with two multiple points, art. 92.

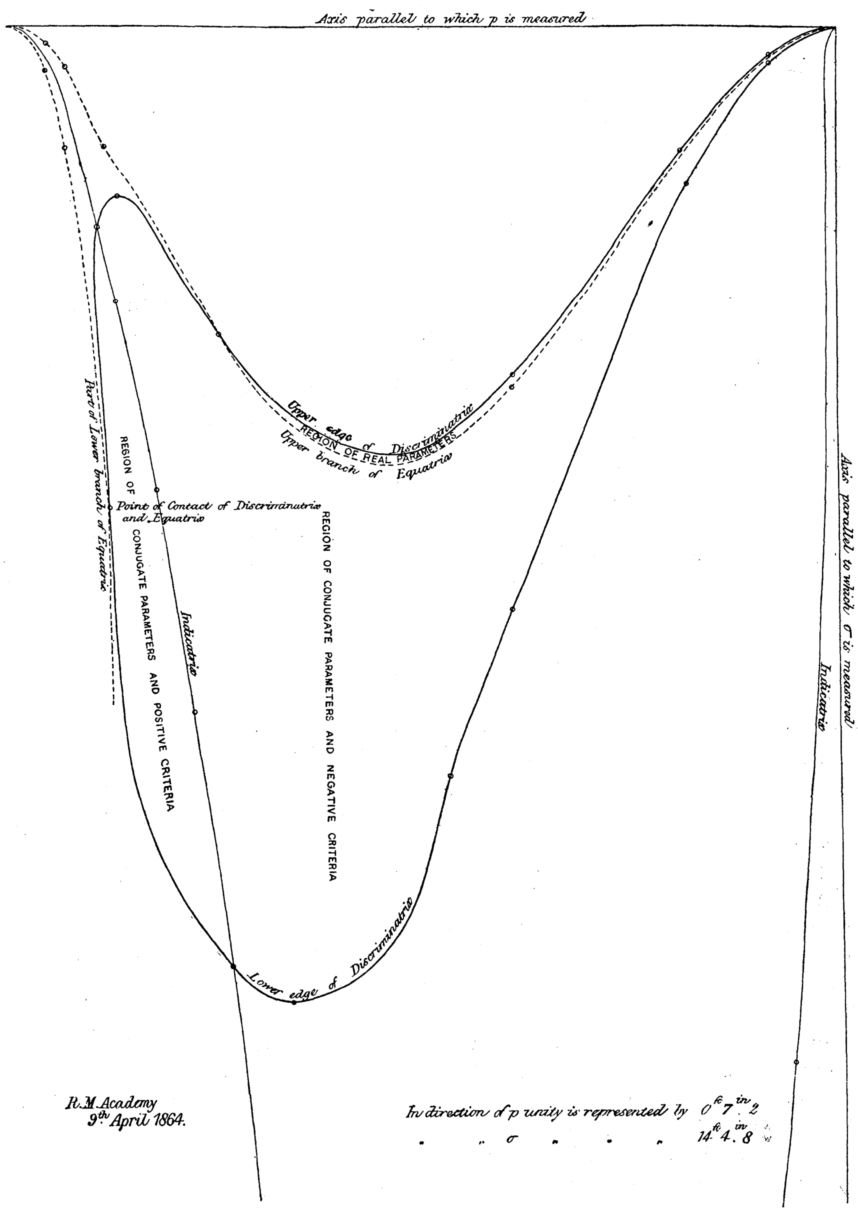

Axis parallel to which p is measured
Upper edge of Discriminatrix
REGION OF REAL PARAMETERS
Upper branch of Equatrix
Part of Lower branch of Equatrix
REGION OF CONJUGATE PARAMETERS AND POSITIVE CRITERIA
Point of Contact of Discriminatrix and Equatrix
REGION OF CONJUGATE PARAMETERS AND NEGATIVE CRITERIA
Indicatrix
Axis parallel to which σ is measured
Indicatrix
Lower edge of Discriminatrix
R. M. Academy
9th April 1864.
In direction of p unity is represented by 0ft 7in 2
" " σ " " " 14ft 4in 8

Figure I.

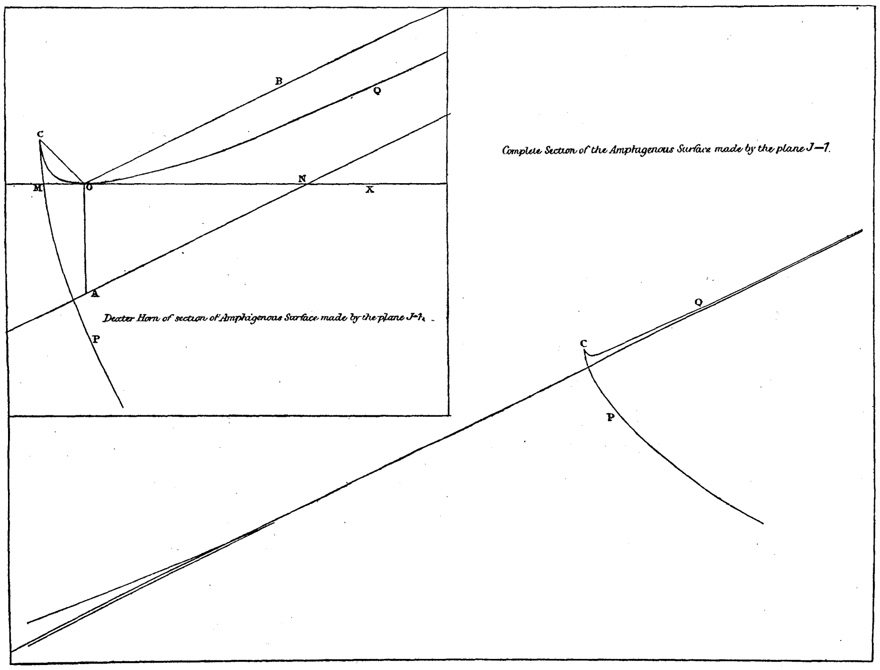

B
Q
C
M
O
N
X
A
Complete Section of the Amphigenous Surface made by the plane J—1.
Dexter Horn of section of Amphigenous Surface made by the plane J=1.
P
C
Q
P

FIGURE II.

75.

ON A SPECIAL CLASS OF QUESTIONS ON THE THEORY OF PROBABILITIES.

[*Birmingham British Association Report* (1865), p. 8.]

AFTER referring to the nature of geometrical or local probability in general, the author of the paper drew attention to a particular class of questions partaking of that character in which the condition whose probability is to be ascertained is one of pure form. The chance of three points within a circle or sphere being apices of an acute or obtuse-angled triangle, or of the quadrilateral formed by joining four points, taken arbitrarily within any assigned boundary, constituting a reentrant or convex quadrilateral, will serve as types of the class of questions in view. The general problem is that of determining the chance that a system of points, each with its own specific range, shall satisfy any prescribed condition of form. For instance, we may suppose two pairs of points to be limited respectively to segments of the same indefinite straight line: the chance of their anharmonic ratio being under or over any prescribed limit will belong to this category of questions, to which, provisionally, the author proposed to attach the name of form-probability. In questions of form-probability, in which all the ranges are either collinear segments or coplanar areas, or defined portions of space, rules may be given for transforming the data, so as to make the required probability depend on one or more probabilities of a simpler kind, leading to summations of an order inferior by two degrees to those required by the methods in ordinary use. Thus Mr Woolhouse's question relating to the chance of a triangle within a circle or sphere being acute can be made to depend upon an easy simple integration, the solutions heretofore given of this problem involving complicated triple integrals. It was shown, as a further illustration, that the form-probability of a group of points all ranging over the same triangle remains unaltered when the range of one of them is limited to any side of the triangle chosen at will, and, again, (for convenience of expression distinguishing the contour into a base and two sides) will be the mean of the two probabilities resulting from limiting one

point to range over either side with uniform probability, and simultaneously therewith a second point of the group over the base, with a probability varying as its distance from that end of the base in which it is met by the side. An analogous rule can be given for transforming the form-probability of a group limited to any the same parallelogram. So again for a group of points ranging over a plane figure bounded by any curvilinear contour. The problem may be transformed by supposing two of the points of the group to range on the contour itself, according to a law which may be expressed by saying that the probability of their being found on any arc shall vary as the product of the segment included between the arc and its chord, multiplied by the time of describing the arc about any centre of force arbitrarily chosen within or upon the contour,—a theorem which, accepting the idea of negative probability, admits also of extension to the case of a centre of force exterior to the contour.

Among other problems which the author readily resolves by aid of his principle of transformation, may be mentioned that of determining the mean value of a triangle whose angles are taken at random anywhere within a given triangle, parallelogram, ellipse, or ellipsoid. In this description of questions a peculiar difficulty arises, from the fact that the figure which is to be integrated in order to determine the numerator of the fraction which gives its mean value must always be taken positive, whereas its algebraical expression will repeatedly change its sign, according to a more or less complicated law. This quality of the analytical exponent of the arithmetical value of the figure constitutes, in fact, a sort of polarization which has to be got rid of; and the depolarizing process is effected with great ease by virtue of the simplified form impressed upon the data by the method set forth in the paper.

The author further took occasion briefly to allude to the form in which his own problem of four and Mr Woolhouse's problem of three points were originally proposed, viz. in each case without a specified boundary, and to express his opinion that the principle which had been applied to them, and in which he had formerly acquiesced, was erroneous, as it could be made to lead to contradictory conclusions, and must be abandoned. He was strongly inclined to believe that, under their original form, these questions do not admit of a determinate solution.

76.

NOTE SUR LES CONDITIONS NÉCESSAIRES ET SUFFISANTES POUR DISTINGUER LE CAS QUAND TOUTES LES RACINES D'UNE ÉQUATION DU CINQUIÈME DEGRÉ SONT RÉELLES.

[*Comptes Rendus de l'Académie des Sciences*, LX. (1865), pp. 759—761.]

DANS une communication [p. 371 above] que j'ai eu l'honneur de faire précédemment à l'Académie, j'ai donné la définition de trois invariants appartenant à une forme binaire du cinquième degré, que j'ai nommés J, D, L, D étant le discriminant.

Quant à J et L, il y a une autre méthode très-nette qui suffit pour les définir.

Si on suppose la fonction donnée mise sous la forme

$$(ax+by)^5+(cx+dy)^5+(ex+fy)^5,$$

et si on écrit

$$(ad-bc)^5=A, \quad (cf-de)^5=B, \quad (eb-fa)^5=C,$$

on aura

$$J=A^2+B^2+C^2-2AB-2AC-2BC,$$
$$L=A^2B^2C^2.$$

Avec J et L on forme un nouvel invariant que j'ai nommé Λ, tel que $\Lambda=2^{11}L-J^3$.

Alors, quand D est positif, on sait que les conditions nécessaires et suffisantes pour que toutes les racines soient réelles, sont* que J et $\Lambda+\mu JD$ soient tous les deux négatifs, μ étant une quantité numérique choisie à volonté, pourvu qu'elle ne sorte pas de l'intervalle compris entre les deux limites 1 et -2.

On voit donc (chose jusqu'ici inouïe dans les recherches de cette nature) que l'un des trois *criteria* est variable entre des limites fixes.

Mais on se forme une idée beaucoup trop restreinte de la nature de cette indétermination en se bornant aux invariants (tels que Λ) du douzième degré par rapport aux coefficients de la fonction donnée pour servir ainsi comme troisième critérium.

Au lieu de $\Lambda+\mu JD$, on peut substituer une fonction rationnelle et entière quelconque de J, K, L, K étant la quantité $A^2BC+AB^2C+ABC^2$

[* Cf. footnote, p. 452.]

homogène par rapport à J, $K^{\frac{1}{2}}$, $L^{\frac{1}{3}}$, à savoir $F(J, K, L)$, pourvu que certaines conditions soient satisfaites que je vais donner. Écrivons

$$J = \theta^2 - 4\theta, \quad K = \theta^2 + 2\theta, \quad L = \theta^2,$$

alors F devient une fonction de θ, et les conditions nécessaires et suffisantes pour que F (pris avec le signe convenable) soit un bon troisième critérium (comme remplaçant de Λ) sont les suivantes, qu'en écartant toutes les racines de F, qui se répètent un nombre pair de fois, une des restantes est égale à -4, mais nulle autre ne sort des limites 0 et 12.

Ainsi, par exemple, on peut se servir (comme critérium) d'un invariant du seizième degré par rapport aux coefficients, dans lequel il entrera deux paramètres variables, et on tombe sur une question très-intéressante d'Algèbre pour trouver les conditions auxquelles ces deux paramètres doivent être assujettis pour que l'invariant soit bon comme critérium, problème qui se résout par des considérations géométriques et sur lequel je prendrai quelque autre occasion de revenir. Comme exemple de la manière de mettre à l'épreuve un critérium quelconque donné, je prendrai la fonction trouvée par M. Hermite par une méthode particulière à lui qu'il a eu la grande bonté de me communiquer.

Cette fonction, exprimée dans ma notation, est

$$18L^2 - JKL - K^3;$$

en faisant les substitutions dont j'ai parlé, cette quantité devient

$$-2\theta^3(\theta^3 + 2\theta^2 - 7\theta + 4) = -2(\theta + 4)(\theta - 1)^2\theta^3,$$

où on voit qu'il existe une racine -4 et que les autres racines d'une multiplicité impaire, c'est-à-dire celles qui appartiennent au facteur θ^3, ne sortent pas des limites 0, 12.

De même, on peut démontrer plus généralement que

$$(2L^2 - K^3) + \mu(16L^2 - JKL)$$

sera bon comme critérium, pourvu que $\mu > -\frac{1}{9}$.

Par exemple, en mettant $\mu = 1$, on retombe sur le critérium de M. Hermite: en mettant $\mu = 0$, on trouve comme critérium $2L^2 - K^3$, et en mettant $\mu = \infty$, on trouve $16L^2 - JKL$, équivalant au seul facteur $16L - JK$, qui à son tour peut s'exprimer sous la forme

$$\frac{\Lambda + JD}{128} \text{ (car } D = J^2 - 128K);$$

on reconnaît immédiatement que 1 étant compris, comme cas extrême, entre les limites 1 et -2, $\Lambda + JD$ et conséquemment $16L - JK$ doit être bon comme critérium.

On comprend aisément que la forme $(2L^2 - K^3) + \mu(16L^2 - JKL)$, avec $\mu > -\frac{1}{9}$, n'est qu'une solution particulière du problème de trouver le critérium le plus général du degré 24 dans les coefficients, lequel contiendra 5 paramètres variables, c'est-à-dire 2 moins que le nombre des compositions du nombre 6 qu'on peut effectuer avec les éléments 1, 2, 3.

77.

THÉORÈME D'ARITHMÉTIQUE.

[*Comptes Rendus de l'Académie des Sciences*, LX. (1865), pp. 1011—1012.]

SOIT $F(a, b, c, d)$ le représentant de la quantité

$$a^2d^2 + 4ac^3 + 4db^3 - 3b^2c^2 - 6abcd;$$

soient b, c deux quantités *positives* qui satisfont à l'équation

$$F(a, b, c, d) = 0;$$

écrivons l'équation cubique en x

$$F(a, x, c, d) = 0,$$

et soient (b, b_1) les deux racines positives de cette équation. De même écrivons

$$F(a, b_1, x, d) = 0,$$

et soient (c, c_1) ses deux racines positives; posons semblablement

$$F(a, x, c_1, d) = 0,$$

dont (b_1, b_2) sont les deux racines positives, et ainsi de suite; on obtiendra de cette façon deux séries infinies $b, b_1, b_2, b_3 \ldots$; $c, c_1, c_2 \ldots$.

Or je dis: 1° que si b est plus grand que b_1, chacune des deux séries sera constamment décroissante, et si au contraire b est moindre que b_1, chacune sera constamment croissante. De plus, je dis: 2° que dans ces deux cas les quantités b tendront vers $\sqrt[3]{(a^2d)}$, et les quantités c vers $\sqrt[3]{(ad^2)}$ comme limite. Nommons $\sqrt[3]{(a^2d)} - b_n = \beta_n$, $\sqrt[3]{(ad^2 - c_n)} = \gamma_n$. Je dis: 3° qu'en même temps que β_n et γ_n deviennent infiniment petits quand n est infini, les différences $\beta_n - \gamma_n$, $\beta_n - \beta_{n-1}$, $\gamma_n - \gamma_{n-1}$ deviendront infiniment petites par rapport à β_n et γ_n.

On remarquera que $F(a, b, c, d)$ est un discriminant binaire du troisième ordre. Il y a un théorème général analogue pour le discriminant binaire d'un ordre quelconque.

78.

RECTIFICATION ET DÉMONSTRATION D'UN THÉORÈME D'ARITHMÉTIQUE DONNÉ DANS LE COMPTE RENDU DU 15 MAI.

[*Comptes Rendus de l'Académie des Sciences*, LX. (1865), pp. 1121—1125.]

UNE erreur s'est glissée dans l'énoncé que j'ai eu l'honneur de donner tout récemment dans les *Comptes rendus*; je me hâte de la corriger en ajoutant en même temps la démonstration du théorème auquel il se rapporte.

Considérons l'équation cubique

$$\phi u = au^3 + 3bu^2 + 3cu + d = 0.$$

Supposons a, b, c, d tous positifs. Il est évident que si les racines sont toutes réelles et distinctes, on peut faire varier à volonté d'une quantité infinitésimale ou b ou c, sans que les racines cessent d'être réelles. Mais quand ϕ possède deux racines égales ρ, en faisant

$$\phi u + 3\delta b \,.\, u^2 = 0,$$

pour déterminer si les racines sont ou non toutes réelles, il faut considérer l'équation

$$\phi''\rho \,.\, \frac{(d\rho)^2}{2} + 3\delta b \,.\, \rho^2 = 0,$$

et les racines resteront réelles ou non, selon que $-\phi''\rho$ et δb auront les mêmes signes ou des signes contraires. De même, la réalité des racines de l'équation

$$\phi u + 3\delta c \,.\, u = 0$$

dépend de la circonstance que $-\phi''\rho$ et $\delta b \,.\, \rho$ aient ou non les mêmes signes, c'est-à-dire, puisque ρ est nécessairement négatif, dans le cas où ϕu possède deux racines égales, il sera toujours possible, ou en diminuant infiniment peu b ou en diminuant infiniment peu c, de conserver la réalité des trois racines. Si en diminuant b cela a lieu, il n'en sera pas de même quand on diminue c, et *vice versâ*, c'est-à-dire en diminuant une des quantités b, c, par exemple b, et en augmentant l'autre c, les racines restent réelles; au contraire, en augmentant b et en diminuant c, deux des racines deviennent imaginaires.

J'ai supposé que deux seulement des racines de ϕ sont égales; si toutes trois sont égales, la chose marche autrement, car dans ce cas on

aura nonseulement $\phi\rho = 0$, $\phi'\rho = 0$, mais aussi $\phi''\rho = 0$, et en faisant varier en même temps b et c, on trouve

$$\phi'''\rho \cdot \frac{(\delta\rho)^3}{6} + 3\delta b \cdot \rho^2 + 3\delta c \cdot \rho = 0.$$

Donc, en faisant ou $\delta b = 0$ ou $\delta c = 0$, il serait impossible d'empêcher que deux des trois racines deviennent imaginaires. Afin de conserver la réalité de ces trois racines, il faut prendre $\delta b \cdot \rho + \delta c = \theta$, où θ est ou zéro ou une quantité infinitésimale d'un certain degré *i au moins*, par rapport à δb ou δc; alors la réalité de ces racines dépendra de la circonstance que $2\delta b \cdot \rho + \delta c$ soit du signe contraire à $\phi'''\rho$, c'est-à-dire négatif, ainsi donc δc, et conséquemment δb sera positif, et en même temps $\frac{\delta b}{\delta c}\rho + 1$ infiniment près de zéro.

Or, commençons avec l'équation $\phi u = 0$, en possession de deux racines réelles, et supposons que c'est b qu'on peut diminuer sans introduire des racines imaginaires: allons toujours en diminuant b tant que cela sera possible, c'est-à-dire jusqu'à ce que ϕu ait deux racines égales; à cet instant, on ne peut plus diminuer b, mais on peut diminuer c sans perdre de racines réelles, et le diminuer jusqu'à ce que deux des racines deviennent égales; alors il faut recommencer avec b, et ainsi de suite pour c et b tour à tour. Je dis qu'en continuant ces opérations, ni b ni c ne peut devenir zéro, car dans ce cas on sait que l'équation en u ne pourrait avoir qu'une seule racine réelle, il faut en effet se rappeler que quand b deviendrait zéro, c serait positif, et *vice versâ*. De plus, il est évident, les variations de b et c n'étant pas simultanées, qu'on ne peut pas tomber *exactement* sur le cas de trois racines réelles. Donc, en commençant avec b et c, on tombe sur une série double prolongée à l'infini bc, b_1c_1, b_2c_2, $b_3c_3, \ldots$, telle, que tous les b décroissent et tous les c décroissent, mais sans que ou b ou c dépasse jamais une certaine limite fixe pour l'une et pour l'autre. J'ai supposé que c'était b qui commençait à décroître; si b ne peut pas être diminué, on sera nécessairement en droit de commencer avec c, et on trouvera la série double

$$cb, \quad {}_1c_1b, \quad {}_2c_2b, \ldots.$$

Ainsi on voit qu'on peut toujours former deux paires de séries

$$b,\ b_1,\ b_2,\ b_3,\ \ldots, \quad c,\ c_1,\ c_2,\ \ldots,$$

$$c,\ {}_1c,\ {}_2c,\ {}_3c,\ \ldots, \quad b,\ {}_1b,\ {}_2b,\ \ldots,$$

et que l'équation

$$au^3 + 3xu^2 + 3yu + d = 0$$

aura deux racines réelles quand

$$x = b_i, \qquad y = c_i,$$

ou bien

$$x = b_{i+1}, \qquad y = c_{i+1},$$

et aussi quand $x = {}_ib, \quad y = {}_ic,$
ou bien $x = {}_ib, \quad y = {}_{i+1}c.$

Dans l'une des deux paires de séries les b et les c croîtront, comme il est facile de démontrer, sans limite; dans l'autre paire, il y aura une limite pour les b et une limite pour les c, vers lesquelles ces quantités tendent continuellement.

La condition que $au^3 + 3xu^2 + 3yu + d = 0$ ait deux racines égales sera

$$a^2d^2 + 4ay^3 + 4dx^3 - 3x^2y^2 - 6adxy = 0,$$

disons $F(x, y) = 0$.

En supposant cette équation satisfaite par les valeurs positives $x = b$, $y = c$, pour obtenir la première paire de séries, on écrit:

$$\begin{aligned}
&F(x,\ c) = 0, \quad \text{qui donnera} \quad x = b,\ x = b_1, \quad b_1 \text{ étant positif,}\\
&F(b_1, y) = 0, \quad \text{qui donnera} \quad y = c,\ y = c_1, \quad c_1 \text{ étant positif,}\\
&F(x,\ c_1) = 0, \quad \text{qui donnera} \quad x = b_1, x = b_2, \quad b_2 \text{ étant positif,}
\end{aligned}$$

et ainsi de suite. De cette manière, on peut trouver les séries b, b_1, b_2, ..., c, c_1, ..., et semblablement l'autre paire.

De plus, on remarquera que ces séries se développent par le moyen de la solution d'équations quadratiques, car dans les équations cubiques $F(x, A) = 0$, ou $F(B, y) = 0$, dont il est question, une des racines est toujours connue d'avance.

Il reste seulement à fixer la valeur de la limite pour chaque série décroissante, ce qui est bien facile. Car si b_n diffère infiniment peu de b_{n+1}, c'est que deux racines de $F(x, c_n)$ seront infiniment près l'une de l'autre, c'est-à-dire que le discriminant de F sera infiniment voisin de zéro.

Or le discriminant du discriminant

$$a^2d^2 + 4ac^3 + 4db^3 - 3b^2c^2 - 6abcd$$

par rapport à b, on le trouve facilement (à un facteur positif numérique près) égal à $(ad^2 - c^3)^3$; donc la limite de c_n, quand n devient infini, sera nécessairement $\sqrt[3]{(ad^2)}$; de même, la valeur limite de b_n sera $\sqrt[3]{(a^2d)}$, de sorte que, comme on aurait pu le deviner *à priori*, la fonction limite de ϕu est la forme pour laquelle toutes ces trois racines deviennent égales.

Ainsi on voit que les valeurs limites des b et des c sont indépendantes de la valeur initiale de l'une ou de l'autre. On voit aussi que par ce théorème on se trouve approcher continuellement de la racine cubique d'un nombre quelconque donné et de son carré *sans tâtonnement* et sans autre procédé que l'extraction de la racine positive d'une suite infinie d'équations *quadratiques*. Pour cela, tout ce qui est nécessaire est de commencer avec l'équation

$$\phi u = (u + \lambda)^2 \left(u + \frac{D}{\lambda^2}\right),$$

λ étant arbitraire. Cela donnera

$$a = 1, \quad d = D, \quad b = 2\lambda + \frac{D}{\lambda^2}, \quad c = \lambda^2 + \frac{2D}{\lambda}.$$

Alors l'une ou l'autre des deux paires de séries, commençant avec les valeurs données pour b, c, aura nécessairement $\sqrt[3]{(D)}$, $\sqrt[3]{(D^2)}$ pour limites respectives.

Puisque b et c décroissent continuellement vers leurs limites respectives, on voit que le théorème suppose que quand l'équation

$$a^2d^2 + 4ac^3 + 4db^3 - 6abcd - 3b^2c^2 = 0$$

est satisfaite par des valeurs positives a, b, c, d, on aura nécessairement

$$b > \sqrt[3]{(a^2d)}, \quad c > \sqrt[3]{(ad^2)}.$$

Cela se confirme très-simplement. Car en traitant cette équation comme une équation en b, puisqu'une racine positive existe, toutes les racines seront réelles; donc le discriminant par rapport à b sera négatif, c'est-à-dire $(ad^2 - c^3)^3$ sera négatif; conséquemment $c^3 > ad^2$, et de même on démontre que $b^3 > a^2d$.

A l'aide des principes expliqués plus haut, on démontre sans difficulté qu'en supposant $\sqrt[3]{(a^2d)}$, $\sqrt[3]{(ad^2)}$ les limites de b_n et c_n, quand on écrit

$$\beta_n = \sqrt[3]{(a^2d)} - b_n, \qquad \gamma_n = \sqrt[3]{(ad^2)} - c_n,$$

$\dfrac{\beta_{n+1} - \beta_n}{\beta_n}$, $\dfrac{\gamma_{n+1} - \gamma_n}{\gamma_n}$ seront tous les deux infiniment petits quand n devient infini; et, de plus, $\dfrac{\sqrt[3]{(d)}\beta_n - \sqrt[3]{(a)}\gamma_n}{\beta_n \text{ ou } \gamma_n}$ sera infiniment petit sous la même supposition.

Je prends la liberté d'ajouter que le théorème ici donné ressort tout naturellement d'une étude approfondie que j'ai eu récemment occasion de faire sur les conditions que la variation d'une fonction rationnelle doit remplir pour qu'elle n'amène pas une perte de racines réelles. C'est M. Hermite qui, à ce qu'il me paraît, a été le premier à se servir du grand principe de la variation des coefficients pour l'étude de la nature des formes algébriques. En poursuivant cette théorie dans ses détails, j'ai déjà réussi avec son aide à établir le théorème de Newton pour la découverte de racines imaginaires jusqu'au septième degré inclusivement, et il est bien probable que dans un court délai on réussira (moi ou quelque autre) à établir ce grand théorème dans toute sa généralité pour les équations d'un degré quelconque.

79.

SUR LES LIMITES DU NOMBRE DES RACINES RÉELLES DES ÉQUATIONS ALGÉBRIQUES.

[*Comptes Rendus de l'Académie des Sciences,* LX. (1865), pp. 1261—1263.]

J'AI l'honneur de soumettre à l'Académie un théorème que j'ai tout récemment réussi à établir par une analyse des plus simples. On verra qu'il comprend comme cas particulier le célèbre théorème de Newton qui, donné sans preuve par son auteur, n'a pas été démontré jusqu'à ce jour, nonobstant les efforts des Maclaurin, des Waring et des Euler. Soit fx une fonction rationnelle et entière de x. Soit c_0, nc_1, $\frac{1}{2}n(n-1)c_2$, ..., c_n les coefficients des puissances successives de x dans $f(x+p)$. Écrivons

$$C_0 = c_0^2, \quad C_1 = c_1^2 - c_0c_2, \quad C_2 = c_2^2 - c_1c_3, \; \ldots, \; C_n = c_n^2.$$

Alors on peut dire qu'à chaque petite lettre c_r est associée une grande lettre C_r, et de même à chaque succession c_r, c_{r+1} de petites lettres est associée une succession de grandes lettres C_r, C_{r+1}. Quand ces successions forment toutes deux des permanences, c'est-à-dire quand les produits $c_r \,.\, c_{r+1}$ et $C_r \,.\, C_{r+1}$ sont tous les deux positifs, on peut dire que la succession composée $\begin{pmatrix} c_r c_{r+1} \\ C_r C_{r+1} \end{pmatrix}$ forme une double permanence ; et en prenant de cette sorte toutes les successions simultanées fournies par ces deux suites, il y aura un certain nombre de ces permanences qu'on peut nommer le nombre de permanences doubles propres à p.

Or, je dis qu'en supposant p plus grand que q, la différence entre le nombre des permanences doubles propres à p et le nombre de ces permanences propres à q ne sera jamais négative, et de plus elle fournira une limite supérieure au nombre de racines réelles comprises entre p et q.

Si l'on prend p égal à zéro et q égal à $-\infty$, il est évident que le nombre de permanences doubles propre à $-\infty$ est zéro, car toutes les successions simples dans $f(-\infty)$ sont des variations.

Ainsi, en donnant aux coefficients de fx, disons c_0, c_1, c_2, ..., c_n, le nom de *suite cartésienne*, et à C_0, C_1, C_2,..., C_n, formés de la manière décrite plus haut, celui de *suite newtonienne* appartenant à fx, on peut affirmer que le nombre des racines négatives dans une équation a pour limite supérieure le nombre des permanences doubles fournies par la combinaison de la suite cartésienne avec la suite newtonienne; et conséquemment, en changeant x en $-x$, on voit également que le nombre des racines positives de la même équation aura pour limite supérieure le nombre des successions simultanées composées d'une permanence newtonienne associée à une variation cartésienne. C'est là, en d'autres termes, le théorème complet de Newton, comme on peut le vérifier en consultant l'*Arithmétique universelle.*

On voit facilement que, pour la forme $f(x+p)$, les éléments c_0, c_1, ..., c_n, au moyen desquels on forme C_0, C_1, ..., C_n, ne sont autre chose (pris en ordre inverse) que les quantités

$$fp,\ \frac{f'p}{n},\ \frac{f''p}{n(n-1)},\ \frac{f'''p}{n(n-1)(n-2)},\ \ldots,\ \frac{f^{(n)}p}{n(n-1)\ldots 1};$$

mais on n'est nullement borné à cette suite déterminée de valeurs pour les éléments. Je trouve qu'on peut prendre pour éléments un système de multiples numériques de $fp, f'p, f''p$, ... dans lesquels il entre deux paramètres arbitraires, dont l'un cependant est limité par la grandeur de n. Par exemple, on peut prendre tout simplement pour les deux séries

$$\begin{array}{ccccc} fp, & f'p, & \ldots, & f^{(n-1)}p, & f^{(n)}p, \\ Tp, & T_1p, & \ldots, & T_{n-1}p, & T_np, \end{array}$$

où T_rp signifie

$$(f^{(r)}p)^2 - f^{(r-1)}p\,.\,f^{(r+1)}p.$$

Alors le nombre de permanences double dans ces deux suites, moins le nombre semblable quand on écrit q pour p, donnera comme auparavant une limite supérieure au nombre des racines réelles de fx compris entre p et q: et l'on doit remarquer que quelquefois l'une des méthodes et quelquefois l'autre donnera la meilleure limite, excepté pour les cas de $n=2$ et $n=3$, cas où la première méthode est toujours préférable.

Ainsi l'on voit qu'on peut substituer à la règle de Fourier une règle où les fonctions qu'il emploie sont associées à des combinaisons quadratiques d'elles-mêmes, formant deux systèmes dont l'un est effectivement fixe, l'autre variable. Je n'entre pas dans les détails sur la loi de variabilité, parce que mon seul but, en faisant cette communication, est de faire connaître les principes sur lesquels repose la démonstration du théorème de Newton, démonstration qui a, depuis près deux siècles, échappé aux recherches des géomètres.

80.

THÉORÈME D'ALGÈBRE ÉLEMENTAIRE.

[*Comptes Rendus de l'Académie des Sciences*, LXI. (1865), pp. 282—283.]

Je demande la permission d'ajouter l'énoncé exact du théorème général auquel (sans le préciser) allusion a été faite dans les *Comptes rendus* de la séance du 19 juin dernier.

Désignons une quelconque des quatre combinaisons de signes

$$\begin{matrix} + + & + + & - - & - - \\ + + & - - & + + & - - \end{matrix}$$

par le mot *double permanence*, et une quelconque des combinaisons

$$\begin{matrix} + - & + - & - + & - + \\ + + & - - & + + & - - \end{matrix}$$

par le mot *varia permanence*.

Soit $fx = 0$ une équation algébrique du degré n; ν une quantité réelle qui n'est pas comprise *en dedans* des limites $0, -n$ (bien entendu que les limites elles-mêmes ne sont pas exclues). Que $f^r x$ représente la quantité $\dfrac{d^r}{dx^r} fx$; $G_r x$ la quantité

$$(f^r x)^2 - \frac{\nu + r - 1}{\nu + r} f^{r-1} x \,.\, f^{r+1} x.$$

Formons la progression simultanée

$$\left.\begin{matrix} fx, & f^1x, & f^2x, & \ldots, & f^nx; \\ Gx, & G_1x, & G_2x, & \ldots, & G_nx. \end{matrix}\right\} \qquad \text{(P)}$$

Alors je dis: 1° qu'en faisant x croître de λ jusqu'à μ, le nombre de doubles permanences dans (P) ne peut pas décroître, et que le nombre de *varia* permanences ne peut pas croître;

2° Que le nombre des racines réelles de fx comprises entre λ et μ ne peut excéder ni le nombre des doubles permanences gagnées, ni le nombre des *varia* permanences perdues par (P) quand x passe de λ à μ.

3° On peut ajouter que la différence entre le premier et le second ou entre le premier et le troisième de ces nombres sera toujours un nombre *pair*.

Pour retrouver le théorème de Newton donné dans le chapitre intitulé *De formâ æquationis,* dans l'*Arithmétique universelle,* en tant qu'il se rapporte à la limite du nombre des racines négatives de fx, on prend $\nu = -n$, $\lambda = -\infty$, $\mu = 0$, et on fait le compte des doubles permanences gagnées; en tant qu'il se rapporte à la limite du nombre des racines positives, on prend $\nu = -n$, $\lambda = 0$, $\mu = \infty$, et on fait le compte des *varia* permanences perdues. Ainsi on obtient une règle qui est en effet identique avec celle de Newton, savoir: qu'en écrivant

$$fx = ax^n + nbx^{n-1} + \tfrac{1}{2}n(n-1)\,cx^{n-2} + \dots,$$

la progression simultanée

$$\left.\begin{matrix} a, & b, & c, \dots, & l, \\ a^2, & b^2 - ac, & c^2 - bd, \dots, & l^2 \end{matrix}\right\} \qquad (Q)$$

fournit, par ses doubles permanences et par ses *varia* permanences, des limites au nombre des racines négatives et positives respectivement de fx.

J'ajoute qu'en écrivant fx dans la forme beaucoup plus générale

$$ax^n + \frac{\nu}{i+1}\,bx^{n-1} + \frac{\nu(\nu+1)}{(i+1)(i+2)}\,cx^{n-2} + \frac{\nu(\nu+1)(\nu+2)}{(i+1)(i+2)(i+3)}\,dx^{n-3} + \dots,$$

ou bien sous la forme

$$ax^n - \frac{\nu}{i+1}\,bx^{n-1} + \frac{\nu(\nu+1)}{(i+1)(i+2)}\,cx^{n-2} - \frac{\nu(\nu+1)(\nu+2)}{(i+1)(i+2)(i+3)}\,dx^{n-3} + \dots,$$

selon que ν est positif ou négatif, alors, pourvu que i soit un entier positif et ν une quantité réelle quelconque qui n'est pas comprise *en dedans* des limites i, $-n$, la progression (Q) sert toujours à limiter, comme auparavant, le nombre total des racines négatives et positives de fx.

Comme corollaire particulier on déduit que, sous les conditions supposées, la fonction hypergéométrique*

$$x^n + \frac{\nu}{i+1}\,x^{n-1} + \frac{\nu(\nu+1)}{(i+1)(i+2)}\,x^{n-2} + \frac{\nu(\nu+1)(\nu+2)}{(i+1)(i+2)(i+3)}\,x^{n-3} + \dots$$

(sauf le cas où, ν étant $-n$ et i étant 0, cette fonction devient une puissance exacte) ne peut jamais avoir plus d'une seule racine réelle.

[* Cf. p. 513.]

81.

ON NEWTON'S RULE FOR THE DISCOVERY OF IMAGINARY ROOTS OF EQUATIONS.

[*Proceedings of the Royal Society of London,* XIV. (1865), pp. 268—270.]

In the first part of my "Trilogy of Algebraical Researches," printed in the *Philosophical Transactions,* will be found a proof of Newton's Rule for the discovery of imaginary roots carried as far as equations of the 5th degree inclusive. The method, however, therein employed offered no prospect of success as applied to equations of the higher degrees. I take this opportunity, therefore, of announcing that I have recently hit upon a more refined and subtle method and idea, by means of which the demonstration has been already extended to the 6th degree, and which lends itself with equal readiness to equations of all degrees. Ere long I trust to be able to lay before the Society a complete and universal proof of this rule—so long the wonder and opprobrium of algebraists. For the present I content myself with stating that the new method consists essentially, first, in the discerption of the question as applied to an equation of any specified degree into distinct cases, corresponding to the various combinations of signs that can be attached to the coefficients; secondly, in the application of the fecund principle of variation of constants, laid down in the third part of my "Trilogy," and, in particular, of the theorem that if a rational function of a variable undergoes a continuous variation flowing in one direction through any prescribed channel, then at the moment when it is on the point of losing real roots, not only must it possess two equal roots (a fact familiar to mathematicians as the light of day), but also its second differential, and the variation, when for the variable is substituted the value of such equal roots, must assume the same algebraical sign*. By aid of the processes afforded by this principle, which admits of an infinite variety of modes of application, according to the form imparted to the channel of variation, and constitutes in effect for the examination of algebraical forms an instrument of analysis as powerful as the

* The above is on the supposition that there is no ternary or higher group of equal roots.

microscope for objects of natural history, or the blowpipe for those of chemical research, the problem in view is resolved with a surprising degree of simplicity; so much so that, as far as I have hitherto proceeded with the inquiry, the computations, algebraical and arithmetical, which I have had occasion to employ may be contained within the compass of a single line. The new method, moreover, enjoys the prerogative of yielding a proof of the theorem in the complete form in which it came from the hands of its author (but which has been totally lost sight of by all writers, without exception, who have subsequently handled the question), namely, in combination with, and as supplemental to, the Rule of Descartes. On my mind the internal evidence is now forcible that Newton was in possession of a proof of this theorem (a point which he has left in doubt and which has often been called into question), and that, by singular good fortune, whilst I have been enabled to unriddle the secret which has baffled the efforts of mathematicians to discover during the last two centuries, I have struck into the very path which Newton himself followed to arrive at his conclusions.

Since the above note was sent in to the Society, I have completed the demonstration for the 7th degree, and in the course of the inquiry have had occasion to consider the conditions to be satisfied in order that a rational function of x, with r equal roots a, may undergo no loss of real roots for any assigned variation imparted to the function: for the theory of the 7th degree the case of three equal roots has to be considered, and the conditions in question are that the variation itself may contain the equal root a, and that its first differential coefficient may have the contrary sign to that of the third differential coefficient of the function which it varies when a is substituted for x—a theorem which is, of course, capable of extension to the case of an equation passing through a phase of any number of equal roots*.

* The above is on the supposition that one of the three equal roots remains unaffected in magnitude by the variation, whilst the other two change. If all three are to change simultaneously, infinitesimals beyond the first order and with fractional indices have to be brought into consideration; in that case, on making $x=a$, the variation need not become absolutely zero, but must contain no infinitesimal of the first order. And a further limitation becomes necessary in addition to the conditions stated in the text, in order that no loss of real roots may be incurred in consequence of the variation.

82.

ON A THEOREM CONCERNING DISCRIMINANTS.

[*Proceedings of the Royal Society of London*, XIV. (1865), pp. 336—337.]

LET $F(a, b, c, d) = a^2d^2 + 4ac^3 + 4db^3 - 3b^2c^2 - 6abcd$, and let a, b, c, d be four quantities all greater than zero, which make this function vanish.

(1) The cubic equation in x, $F(a, x, c, d) = 0$, will have two positive roots (b, b_1); so $F(a, b_1, x, d)$ will have two such roots (c, c_1), $F(a, x, c_1, d)$ two such (b_1, b_2), $F(a, b_2, x, d)$ two such (c_1, c_2), and so on *ad infinitum*; we may thus generate the infinite series $b_1c_1b_2c_2$.......

Similarly, beginning with the equation $F(a, b, x, d)$, and proceeding as above, we shall obtain a similar series, c', b', c'', b'' ...; and combining the two together, and with the initial quantities b, c, we obtain a series proceeding to infinity in both directions ...... $b''\, c''\, b'\, c'\, b\, c\, b_1\, c_1\, b_2\, c_2$ ....

(2) The four quantities

$$\frac{\delta F}{\delta a},\ \frac{\delta F}{\delta b},\ \frac{\delta F}{\delta c},\ \frac{\delta F}{\delta d},$$

where F represents $F(a, b, c, d)$, will present one or the other of the three following successions of sign,

$$\begin{array}{cccc} + & - & + & - \\ - & + & - & + \\ 0 & 0 & 0 & 0 \end{array}$$

(3) When the last is the case, that is, when the differential derivatives all vanish, the quantities b, c remain stationary in the above double infinite series; in the two other cases, the b quantities and c quantities *continually* increase in one direction and *continually* decrease in the other, the increase taking place in that direction in which we must read the successions of sign of the derivatives of F so as to begin with passing from plus to minus.

(4) To the increase of b and c there is no limit, but to the decrease of each there is a limit, namely $a^{\frac{2}{3}}\, d^{\frac{1}{3}}$ and $a^{\frac{1}{3}}\, d^{\frac{2}{3}}$ are the limits towards which the b and the c terms respectively converge.

I conclude with remarking that the above theorem is only a particular illustration, and the most simple that can be given, of a very wide theory relating to discriminants of all orders which springs as an immediate consequence from the principles involved in the theory of variation of algebraical forms referred to in the note which I had recently the honour of laying before the Society.

83.

ON LAMBERT'S THEOREM FOR ELLIPTIC MOTION.

[*Monthly Notices of the Royal Astronomical Society*, XXVI. (1865), pp. 27—29.]

THE original demonstration by Lambert of the celebrated theorem which bears his name was a geometrical one, see *Monthly Notices*, vol. XXII. p. 238, where this demonstration is reproduced by Mr Cayley. Lagrange has given no less than three distinct demonstrations of the same: one a sort of verification by aid of trigonometrical formulae, another founded on a property of integrals, and a third, perhaps the most remarkable of all, derived from the general expressions for the time in an orbit described about two centres of force varying according to the law of nature by supposing one of them to be situated in the orbit itself, and to become zero. Notwithstanding this plethora of demonstration, the following direct algebraical method of proving from the ordinary formulae for the time of a planet passing from one point to another, that, when the period is given, the time is a function only of the sum of the distances of these points from the centre of force, and of their distance from one another, may be deemed not wholly undeserving of notice.

Let ρ, ρ' be the distances of the two positions from the Sun, c their distance from one another, v, v' the true, u, u' the excentric, m, m' the mean anomalies thereunto corresponding, e the excentricity,

$$\omega = m - m', \quad s = \rho + \rho', \quad \Delta = \tfrac{1}{2}(s^2 - c^2):$$

then

$$\rho = 1 - e\cos u, \quad \rho' = 1 - e\cos u', \quad m = u - e\sin u, \quad m' = u' - e\sin u',$$

$$\rho\cos v = \cos u - e, \qquad \rho\sin v = \sqrt{(1-e^2)}\sin u,$$

$$\rho'\cos v' = \cos u' - e, \quad \rho'\sin v' = \sqrt{(1-e^2)}\sin u',$$

$$c^2 = \rho^2 + \rho'^2 - 2\rho\rho'\cos(v' - v).$$

Writing for brevity c, c', s, s', for $\cos u$, $\cos u'$, $\sin u$, $\sin u'$, and to avoid confusion putting also for the moment $\bar{s}, \bar{c}$ in place of the original s and c, we have

$$\bar{s} = 2 - ec - ec', \quad \omega = u - u' - es + es',$$

$$\Delta = \rho\rho' + \rho\rho' \cos(v' - v) = 1 + cc' + ss' - 2e(c + c') + e^2(1 + cc' - ss').$$

Let $J = \dfrac{d(\Delta, \bar{s}, \omega)}{d(e, u, u')}$; then J is the determinant

$$\begin{vmatrix} -2(c+c')+2e(1+cc'-ss'); & cs'-c's+2es-e^2(cs'+c's); & c's-cs'+2es'-e^2(cs'+c's) \\ -c-c' & ; \quad es & ; \quad es' \\ -s+s' & ; \quad 1-ec & ; \quad -1+ec' \end{vmatrix}.$$

Denoting this determinant by

$$\begin{vmatrix} A, & B, & C \\ D, & E, & F \\ G, & H, & K \end{vmatrix},$$

we find

$$(A, B, C) - 2H(D, E, F) + 2E(G, H, K) = (0, B, -B),$$

$$(A, B, C) - 2K(D, E, F) + 2F(G, H, K) = (0, -C, C),$$

so that
$$J = \begin{vmatrix} A, & B, & C \\ 0, & B, & -B \\ 0, & -C, & C \end{vmatrix} = 0.$$

Hence restoring s, c, instead of $\bar{s}$, $\bar{c}$, it appears that $d\omega$ is a linear function of ds and $d\Delta$; that is, ω is a function of s and Δ, or what is the same thing of s and c, independent of e. If then, when $e = 1$, the corresponding values of $\rho, \rho', v, v', u, u'$ are $r, r', \theta, \theta', \phi, \phi'$, we have $\cos\theta = -1$, $\cos\theta' = -1$, $\sin\theta = 0$, $\sin\theta' = 0$, $r - r' = c$, $r + r' = s$, whence writing

$$1 - \cos\phi = \frac{s+c}{2}, \quad 1 - \cos\phi' = \frac{s-c}{2},$$

we have finally $\omega = \phi - \phi' - \sin\phi + \sin\phi'$ as was to be proved.

Essentially this demonstration is of the same value as the first of Lagrange's three methods of proof above referred to, but with the difference that it leads up to and accounts beforehand for the success of the transformations therein employed.

84.

ON AN ELEMENTARY PROOF AND GENERALIZATION OF SIR ISAAC NEWTON'S HITHERTO UNDEMONSTRATED RULE FOR THE DISCOVERY OF IMAGINARY ROOTS*.

[Syllabus of Lecture delivered at King's College, London†, June 28, 1865. *Proceedings of the London Mathematical Society*, I. (1865—1866), pp. 1—16.]

LET $fx = 0$ be an algebraical equation of degree n.

Suppose $fx = a_0x^n + na_1x^{n-1} + \frac{1}{2}n(n-1)a_2x^{n-2} + \ldots\ldots\ldots + na_{n-1}x + a_n$;

$a_0,\ a_1,\ a_2 \ldots\ldots\ldots a_n$ may be termed the simple elements of fx.

Suppose

$$A_0 = a_0^2,\quad A_1 = a_1^2 - a_0a_2,\quad A_2 = a_2^2 - a_1a_3, \ldots\ A_{n-1} = a^2_{n-1} - a_{n-2}a_n,\quad A_n = a_n^2;$$

$A_0,\ A_1,\ A_2,\ \ldots\ldots\ldots\ A_n$ may be termed the quadratic elements of fx. $a_r,\ a_{r+1}$ is a succession of simple elements, and $A_r,\ A_{r+1}$ of quadratic elements.

$\left.\begin{matrix} a_r \\ A_r \end{matrix}\right\}$ is an *associated* couple of elements;

$\left.\begin{matrix} a_r\ a_{r+1} \\ A_rA_{r+1} \end{matrix}\right\}$ is an associated couple of *successions*.

A succession may contain a permanence or a variation of signs, and will be termed for brevity a permanence or variation, as the case may be. Each succession in an associated couple may be respectively a *permanence* or a *variation*. Thus an associated couple may consist of two permanences or two variations, or a superior permanence and inferior variation, or an inferior permanence and superior variation; these may be denoted respectively by the symbols pP, vV, pV, vP, and termed *double* permanences, *double* variations, permanence variations, variation permanences. The meaning of the simple symbols p, v, P, V speaks for itself.

* In chap. 2 of part 2 of the *Arithmetica Universalis*, entitled "De Formâ Æquationis."

† The substance of this lecture was communicated to the Mathematical Society of London (Professor De Morgan in the Chair), June 19, 1865.

Newton's rule in its complete form may be stated as follows:—On writing the complete series of quadratic under the complete series of simple elements of fx in their natural order, the number of double permanences in the associated series, or pair of progressions so formed, is a superior limit to the number of negative roots, and the number of variation permanences in the same is a superior limit to the number of positive roots in fx.

Thus the number of negative roots = or $< \Sigma pP$ } This is the Complete Rule as
„ „ „ „ positive roots = or $< \Sigma vP$ } given in other terms by Newton.

The rule for negative roots is deducible from that for positive, by changing x into $-x$.

As a corollary, the total number of real roots = or $< \Sigma pP + \Sigma vP$, that is, = or $< \Sigma P$.

Hence, the number of imaginary roots

$$= \text{or} > n - \Sigma P, \text{ that is, } = \text{or} > \Sigma V.$$

This is Newton's incomplete rule, or *first part* of complete rule, the rule as stated by every author whom the lecturer has consulted except Newton himself*.

By a group of negative signs, or a negative group, if we understand a sequence of negative signs, with no positive sign intervening, this incomplete rule may be stated otherwise, as follows:—

The number of imaginary roots of an algebraic function cannot be less than the number of negative groups in the complete series of its quadratic elements.

Arithmetical illustrations:—

Relation of Newton's complete rule to rule of Descartes. Newton's "Imaginary positives," "imaginary negatives" equivalent to ΣpV, ΣvV.

Newton's complete rule may be made to undergo its first step of generalization as follows:—

Let the two series of simple and quadratic elements of $f(x+\lambda)$ be formed, and the double permanences due to this transformation, say $\Sigma pP(\lambda)$, or more briefly, $pP(\lambda)$, be called the number of double permanences *proper* to λ, and in like manner $pP(\mu)$ the number of the same *proper* to μ.

* In some cases this rule appears to give very little information. For example, If we take the equation $x^3+3qx+r=0$, its immediate application will only show that if all the roots are real $q<0$. If, however, we make $y=x^\lambda$ and apply the criteria to the transformed equation in y, giving λ successive values between 1 and ∞, the rule will in fact lead (though by a difficult process) to the true discriminative criterion; this example serves to show that there is a deeper significance seated in the Newtonian criteria than does at first sight appear.

[N.B. $pP(0)$ becomes the notation for what has been termed above ΣpP.]

Then we have the theorem following:—

Supposing $\mu > \lambda$,

$$pP(\mu) = \text{or} > pP(\lambda),$$

or more exactly,

$$pP(\mu) - pP(\lambda) = (\mu, \lambda) + 2k,$$

where (μ, λ) denotes the number of real roots included between μ and λ, and k is zero or any positive integer.

Statement of this theorem in general terms.

This is to Newton's what Fourier's is to Descartes's.

Fourier's theorem recalled; it may be stated as follows:—

Form the *simple* elements appertaining to $f(x+\lambda)$ and to $f(x+\mu)$; then

$p(\mu) - p(\lambda) = (\mu, \lambda) + 2k$, briefly $p(\mu) - p(\lambda) = \text{or} > (\mu, \lambda)$; as a consequence

$p(0) - p(-\infty) = p(0)$ for $p(-\infty) = 0$; hence permanences in fx = superior limit to number of negative roots;

$p(\infty) - p(0) = v(0)$ for $p(\infty) = n$; hence variations in fx = superior limit to number of positive roots.

So from the new theorem, briefly No. 1 Theorem (presently to be established in a more general form), namely,

$$pP(\mu) - pP(\lambda) = \text{or} > (\mu, \lambda),$$

since

$$pP(0) - pP(-\infty) = pP(0); \text{ for } p(-\infty) = 0,\ pP(-\infty) = 0,$$

we obtain

$$pP(0) = \text{or} > (0, -\infty).$$

On examination it will be found that $P(+\infty) = P(-\infty) = n$ or $(n-2)$ according as the second quadratic element in $P(0)$ is positive or negative *. Thus

$$pP(\infty) - pP(0) = n - pP(0) = vV(0) + vP(0) + pV(0) = \text{or} > (\infty, 0),$$

or else

$$pP(\infty) - pP(0) = n - 2 - pP(0) = vV(0) + vP(0) + pV(0) - 2 = \text{or} > (\infty, 0),$$

which is not what is wanted; but by changing x into $-x$, and thereby commuting the variations and permanences of the simple elements of $f(x)$ one

* If $fx = (a_0, a_1, a_2, \ldots a_n)(x, 1)^n$, and if $B = a_1^2 - a_0 a_2$, and $h = \pm\infty$, the series of quadratic elements (when in each term only the highest power of h is retained) will be found to be a^2; Bh^2; Bh^4; $\ldots Bh^{2n-2}$; a^2h^{2n}.

into the other, the rule for the negative roots gives the rule required, namely, $vP(0) = \text{ or } > (\infty, 0)$ *.

No. 1 theorem may be transformed or, rather, otherwise stated as follows:—

The simple elements of $f(x+\lambda)$ are in fact

$$\frac{f^{(n)}\lambda}{1.2...n};\ \frac{1}{n}\frac{f^{(n-1)}\lambda}{1.2...(n-1)};\ \frac{1.2}{n(n-1)}\frac{f^{(n-2)}\lambda}{1.2...(n-2)};\ ...;\ \frac{1}{n}\frac{f'\lambda}{1};\ f\lambda.$$

It will not affect the successions either in this series itself or the derived series of quadratic elements if we reject the common factor $\frac{1}{\Pi(n)}$ from each term. It then becomes

$$f^n\lambda;\ f^{n-1}\lambda;\ 1.2f^{n-2}\lambda;\ 1.2.3f^{n-3}\lambda;\ 1.2.3.4f^{n-4}\lambda;\ ...;\ 1.2...nf\lambda;\ \text{etc.} \quad (A)$$

and similarly rejecting the positive factors 1^2, $(1.2)^2$, $(1.2.3)^2$, etc., from the second, third, fourth derived terms respectively, the derived series may be written

$$G_n\lambda;\ G_{n-1}\lambda;\ G_{n-2}\lambda;\ ....;\ G\lambda, \quad (B)$$

where in general

$$G_r\lambda = (f^r\lambda)^2 - \gamma_r f^{r-1}\lambda . f^{r+1}\lambda,$$

γ_r denoting the fraction $\frac{n-r+1}{n-r}$.

Theorem No. 2 stated. If γ_r in the above association of series $\begin{Bmatrix}A\\B\end{Bmatrix}$ be subject to satisfy the equation in differences $2-\gamma_r = \frac{1}{\gamma_{r+1}}$, provided γ_r remains always positive from $r=n$ to $r=1$ inclusive †, and if we call the number of double permanences in $\begin{Bmatrix}A\\B\end{Bmatrix}$, $pP(\lambda)$, then it is still true that

$$pP(\mu) - pP(\lambda) = (\mu, \lambda) + 2k.$$

The theorem will be subsequently simplified by integrating the above equation, and will be shown to include theorem No. 1.

* So in general $vP(\lambda) - vP(\mu) = (\mu, \lambda) + 2k'$, where k' is zero or a positive integer. We have thus a second theorem as general as the first, and the two will give different limits unless $k=k'$, that is, unless $P(\mu) = P(\lambda)$, for $2(k-k') = P(\mu) - P(\lambda)$. There is nothing corresponding to this in Fourier's theorem; for the two inequalities

$$p(\mu) - p(\lambda) = \text{ or } < (\mu, \lambda), \quad v(\lambda) - v(\mu) = \text{ or } < (\mu, \lambda)$$

constitute not distinct but identical assertions.

† As regards the necessity of the extreme limits n and 1, observe that unless γ_n were made positive the product of $\begin{bmatrix} f^{r+1}\epsilon & f^r\epsilon \\ G_{r+1}\epsilon & G_r\epsilon \end{bmatrix}$ (see C, p. 502) would not follow the sign of ϵ for the case of $r=n-1$, $G_{n-1}=0$; and unless γ_1 were made positive, $f''.f$ would not be positive (see p. 504) when $G_1=0$; consequently the three final associated pairs of signs as x increases might pass through the successive phases

$$\left.\begin{matrix} + & - & - \\ + & + & + \end{matrix}\right| \left.\begin{matrix} + & - & - \\ + & 0 & + \end{matrix}\right| \left.\begin{matrix} + & - & - \\ + & - & + \end{matrix}\right|,$$

and thus a double permanence would be *lost* in the ascending transit.

A theorem No. 3 exists derived from an order of considerations into which this Lecture will not enter. It gives much greater generality to the value of γ_r by the introduction of a second arbitrary parameter into the criteria: see footnote *, p. 508.

I proceed to establish theorem No. 2 by a method precisely analogous to that used in establishing Fourier's simpler one.

For brevity, by f^r understand $f^r x$; by $f^r(\pm\epsilon)$, understand $f^r(x\pm\epsilon)$; and so by G_r, $G_r(\pm\epsilon)$ understand $G_r x$, $G_r(x\pm\epsilon)$.

ϵ will denote an infinitesimal.

$$\left.\begin{array}{l}\text{When } f^r=0,\ f^r\epsilon=\epsilon f^{r+1}.\\ \text{When } f^r=0,\ f^{r+1}=0\ldots f^{r+i-1}=0, \text{ then } f^r\epsilon=\dfrac{\epsilon^i}{1.2\ldots i}.f^{r+i}.\end{array}\right\}$$

When $G_r=0$, $G_r(\epsilon)=\epsilon\dfrac{d}{dx}G_r$.

But
$$(f^r)^2-\gamma_r(f^{r-1})(f^{r+1})=0$$
and
$$\frac{d}{dx}G_r=(2-\gamma_r)f^rf^{r+1}-\gamma_r f^{r-1}f^{r+2}$$
$$=(2-\gamma_r)\frac{f^r}{f^{r+1}}\left\{(f^{r+1})^2-\frac{\gamma_r}{2-\gamma_r}\cdot\frac{f^{r-1}.f^{r+1}}{f^r}f^{r+2}\right\}$$
$$=\frac{f^r}{\gamma_{r+1}f^{r+1}}\left\{(f^{r+1})^2-\gamma_{r+1}f^rf^{r+2}\right\},$$
or
$$G_r(\epsilon)=\frac{\epsilon}{\gamma_{r+1}}\cdot\frac{f^r}{f^{r+1}}G_{r+1}{}^*. \tag{C}$$

Similarly, when
$$G_r=0,\quad G_{r+1}=0\ldots G_{r+i-1}=0,$$
it may be proved that
$$G_r(\epsilon)=\frac{\epsilon^i}{\Pi(i)\,\gamma_{r+1}\gamma_{r+2}\ldots\gamma_{r+i}}\frac{f^r}{f^{r+i}}G_{r+i}{}^\dagger.$$

* Consequently if we write $\begin{bmatrix}f^{r+1}\epsilon & f^r\epsilon\\ G_{r+1}\epsilon & G_r\epsilon\end{bmatrix}$ the product of these four quantities obeys the sign of ϵ.

† For example, If $G_r=0$ and $G_{r+1}=0$, we have

$$\frac{dG_r}{dx}=0,\ \text{and}\ \frac{d^2G_r}{dx^2}=(2-2\gamma_r)f^rf^{r+2}+(2-\gamma_r)(f^{r+1})^2-\gamma_rf^{r-1}f^{r+3}$$
$$=(2-2\gamma_r+2\gamma_{r+1}-\gamma_r\cdot\gamma_{r+1})f^rf^{r+2}-\gamma_rf^{r-1}.f^{r+3}.$$

But
$$2-2\gamma_r+2\gamma_{r+1}-\gamma_r\cdot\gamma_{r+1}=3-2\gamma_r=\frac{2}{\gamma_{r+1}}-1=\frac{1}{\gamma_{r+1}\cdot\gamma_{r+2}}.$$

Thus
$$\frac{d^2G_r}{dx^2}=\frac{1}{\gamma_{r+1}\cdot\gamma_{r+2}}\left\{f^rf^{r+2}-\gamma_r\gamma_{r+1}\gamma_{r+2}f^{r-1}.f^{r+3}\right\}$$
$$=\frac{f^r}{\gamma_{r+1}\gamma_{r+2}f^{r+2}}\left\{(f^{r+2})^2-\gamma_{r+2}f^{r+1}.f^{r+3}\right\},$$

and consequently
$$G_r(\epsilon)=\frac{f^r}{\gamma_{r+1}\gamma_{r+2}f^{r+2}}\cdot G_{r+2}\frac{\epsilon^2}{1.2}.$$

We can now trace the law of the change in the number of double permanences in the associated pair of series,

$$\left.\begin{matrix} f^n, & f^{n-1}, & f^{n-2}, \ldots f^1, & f, \\ G_n, & G_{n-1}, & G_{n-2}, \ldots G_1, & G, \end{matrix}\right\}, \text{ where } G_r = (f^r)^2 - \gamma_r f^{r-1} \cdot f^{r+1},$$

as x increases *continuously*.

No change can take place except at the instant when one or more of the terms in the inferior or superior series, or in both, simultaneously become zero.

1°. Suppose a single term in the upper series as f^r to become zero.

Writing down the sequence f^{r+1}, f^r, f^{r-1} in conjunction with the associated terms,

$$G_{r+1},\ G_r,\ G_{r-1},$$

G_{r+1}, G_{r-1} are seen to be necessarily positive, and G_r of the contrary sign to f^{r+1}, f^{r-1}.

The number of cases depending on the signs of f^{r+1} and f^{r-1} are four, but reducible to two essentially distinct ones as below *,

$$\left\{\begin{matrix} + & 0 & + \\ + & - & + \end{matrix}\ \middle|\ \begin{matrix} + & 0 & - \\ + & + & + \end{matrix}\right\} \qquad \text{(D)}$$

For $(x-\epsilon)$ these become respectively

$$\begin{matrix} + & - & + \\ + & - & + \end{matrix}\ \middle|\ \begin{matrix} + & - & - \\ + & + & + \end{matrix}$$

For $(x+\epsilon)$ they become

$$\begin{matrix} + & + & + \\ + & - & + \end{matrix}\ \middle|\ \begin{matrix} + & + & - \\ + & + & + \end{matrix}$$

And there is neither gain nor loss of double permanences.

2°. Suppose a single term in the lower series to become zero. From the value of G_r in terms of f^{r+1}, f^r, f^{r-1}, it follows that the two extremes of these three must have the same sign; and the signs of f^{r+1}, f^r, G_{r+1}, G_{r-1} give rise to sixteen cases reducible to the four following essentially distinct ones †:—

$$\begin{matrix} + & + & + \\ + & 0 & + \end{matrix} \qquad \begin{matrix} + & - & + \\ + & 0 & + \end{matrix} \qquad \begin{matrix} + & + & + \\ + & 0 & - \end{matrix} \qquad \begin{matrix} + & - & + \\ + & 0 & - \end{matrix}\ ‡$$

* For the simultaneous reversal of the signs of the upper line will not affect the reasoning.

† For neither the reversal of the signs in the upper nor in the lower line will affect the reasoning.

‡ The number of occurrences of the combination $\begin{matrix} + & + & + \\ + & 0 & + \end{matrix}$ as x ascends from λ to μ will, except for special cases, be the value of k, and the number of occurrences of $\begin{matrix} + & - & + \\ + & 0 & + \end{matrix}$, in like manner, the value of k'; k, k' having the meanings attributed to them at footnote *, p. 501.

Immediately *after* the transit of the root, see footnote *, p. 502, these become of the form

$$\begin{array}{ccc} + & + & + \\ + & + & + \end{array} \qquad \begin{array}{ccc} + & - & + \\ + & - & + \end{array} \qquad \begin{array}{ccc} + & + & + \\ + & + & - \end{array} \qquad \begin{array}{ccc} + & - & + \\ + & - & - \end{array}$$

And immediately *before* the transit they were of the form

$$\begin{array}{ccc} + & + & + \\ + & - & + \end{array} \qquad \begin{array}{ccc} + & - & + \\ + & + & + \end{array} \qquad \begin{array}{ccc} + & + & + \\ + & - & - \end{array} \qquad \begin{array}{ccc} + & - & + \\ + & + & - \end{array}$$

In the first of the four cases there is a *gain* of two double permanences in ascending from $x-\epsilon$ to $x+\epsilon$; in the other three cases there is neither gain nor loss.

Thus for a single vanishing of an *intermediate* term in the upper or lower series double permanences may be gained, as x continuously *increases*, but can never be lost.

The same conclusion may be also established in a similar manner when several consecutive terms of the lower series forming a group vanish simultaneously, without the associated upper terms any of them vanishing; and also when *two* or more consecutive terms in the upper series vanish, which necessitates all the associated terms in the lower series also vanishing; and the law which limits the increase may be ascertained, and such increase may be shown to be always an even number.

But these cases are singular cases, and may be met at once by the consideration (equally applicable to the proof of Fourier's simpler theorem) that if two or more functions of x depending on fx vanish simultaneously, this must be by virtue of one or more relations existing between the coefficients of fx; and by giving infinitesimal variations to the coefficients, we shall leave the criteria [the two sets of terms corresponding to given limits] virtually undisturbed, and may manage that the coincidence of the transits will no longer take effect; at the same time *in general* the character of the roots will remain unaltered as regards the number of real and imaginary ones; so that the singular cases come under the operation of the same law as the general case. There is, however, an apparent possible exception to this reasoning, namely, when fx possesses one or more groups of equal roots, in which case an infinitesimal variation in the coefficients *may* be accompanied with a change of character in the roots, such as a passage from real equal to imaginary pairs: to meet this objection without embarrassing oneself with intricate considerations as to the signs to be given to the infinitesimal variations in order to avoid liability to such change, it is better to adopt the same kind of proof as is usual with Fourier's method, which there is no difficulty in doing by aid of the expressions given above with reference to the value of $f^n(\epsilon)$, and consequently also of $f^{n+1}(\epsilon)$, $f^{n+2}(\epsilon)$, $\ldots f^{n+i-2}(\epsilon)$, when f^n, f^{n+1}, $\ldots f^{n+i-1}$ are all zeros, from which may easily be deduced also the special

expressions for $G_n(\epsilon)$, $G_{n+1}(\epsilon)$, ... $G_{(n+i-2)}(\epsilon)$ applicable to this case; and again, as regards the case when a group of lower terms vanish without the associated upper ones so doing, by aid of the general expressions given for the G functions last above written. But time would not suffice for going into these details in a single lecture*.

We must, lastly, consider what happens when one or more of the terms at either extremity vanish.

f^n and G_n are constants, and can never vanish, and G is a square, and essentially positive.

But suppose x to become a root of fx, so that $f=0$, then the last pair of couples in the associated series immediately before the transit will be $\begin{matrix} f'; & -\epsilon f' \\ f'^2; & \epsilon^2 f'^2 \end{matrix}$, and immediately subsequent to the transit $\begin{matrix} f'; & \epsilon f' \\ f'^2; & \epsilon^2 f'^2 \end{matrix}$; thus there will be one double permanence *gained* when a simple root is passed over.

Suppose now x to become a root of the ith order of repetition, equivalent, that is, to i roots passed over, so that $f=0$, $f^1=0$, $f^2=0$, ... $f^{i-1}=0$, then the last $i+1$ superior terms of the Association become, immediately after the transit,

$$f^i;\ \frac{\epsilon}{1}f^i;\ \frac{\epsilon^2}{1.2}f^i;\ \frac{\epsilon^3}{1.2.3}f^i;\ \ldots\ldots\ \frac{\epsilon^i}{1.2\ldots i}f^i,$$

tantamount to

$$1;\ \epsilon;\ \frac{\epsilon^2}{1.2};\ \frac{\epsilon^3}{1.2.3};\ \ldots\ldots\ \frac{\epsilon^i}{1.2\ldots i};$$

and the lower terms associated therewith, rejecting the common factor f^i, and certain obviously superfluous positive numerical factors besides, become

$$1;\ \left(1-\frac{\gamma_{i-1}}{2}\right)\epsilon^2;\ \left(1-\frac{2\gamma_{i-2}}{3}\right)\epsilon^4;\ \ldots\left(1-\frac{(1-i)\gamma_1}{i}\right)\epsilon^{2i-2};\ \epsilon^{2i}.$$

By hypothesis $2-\gamma_{i-1}>0$. Hence there is obviously *one* double permanence, namely at the first couple of pairs above written, corresponding to a value of x immediately greater than a multiple root, whereas for a value of x immediately less than such root, there is no double permanence at all corresponding, for the upper series will consist exclusively of variations of sign when ϵ is changed into $-\epsilon$; as regards the other terms in the lower series, they will not necessarily be all positives, unless, *in order to meet the case of* equal *roots in fx*, we determine the value of γ_r in the equation $2-\gamma_r=\dfrac{1}{\gamma_{r+1}}$, subject to the condition that γ_r shall be not only positive, but also subject to the limitation $\gamma_r<\dfrac{i+1-r}{i-r}$ for all values of i not superior to n

* See Appendix for summary of demonstration applicable to these hypotheses.

[r of course being supposed less than i]. This latter limitation, however, will eventually be seen to be included in the former*. This being the case, it follows that the number of double permanences appertaining to the associated series

$$f^n, \quad f^{n-1}, \quad f^{n-2}, \ldots f', \quad f$$
$$G_n, \quad G_{n-1}, \quad G_{n-2}, \ldots G_1, \quad G$$

will be increased by at least as many units as there are real roots, equal† or unequal, passed over as we ascend from λ to μ, and that the excess, if any, of the increase of such number over the number of real roots will be an even integer.

To determine the value of γ_r, make $\gamma_r = \frac{U_r}{U_{r+1}}$, then

$$2 - \frac{U_r}{U_{r+1}} = \frac{U_{r+2}}{U_{r+1}},$$

or

$$U_{r+2} - 2U_{r+1} + U_r = 0,$$

of which the general solution is $U_r = A + B(r-1)$; so that

$$\gamma_r = \frac{A + B(r-1)}{A + Br}.$$

If we write $A = n$, $B = -1$, we obtain

$$\gamma_{n-1} = \frac{2}{1}, \quad \gamma_{n-2} = \frac{3}{2}, \;\ldots\ldots\ldots\; \gamma_2 = \frac{n-1}{n-2}, \quad \gamma_1 = \frac{n}{n-1}.$$

These are the values of γ_r in the theorem No. 1, and they satisfy the *two* conditions above stated. For (1) γ_r is positive, and (2)

$$\gamma_r = \frac{n-r+1}{n-r} < \frac{i-r+1}{i-r}$$

for all values of $i < n$‡.

Hence theorem No. 1 is contained in theorem No. 2, and Newton's theorem is contained in theorem No. 1, so that it is a corollary in the

* It is shown at p. 507, that $\gamma_r = \frac{\nu+r-1}{\nu+r}$, ν having any value not intermediate between $-n$ and 0. When ν is positive $\gamma_r < 1 < \frac{i+1-r}{i-r}$.

Again, when ν is negative let $\nu = -\nu'$, then $\nu' =$ or $> n =$ or $> i$.

Hence $\gamma_r = \frac{\nu'-r+1}{\nu'-r} =$ or $< \frac{i+1-r}{i-r}$, and the required condition is still satisfied. The sole exception is when $\nu = -n$ and $i = n$, that is, when the original Newtonian criteria are those employed, and the equation has all its roots equal to one another, for which case see footnote ‡, below.

† Each root repeated m times counts as m roots.

‡ The degree of fx being n, i is necessarily less than n, unless all the roots are equal, a case which may be considered excluded, as then all the Newtonian criteria become zero.

second order of derivation from our theorem No. 2. The general value of γ_r is $\dfrac{\nu+(r-1)}{\nu+r}$, ν being any real quantity whatever not *intermediate* between 0 and $-n$. To obtain theorem No. 1 we must make $\nu=-n$. In that case $\gamma_n=\dfrac{1}{0}$ and accordingly when $G_{n-1}=0$, $G_{n-1}(\epsilon)$ by the formula at p. 502 should vanish. It will easily be found that for this peculiar value $G_{n-1}(x)$ becomes independent of x, in fact a constant*.

Theorem No. 2 may also be stated as follows:—

If $$c_0+c_1x+c_2x^2+\ldots\ldots+c_nx^n=f(x),$$

and ν be any real quantity not *intermediate* between 0 and $-n$, and if

$$c_0,\ \frac{c_1}{\nu},\ \frac{1.2c_2}{\nu(\nu+1)},\ \frac{1.2.3c_3}{\nu(\nu+1)(\nu+2)},\ \ldots\ldots\ldots$$

say, $$\alpha_0,\ \alpha_1,\ \alpha_2,\ \alpha_3,\ \ldots\ldots,$$

be taken as the simple elements of fx, and

$$\alpha_0^2,\ \alpha_1^2-\alpha_0\alpha_2,\ \alpha_2^2-\alpha_1\alpha_3,\ \ldots\ldots\ \alpha^2_{n-1}-\alpha_{n-2}\alpha_n,\ \alpha_n^2,$$

say, $$A_0,\ A_1,\ A_2\ \ldots\ldots\ldots\ldots\ldots\ A_{n-1},\ A_n,$$

as the quadratic elements of the same; and if we understand by the (cA) association the paired series

$$\begin{Bmatrix} c_0\ c_1\ldots\ldots\ldots c_n \\ A_0A_1\ldots\ldots\ldots A_n\end{Bmatrix},$$

and if $pP(\lambda)$ signifies the number of double permanences in the (cA) association corresponding to $f(x+\lambda)$, then

$$pP(\mu)-pP(\lambda)=(\mu,\lambda)+2k\dagger,$$

where k is zero or some positive integer.

When $\nu=-n$, this theorem becomes effectively identical with theorem No. 1 in its original form.

The existence of a theorem No. 3, including No. 2, and containing two arbitrary parameters, becomes apparent from the consideration that $fx=0$ will have the same *finite* roots as $Fx=0$, where

$$Fx=\epsilon x^{n+i+j}+x^ifx+\eta,$$

* Analogous observations apply to the other extreme or limiting case, namely, that where ν is made equal to zero, for then $\gamma_1=0$ so that $G_1=(f^1)^2$, and is always positive, so that when f^1 becomes 0, for this particular system of γ's, the three last couples in the pair of progressions before such transit may be $\begin{matrix}+&-&+\\+&+&+\end{matrix}$, and after transit $\begin{matrix}+&+&+\\+&+&+\end{matrix}$; and thus contrary to what usually happens (see p. 503), there may on the above hypothesis be an *extra* gain of two double permanences and loss of two variation permanences, thus increasing the values of k and k' in footnote *, p. 501, which of course does not affect the validity of the theorem.

† And of course also $vP(\lambda)-vP(\mu)=(\mu,\lambda)+2k'$, where k' is zero, or some positive integer.

ϵ, η being two infinitesimals: this brings in two parameters, which, so far as this particular method of demonstration of their existence applies, would seem to be necessarily integer, but, from the nature of the question, must obviously admit of some wider definition*.

The labours of all preceding writers on this subject have been confined exclusively to the *imperfect* form of Newton's theorem; nor previously to the lecturer's communication of his Trilogy of Algebraical researches to the Royal Society of this year was anything more made out of it than to show that if *any* negative terms occur in the quadratic elements, there must be *some* imaginary roots†; but this becomes immediately apparent if we consider f a homogeneous function of the nth degree in x and y; for then the number of imaginary roots of $\frac{x}{y}$ or $\frac{y}{x}$ in $f(x, y)$ cannot be fewer than in

* If we use theorem No. 2, j may be made zero (it will be found) without any loss of generality; in fact if it be made greater than zero, the result obtained will be included in that obtained by making it equal to zero. On the other hand, if we use theorem No. 1, retaining for i, j their general values, the result obtained will be the same as that which flows from the use of theorem No. 2, with $j=0$, except that it will be specialized by ν being restricted to integer values only. By aid of the method above indicated, we may substitute as the values of $a_0, a_1, a_2, \ldots a_n$ in p. 507, [see *Erratum*, p. 513]

$$c_0;\quad \frac{i+1}{\nu}c_1;\quad \frac{(i+1)(i+2)}{\nu(\nu+1)}c_2;\quad \frac{(i+1)(i+2)(i+3)}{\nu(\nu+1)(\nu+2)}c_3;\ \ldots\ldots$$

provided $\nu=$ or $>i$ or $\nu=$ or $<-n$ and $i=$ or >0. The particular form of demonstration indicated requires i to be integer; but this restriction I have great reason to believe, indeed have scarcely a doubt, is unnecessary and may be neglected.

$$1;\quad \frac{\nu}{i+1};\quad \frac{\nu(\nu+1)}{(i+1)(i+2)};\ \ldots\ldots$$

may be termed a divestible system of coefficients; such system is a generalization of the ordinary binomial system

$$1;\quad n;\quad \frac{n(n-1)}{2};\ \ldots\ldots$$

If we call $\alpha, \beta, \gamma \ldots$ any system of the like nature, and form the equation

$$\alpha x^n+\beta x^{n-1}+\gamma x^{n-2}+\ \ldots\ldots\ldots=0,$$

it is obvious, or at least very readily proved, that the *simple* roots of this equation must all, or all but one, be imaginary. It is not unlikely that every system $\alpha, \beta, \gamma \ldots$ which satisfies the above condition and one or more other general conditions, may be employed as a *divestible system*. The particular system, involving two arbitrary parameters, above given will be found to satisfy the further condition that *all* its Newtonian non-trivial criteria (the $A_1, A_2, \ldots\ldots A_{n-1}$ of p. 488) become negative. When $n=2$ or $n=3$, the second condition implies the first; and for these cases it is easily proved that every system of quantities which satisfies the second (and therefore the first condition) forms a valid system of divestible factors. If we make $i=0$ and $\nu=1$, we learn that the equation

$$1+x+x^2+\ \ldots\ +x^n=0$$

can never have more than one real root; if we make $i=0$ and $\nu=\infty$, we learn the same of the equation

$$1+x+\frac{x^2}{1\,.\,2}+\frac{x^3}{1\,.\,2\,.\,3}+\ \ldots\ldots\ +\frac{x^n}{1\,.\,2\ldots n}=0.$$

Newton's own rule would only, in these and such like examples, reveal the existence of a single pair of imaginary roots.

† *See Postscript.*

$\left(\frac{d}{dx}\right)^i\left(\frac{d}{dx}\right)^j f(x, y)$, i and j being *any* two integers: making $i+j=n-2$, and giving j all values from 0 to $n-2$ in succession, it will readily be seen that Newton's criteria are the quantities which respectively serve to determine whether the quadratic *functions* obtained by these various substitutions have real or imaginary roots. If any one of them is negative, one of those derivatives has imaginary roots, and therefore the primitive function $f(x, y)$ has at least one such pair*.

Newton's assertion (if it is his own assertion), that his rule will in *general* give the *actual* number, and not merely a superior limit to such number of real roots, is certainly more than questionable. For if $f'x$, $f''x$, $f'''x$, $f''''x$, $f'''''x$ have respectively not more than i_1, i_2, i_3, i_4, i_5 pairs of imaginary roots, this rule *cannot* reveal the existence of more than j pairs, where j is the *least* number of the set of numbers i_1+1, i_2+1, i_3+2, i_4+2, i_5+3, ...

Geometrical illustration.

Ortæ à Cartesio, quam Newtonus insuper auxit,
Doctrinae, en! demum, fons et origo patent.

Postscript.

Dr J. R. Young contests the accuracy of the assertion on p. 508, and claims to have demonstrated Newton's rule twenty years ago. Call the equation

$$(a, b, c, d, e, \ldots\ldots \between x, 1)^n = 0\ ;$$

the derived cubics are of course to a factor *près* respectively

$$(a, b, c, d \between x, 1)^3;\ (b, c, d, e \between x, 1)^3;\ (c, d, e, f \between x, 1)^3;\ \&c.$$

* In like manner, if $[a, b, c, d]$ represent the discriminant with its sign reversed of

$$(a, b, c, d \between x, y)^3,$$

the fact of any of the quantities

$$[a, b, c, d],\quad [b, c, d, e],\quad [c, d, e, f], \&c.$$

becoming negative will imply the existence of some imaginary roots in

$$(a, b, c, d, e, f \ldots \between x, y)^n,$$

and so in general. One would be glad to know whether by aid of a complete table of the discriminants of a function of the nth order and of its successive derivatives (respectively 2, 3, ... $(n-1)$ in number), it is possible or not to ascertain the exact number of its real roots.

When n is 4 the only dubious case arising under such table is that where the discriminant of the quartic itself is positive but those of its two derived cubics and three derived quadratics each negative. In such case it remains to be ascertained whether it is true or not that the roots of the quartic cannot all be imaginary, or (which is here the same thing) must all of necessity be real.

It seems desirable to show *à priori* that when the roots of

$$fx = x^n + npx^{n-1} + \tfrac{1}{2}n(n-1)qx^{n-2} + \&c.$$

are all real, the criteria p^2-q, q^2-pr, &c. are necessarily all positive. This endoscopic method of proof I have not yet been able to complete; but I have noticed that if $\alpha, \beta, \gamma, \delta \ldots$ are the roots of fx, and

$$\frac{fx}{(x-\alpha)(x-\beta)} = x^{n-1} + \lambda x^{n-2} + \mu x^{n-3} + \ldots,$$

the following somewhat curious relations obtain, namely,

$$p^2 - q = \Sigma[(\alpha-\beta)^2(\lambda^2-\mu)];\quad q^2 - pr = \Sigma[(\alpha-\beta)^2(\mu^2-\lambda\nu)], \&c.$$

It is with these cubics that Dr Young, in his argument (if it may be called so) exclusively deals. Omitting merely superfluous observations his ratiocination runs as follows:—"If any of these limiting cubics indicate imaginary roots when submitted to the criteria, such indications will imply imaginary roots in the proposed equation. But several indications, apparently distinct, may offer themselves in those equations which, upon closer examination, may be found to be necessarily dependent or concurrent. Distinct imaginary pairs can of course be inferred only from independent and non-concurring conditions. We have therefore to inquire how these are to be discovered in the above series of equations. And first we may remark that since only one imaginary pair can enter into a cubic equation, it follows that whether the criterion of imaginary roots is satisfied by the three leading terms of any of the above cubics or by the three final terms, or simultaneously by both sets of three, one imaginary pair, and one only, is implied. Hence when both sets of three terms, furnished by any cubic, fulfil the proposed conditions, these conditions, though really independent, that is, not necessarily implied one in the other, nevertheless necessarily concur in indicating the same thing. Thus only a single imaginary pair can be inferred from any one of the limiting cubics, whether the criterion is satisfied for one set of three terms or for the two consecutively.

. . . .

"If the first set of three (that is, the leading terms in the first set) satisfy the criterion, we can immediately infer the existence of one imaginary pair. If the next set (the final terms of the same cubic) also satisfy it, the preceding condition merely recurs, and supplies no additional information. In this case the following set of three, the leading terms of the next cubic..., furnish the same concurring condition..., and so on, until we arrive at a set of three terms for which the condition fails, thus putting a stop to the series of concurring indications, and preparing the way for new and distinct conditions altogether unconnected with the former. As soon as the criterion again holds, the condition being thus entirely independent of, and unconnected with, the former, must imply another and distinct imaginary pair, and so on to the end of the series." Dr Young subsequently goes on to observe that the criteria of the successive cubics (discarding repetitions) are identical with those of the original equation, which he says, "we now know to be so connected together, that if, when proceeding from one set of three terms in the equation to the three next in order, the consecutive criteria both have place, the recurrence is to be regarded merely as a second indication of the same thing—the existence of a single imaginary pair; and that as soon as the condition fails, preparation is made for a new and independent indication, and so on, until all the sets of three have been examined. Hence the indications that are really non-concurrent, and consequently the number of imaginary

roots inferrible from them, may be noted" according to a rule "which is virtually the same as that of Newton."

This is the sum and substance, in a simplified form, of Dr Young's so-called proof. "It is such stuff as dreams are made of," and, culminating as it does in a palpable *petitio principii*, does not need a detailed refutation at the hands of the author of this lecture. It is not by such vague rhetorical processes, but by quite a different kind of mental toil, that the truths of science are to be won, or a way opened to the inner recesses of the reason.

Appendix on the singular cases referred to in the text foregoing, *see p.* 504.

It may easily be shown that there are only four hypotheses admissible concerning vanishing f's and G's.

1°. f^r may vanish, but not the adjacent terms nor the associated term G_r.

2°. G_r may vanish, but not the adjacent terms nor the associated term f^r.

3°. $G_r, G_{r+1}, \ldots G_{r+i-1}$ (i being greater than unity) may vanish *without any of* the associated terms vanishing.

4°. $f^r, f^{r+1}, \ldots f^{r+i-1}$ (i being greater than unity) may vanish, and *consequently* also $G_r, G_{r+1}, \ldots G_{r+i-1}$ *all* vanish.

Hypotheses (1) and (2), which are the only cases that can happen in general, have been discussed in the text preceding.

Now consider the 3rd hypothesis.

1st. Suppose i any even number (say 4), so that

$$G_r = 0,\ G_{r+1} = 0,\ G_{r+2} = 0,\ G_{r+3} = 0,$$

then $\qquad f^{r-1}, f^{r+1}, f^{r+3}$ have the same sign *inter se*,

and $\qquad f^r, f^{r+2}, f^{r+4}$ have the same sign *inter se*,

and there will be four essentially distinct cases as below, representing the partial association

$$\begin{Bmatrix} f^{r-1}, & f^r, & f^{r+1}, & f^{r+2}, & f^{r+3}, & f^{r+4} \\ G_{r-1}, & G_r, & G_{r+1}, & G_{r+2}, & G_{r+3}, & G_{r+4} \end{Bmatrix}$$

at the moment of x taking the critical value which causes the G's to vanish, namely—

$$\begin{array}{cccccc|cccccc|cccccc|cccccc|} + & + & + & + & + & + & + & + & + & + & + & + & + & - & + & - & + & -. & + & - & + & - & + & - \\ + & 0 & 0 & 0 & 0 & + & + & 0 & 0 & 0 & 0 & - & + & 0 & 0 & 0 & 0 & +. & + & 0 & 0 & 0 & 0 & - \end{array}$$

Immediately *after* the passage of the critical value these signs become

$$\begin{array}{cccccc|cccccc|cccccc|cccccc|}
+ & + & + & + & + & + & + & + & + & + & + & + & + & - & + & - & + & - & + & - & + & - & + & - \\
+ & + & + & + & + & + & + & + & + & + & + & - & + & - & + & - & + & + & + & - & + & - & + & -
\end{array}$$

as appears from the expression for $G_r(\epsilon)$ in terms of G_{r+i}, f^r, f^{r+i} given at p. 502; and immediately *before* the passage these partial associations are of the form

$$\begin{array}{cccccc|cccccc|cccccc|cccccc|}
+ & + & + & + & + & + & + & + & + & + & + & + & + & - & + & - & + & - & + & - & + & - & + & - \\
+ & - & + & - & + & + & + & - & + & - & + & - & + & + & + & + & + & + & + & + & + & + & + & -
\end{array}$$

There will thus be a gain of 4 double permanences in passing upwards through the critical values in the two first cases, and neither gain nor loss in the two last.

2nd. If i is any odd number, say 3, there will still be 4 cases, for which respectively the signs after transit will be

$$\begin{array}{ccccc|ccccc|ccccc|ccccc}
+ & + & + & + & + & + & + & + & + & + & + & - & + & - & + & + & - & + & - & + \\
+ & + & + & + & + & + & + & + & + & - & + & - & + & - & + & + & - & + & - & -
\end{array}$$

and before transit,

$$\begin{array}{ccccc|ccccc|ccccc|ccccc}
+ & + & + & + & + & + & + & + & + & + & + & - & + & - & + & + & - & + & - & + \\
+ & - & + & - & + & + & - & + & - & - & + & + & + & + & + & + & + & + & + & -
\end{array}$$

showing a gain in the upward passage of 4, 2, 0, 0 double permanences respectively.

Lastly, let us consider the 4th hypothesis.

1st. Suppose i is any even number (say 4), and that

$$f^r = 0,\ f^{r+1} = 0,\ f^{r+2} = 0,\ f^{r+3} = 0,$$

then, after the transit, the f and G associated series from $r+4$ to $r-1$ become, writing for brevity's sake ϕ for f^{r+4} and ψ for f^{r-1},

$$\phi;\quad \epsilon\phi;\quad \frac{\epsilon^2}{1\,.\,2}\phi;\quad \frac{\epsilon^3}{1\,.\,2\,.\,3}\phi;\quad \frac{\epsilon^4}{1\,.\,2\,.\,3\,.\,4}\phi;\quad \psi;$$

$$\phi^2;\ k_1\epsilon^2\phi^2;\ k_2\epsilon^4\phi^2;\quad k_3\epsilon^6\phi^2;\quad -\frac{\epsilon^3}{1\,.\,2\,.\,3}\gamma_r\phi\psi;\ \psi^2;$$

where γ_r is positive, and, as follows from what is proved in footnote *, p. 502, k_1, k_2, k_3 also will be all positive. Thus there will be essentially only two cases, according as ϕ, ψ have the same or contrary signs; after transit, the association will be of the form

$$\left|\begin{array}{cccccc} + & + & + & + & + & + \\ + & + & + & + & - & + \end{array}\right| \text{ or } \left|\begin{array}{cccccc} + & + & + & + & + & - \\ + & + & + & + & + & + \end{array}\right|$$

and before transit (changing ϵ into $(-\epsilon)$) of the form

$$\left|\begin{array}{cccccc} + & - & + & - & + & + \\ + & + & + & + & + & + \end{array}\right| \text{ or } \left|\begin{array}{cccccc} + & - & + & - & + & - \\ + & + & + & + & - & + \end{array}\right|$$

showing a gain of 2 double permanences, or else of 4 such.

2nd. If i is any odd number, as 3, the associated series corresponding to the values of r, from $r+3$ to $r-1$, using ϕ to denote f^{r+3}, will be

$$\phi; \quad \epsilon\phi; \quad \frac{\epsilon^2}{1.2}\phi; \quad \frac{\epsilon^3}{1.2.3}\phi; \quad \psi;$$

$$\phi^2; \; k_1\epsilon^2\phi^2; \; k_2\epsilon^2\phi^2; \; -\frac{\epsilon^2}{1.2}\gamma_r\phi\psi; \; \psi^2,$$

and thus the signs after transit will be

$$\left|\begin{matrix} + & + & + & + & + \\ + & + & + & - & + \end{matrix}\right| \text{ or } \left|\begin{matrix} + & + & + & + & - \\ + & + & + & + & + \end{matrix}\right|$$

and before transit

$$\left|\begin{matrix} + & - & + & - & + \\ + & + & + & - & + \end{matrix}\right| \text{ or } \left|\begin{matrix} + & - & + & - & - \\ + & + & + & + & + \end{matrix}\right|$$

showing a gain of 2 double permanences on either supposition in the ascent from $x-\epsilon$ to $x+\epsilon$. And so in general from the 3rd and 4th (the two singular) hypotheses, whatever the value of i may be, an even number of double permanences may be gained in passing upwards through a critical value of x, but none can ever be lost *: this completes the demonstration.

* The number of double permanences gained on the 3rd hypothesis will be in the 4 subcases respectively $2i$, $2i$, 0, 0 if $2i$ consecutive G's vanish, and $2i+2$, $2i$, 0, 0 if $2i+1$ of them vanish; on the 4th hypothesis, in the 2 subcases the respective numbers gained will be $2i-2$, $2i$ if $2i$ consecutive f's vanish, and $2i$, $2i$ if $2i+1$ of them vanish: this statement requires a slight modification for the particular form of the theorem No. 2 corresponding to $\nu=0$. It may be noticed that in an exhaustive study of this theorem two sorts of singularities occur separately or in combination, namely, those arising out of the form of the equation, and those imparted to the criteria by giving critical values to the limited arbitrary parameter.

ERRATUM.

Theorem 3 is stated erroneously in footnote (*), p. 508. Correctly stated it furnishes a wider generalization of Newton's own theorem than can be obtained directly from the theorem 1 or 2 of the text, but not a generalization of those theorems themselves. It should be as follows:—

If $$fx = a_0 + \frac{\nu}{i+1}a_1x + \frac{\nu(\nu+1)}{(i+1)(i+2)}a_2x^2 + \&c.,$$

and we form the pair of progressions

$$\begin{matrix} a_0; & \nu a_1; & \nu^2 a_2; & \nu^3 a_3; & \ldots\ldots \\ a_0^2; & a_1^2 - a_0a_2; & a_2^2 - a_1a_3; & a_3^2 - a_2a_4; & \ldots\ldots \end{matrix}$$

where $\nu =$ or $> i$ or else $=$ or $< -n$,

then if i is a positive *integer* the number of double permanences is a limit to the number of the negative roots, and the number of variation permanences to that of the positive roots in fx. *Possibly* this theorem continues to subsist when i is *any* positive quantity.

85.

OBSERVATIONS SUR UN ARTICLE DE M. POULAIN.

[*Les Mondes*, XI. (1866), pp. 435—437.]

C'EST avec une vive satisfaction que j'ai trouvé dans *les Mondes* le compte rendu fait par M. Poulain de mon travail sur le théorème de Newton. Je m'estime fort heureux d'avoir rencontré dans le savant membre de la compagnie de Jésus un interprète aussi fidèle. L'exposition ne laisse rien à désirer en fait de précision et de clarté. Permettez-moi de prendre moi-même la parole dans votre journal, pour ajouter quelques nouveaux développements propres à faire voir toute l'étendue de la théorie en question.

Et tout d'abord, ne voulant pas m'attribuer ce qui appartient à autrui, je tiens à faire remarquer que le théorème nommé par moi *Newton's complete rule* a été donné par Newton lui-même. Je l'appelle ainsi pour le distinguer de l'autre, que je nomme par opposition *Newton's incomplete rule.* C'est ce dernier théorème seul qui a occupé l'attention de Maclaurin, Waring, Euler, et de tous les autres auteurs qui ont voulu traiter cette question. Voilà pourquoi on l'a désigné de préférence sous le nom de *théorème de Newton.* Newton commence par énoncer la règle imparfaite qui, suivant la remarque de M. Poulain, n'est que le corollaire du théorème premier. Mais après cet énoncé et quelques applications numériques, il ajoute le théorème dans sa forme complète. Il est très-curieux de remarquer comment ce théorème paraît se lier dans l'esprit de Newton au théorème de Descartes. Il semble avoir eu l'idée que, dans un certain sens transcendant, chaque variation de signes des coëfficients simples peut être regardée comme indiquant une racine positive, et chaque permanence, comme indiquant une racine négative. Seulement il distingue chacune de ces espèces de racines en réelles et imaginaires. Les racines *positives-imaginaires* correspondent aux *doubles-variations,* et les racines *négatives-imaginaires* aux *permanences-variations.*

2°. En consultant le *syllabus* cité par M. Aug. Poulain, on trouvera que j'ai donné une démonstration rigoureuse du théorème 2^e, non-seulement pour le cas auquel se borne mon habile commentateur, c'est-à-dire le cas où γ_r désigne la fraction $\dfrac{m-r+1}{m-r}$, mais, en général, pour chaque valeur de γ_r qui satisfait à l'équation $2-\gamma_r=\dfrac{1}{\gamma_{r+1}}$ à la condition toutefois que γ_r reste

positif pour toutes les valeurs de r comprises entre 1 et m, ces limites étant exclues. Cela revient à dire qu'on peut poser $\gamma_r = \frac{\mu - r + 1}{\mu - r}$, pourvu que μ ait une valeur réelle quelconque, non comprise entre 0 et m, mais pouvant d'ailleurs être une de ces valeurs limites. On voit donc qu'en réalité je démontre un théorème troisième, qui devient le deuxième si l'on attribue à μ une de ses valeurs limites, la valeur m.

Dans la troisième partie d'un travail inséré par moi dans les *Philosophical Transactions* de l'année dernière, j'avais à traiter les caractères invariantifs qui servent à distinguer exactement les trois cas offerts par les équations du cinquième degré. Cette fois encore je suis tombé sur des formules renfermant un paramètre pouvant prendre des valeurs arbitraires entre certaines limites. Il y a là, en ce qui concerne les *criteria*, un phénomène jusqu'alors inconnu dans les fastes de l'algèbre.

On trouvera un exemple de l'utilité du théorème troisième dans la partie mathématique de l'*Educational Times* du mois d'avril de cette année.

3°. Il reste encore une remarque importante à faire sur l'application des théorèmes deuxième et troisième; c'est que, à la formule

$$pP(\mu) - pP(\lambda) = (\lambda, \mu) + 2K, \qquad (\beta)$$

on doit ajouter la formule également importante

$$vP(\lambda) - vP(\mu) = (\lambda, \mu) + 2K', \qquad (\gamma)$$

qui se déduit de la précédente quand on change x en $-x$ dans l'équation donnée. Ces deux formules (β) et (γ) donnent des limites tout à fait indépendantes l'une de l'autre, de sorte qu'on peut comparer le théorème ainsi présenté à un fusil à deux coups: si l'un des canons rate, l'autre peut atteindre le but. Il va sans dire que la formule (γ) peut se démontrer directement sans se servir de (β).

4°. Il existe une méthode pour modifier les énoncés des formules (β) et (γ). Au point de vue théorique elle me paraît utile parce qu'elle fournit le moyen de se passer du mot gênant *variation-permanence*, en remplaçant cette combinaison mêlée par une *double-variation*.

Remarquons que si la succession $\begin{vmatrix} a, & b \\ A, & B \end{vmatrix}$ est une *double-permanence*, $\begin{vmatrix} a, & b \\ aA, & bB \end{vmatrix}$ le sera aussi. Mais si $\begin{vmatrix} a, & b \\ A, & B \end{vmatrix}$ est une variation-permanence, $\begin{vmatrix} a, & b \\ aA, & bB \end{vmatrix}$ change de caractère et devient une *double-variation*. Donc si l'on substitue des éléments cubiques aux éléments quadratiques, en écrivant

$$\psi^r y = f^r y \,.\, \phi^r y = (f^r y)^3 - \gamma_r f^{r-1} y \,.\, f^r y \,.\, f^{r+1} y,$$

et si l'on fait porter les symboles V, P, non plus sur la série des ϕ, mais sur celle des ψ, les formules (β) (γ) deviendront

$$pP(\mu) - pP(\lambda) = (\lambda, \mu) + 2K, \tag{B}$$

$$vV(\lambda) - vV(\mu) = (\lambda, \mu) + 2K'. \tag{C}$$

5°. La série ψ donne lieu à un nouveau théorème. Remarquons que le théorème de Budan s'exprime indifféremment par la formule

$$p(\mu) - p(\lambda) = 2A,$$

ou bien

$$v(\lambda) - v(\mu) = 2A. \tag{D}$$

A l'aide de la série des ψ, on peut modifier ces formules; car en rapportant les signes P et V à ces nouvelles quantités, on peut établir les formules

$$\frac{p(\mu) + P(\mu) - p(\lambda) - P(\lambda)}{2} = (\lambda, \mu) + L,$$

ou bien

$$\frac{v(\lambda) + V(\lambda) - v(\mu) - V(\mu)}{2} = (\lambda, \mu) + L, \tag{E}$$

L étant un nombre entier positif quelconque, pair ou impair.

La démonstration et quelques corollaires simples de cette proposition (E) sont donnés dans le *Philosophical Magazine* du mois de mai de cette année, [p. 542, below].

6°. Les quatre formules (β), (γ), (B), (C) peuvent être réunies avec avantage dans la pratique, quand on opère avec la méthode dite de Fourier servant à séparer les racines d'une équation.

P.S. J'ajoute un exemple de l'importance du paramètre arbitraire que j'ai introduit parmi les *criteria*.

Prenons l'équation

$$1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \dots + \frac{x^n}{1.2.3\dots n} = 0,$$

ou bien

$$x^n + nx^{n-1} + n(n-1)x^{n-2} + \dots + 1.2.3\dots n = 0.$$

Avec la règle de Newton, on trouvera pour la série des éléments quadratiques

$$1;\ -1;\ 1;\ 1;\dots;\ 1,$$

et de cette série on ne peut conclure qu'à l'existence d'une seule paire de racines imaginaires. Mais avec l'aide du théorème général, on obtient

$$1;\ 0;\ 0;\dots;\ 0;\ 1,$$

et cette nouvelle série prouve que toutes les racines, à l'exception d'une seule, quand n est impair, sont imaginaires.

86.

ON AN ADDITION TO POINSOT'S ELLIPSOIDAL MODE OF REPRESENTING THE MOTION OF A RIGID BODY TURNING FREELY ROUND A FIXED POINT, WHEREBY THE TIME MAY BE MADE TO REGISTER ITSELF MECHANICALLY.

[*Proceedings of the London Mathematical Society*, I. (1866), No. VI.]

THE author of this communication alluded to the well-known geometrical representation invented by Poinsot for exhibiting the motion of a rigid body acted on by no forces, through the medium of an ellipsoid, whose principal moments of inertia are equal to those of the rotating body, and which is supposed to roll without swinging upon a fixed plane, its centre remaining fixed. He pointed out, as an imperfection in this mode of representation, that whilst it exhibits the geometrical path of the body, it gives no image to the mind of the time in which any portion of such path is performed; the velocity of rotation in any position, it is true, is in a manner represented proportionally by the length of the radius vector drawn from the fixed centre to the point of contact of the ellipsoid with the fixed plane; but of the time, there is no indication afforded other than what can be inferred analytically from this law of velocity by means of an integration. The author explained how this imperfection could be remedied and the time put distinctly in evidence, and conceived as registering itself on a dial plate as the ellipsoid continues to roll. For this purpose, the part of the ellipsoid remote from the fixed plane may be conceived to be pared away until its form becomes that of a portion of an ellipsoid confocal with the surface in contact with the fixed plane. This confocal part of the surface may be conceived to be brought into contact with a fixed plate parallel to the fixed plane, and capable of only one motion, namely, of revolution round an axis, which, if produced, would pass through the fixed centre, and perpendicular to the fixed plane. It was explained how the ellipsoid in the act of rolling upon the last-named plane would roll upon the plate parallel to it, and at the same time by the friction drive the plate round its axis: this angular rotation could be measured upon

a fixed dial-face immediately above the plate, and the amount of it would be always *precisely proportional* to the time that would be occupied by the body which the ellipsoid represents, if perfectly free, in passing from any given position to any other; that is to say, the representative ellipsoid passing from a position A to a position B, and in the act of passing from A to B, driving the rotating plate through the angle U, U would measure the time in which the free body could pass from A to B. If a^2, b^2, c^2 be the semi-axes squared of Poinsot's ellipsoid, $a^2-\lambda$, $b^2-\lambda$, $c^2-\lambda$ of the confocal one above described, L the initial impulsive moment of the free body, the time between the position A and position B will be measured by $\frac{U}{L\lambda}$; so that the initial impulse being supposed given, the relation between the time and the magnitude of the divisions in the dial-plate is invariable; and thus by supposing the fixed plane and the parallel plate to admit, by a preliminary adjustment, of being set at any required distance from one another, the same instrument would continue to measure on the same scale the time of the free body passing from any initial position whatever to any other position into which it could turn.

87.

ASTRONOMICAL PROLUSIONS: COMMENCING WITH AN INSTANTANEOUS PROOF OF LAMBERT'S AND EULER'S THEOREMS, AND MODULATING THROUGH A CONSTRUCTION OF THE ORBIT OF A HEAVENLY BODY FROM TWO HELIOCENTRIC DISTANCES, THE SUBTENDED CHORD, AND THE PERIODIC TIME, AND THE FOCAL THEORY OF CARTESIAN OVALS, INTO A DISCUSSION OF MOTION IN A CIRCLE AND ITS RELATION TO PLANETARY MOTION*.

[*Philosophical Magazine*, XXXI. (1866), pp. 52—76.]

THE original demonstration by Lambert of the celebrated theorem which bears his name was a geometrical one. See *Monthly Notices* of the Astronomical Society, Vol. XXII. p. 238, where this demonstration is reproduced, or rather recapitulated, by Mr Cayley. See also Lambert's own *Insigniores Orbitæ cometarum proprietates,* Augusta Vindelicorum (Augsburg), 1761. It occupies seven or eight pages of this celebrated tract, and, elegant as may be considered the chain of geometrical enunciations from which it is deduced, is, as a specimen of geometrical style, little worthy of the inconsiderate commendations which have been heaped upon it, containing, as it does, a hybrid mixture of algebraical, geometrical, and trigonometrical ratiocination. The late Professor MacCullagh, as I am informed by my ingenious coadjutor, Mr Crofton, one of his hearers at Trinity College, Dublin, greatly improved upon Lambert's method, and succeeded in reducing it to a purely geometrical form. Lagrange has given no less than four distinct demonstrations of the same,—one a sort of verification by aid of trigonometrical formulæ in which

* Communicated by the author. A portion of this paper has appeared in the *Monthly Notices* of the Astronomical Society of London for December last [p. 496, above], namely, so much of it as relates to Lambert's theorem *proper*. The portion concerning circular motion formed the subject of a communication to the London Mathematical Society at the meeting of December 18, 1865. The part which presented itself last to the author's mind, and is consequently the least developed, is that which relates to the determination of the forces in any orbit to any two (or more) centres of force. The general expression for such forces will be found stated further on in a footnote, where the *equation of radial work* is defined and employed to obtain the solution in a form of unexpected simplicity.

the eccentric anomalies are introduced; a second of a similar nature, but dealing only with the true anomalies; a third founded on a property of integrals*; and a fourth, perhaps the most remarkable of any, derived from the general expressions for the time in an orbit described about two centres of force varying according to the law of nature, but one of them supposed to be situated in the orbit itself, and to become zero. Notwithstanding this plethora of demonstrations I venture to add a seventh, the simplest, briefest, and most natural of all, in which I employ a direct method to prove, from the ordinary formulæ for the time of a planet passing from one point to another, that, when the period is given, the time is a function only of the sum of the distances of these points from the centre of force, and of their distance from one another.

Let ρ, ρ' be the distances of the two positions from the sun, c their distance from one another, v, v' the true, u, u' the eccentric, m, m' the mean anomalies thereunto corresponding, e the eccentricity,

$$\omega = m - m', \quad s = \rho + \rho', \quad \Delta = \tfrac{1}{2}(s^2 - c^2);$$

then

$$\rho = 1 - e\cos u, \quad \rho' = 1 - e\cos u', \quad m = u - e\sin u, \quad m' = u' - e\sin u',$$

$$\rho\cos v = \cos u - e, \qquad \rho\sin v = \sqrt{(1-e^2)}\sin u,$$

$$\rho'\cos v' = \cos u' - e, \quad \rho'\sin v' = \sqrt{(1-e^2)}\sin u',$$

$$c^2 = \rho^2 + \rho'^2 - 2\rho\rho'\cos(v' - v).$$

Writing for brevity p, p', q, q' for $\cos u$, $\cos u'$, $\sin u$, $\sin u'$, we have

$$s = 2 - ep - ep', \quad \omega = u - u' - eq + eq',$$

$$\Delta = \rho\rho' + \rho\rho'\cos(v' - v) = 1 + pp' + qq' - 2e(p + p') + e^2(1 - qq' + pp').$$

Let
$$J = \frac{d(\Delta, s, \omega)}{d(e, u, u')};$$

then J is the determinant

$$\left|\begin{matrix} \left\{\begin{matrix} -2(p+p') \\ +2e(1+pp'-qq') \end{matrix}\right\}; & \left\{\begin{matrix} -p'q+pq'+2eq \\ -e^2(pq'+p'q) \end{matrix}\right\}; & \left\{\begin{matrix} -pq'+p'q+2eq' \\ -e^2(pq'+p'q) \end{matrix}\right\} \\ -p-p' \quad ; & eq \quad ; & eq' \\ -q+q' \quad ; & 1-ep \quad ; & -1+ep' \end{matrix}\right|.$$

* The property in question, discovered by Lagrange, is that the integral

$$\int \frac{rdr}{\sqrt{(L + Mr + Nr^2)}}$$

may be transformed into

$$\int \frac{(x^2 + h)\,dx}{\sqrt{(a + bx + cx^2 + dx^3 + ex^4)}} - \int \frac{(y^2 + h)\,dy}{\sqrt{(a + by + cy^2 + dy^3 + ey^4)}};$$

in applying it to Lambert's theorem a, b, c are made to vanish. This transformation and its consequences appear to us to deserve further study; as far as I know it has not been touched upon by the writers on elliptic functions.

Denoting this determinant by

$$\begin{vmatrix} A, & B, & C \\ D, & E, & F \\ G, & H, & K \end{vmatrix},$$

we find

$$(A, B, C) - 2H(D, E, F) + 2E(G, H, K) = (0, B, -B),$$

$$(A, B, C) - 2K(D, E, F) + 2F(G, H, K) = (0, -C, C),$$

so that

$$J = \begin{vmatrix} A, & B, & C \\ 0, & B, & -B \\ 0, & -C, & C \end{vmatrix} = 0.$$

Hence it appears that $d\omega$ is a linear function of ds and $d\Delta$; that is, ω is a function of s and Δ, or, what is the same thing, of s and c, and independent of e. If then, when $e=1$, the corresponding values of $\rho, \rho', v, v', u, u'$ are $r, r', \theta, \theta', \phi, \phi'$, we have

$$\cos\theta = -1, \quad \cos\theta' = -1, \quad \sin\theta = 0, \quad \sin\theta' = 0, \quad r - r' = c, \quad r + r' = s,$$

whence, writing

$$1 - \cos\phi = \frac{s+c}{2}, \quad 1 - \cos\phi' = \frac{s-c}{2},$$

we have finally

$$\omega = \phi - \phi' - \sin\phi + \sin\phi',$$

as was to be proved.

Essentially this demonstration is of the same nature as the first of Lagrange's four methods of proof above referred to, but with the difference that it leads up to and accounts beforehand for the success of the transformations therein employed.

Alluding to Lambert's cumbrous demonstration, Lagrange says of it, "His theorem merits the especial notice of mathematicians, both on its own account, and because it appears difficult to arrive at it by algebraical processes (*calcul*); so that it may be ranked among the small number of those in which geometrical seems to have the advantage over algebraical analysis." In the nature of things such advantage can never be otherwise than temporary. Geometry may sometimes appear to take the lead of analysis, but in fact precedes it only as a servant goes before his master to clear the path and light him on his way. The interval between the two is as wide as between empiricism and science, as between the understanding and the reason, or as between the finite and the infinite.

The result so simply obtained above is of course not restricted to the case of the ellipse, but applies to motion generally about a centre of force according to the law of nature.

Calling t the time, the syzygy shown to exist between δt, δs, δc, being independent of any supposition as to the value of e, or as to the reality of the functions employed, will of necessity continue to obtain where, e being greater than 1, the motion becomes hyperbolic. If μ be the absolute force, and 1, as before, the semi-major axis, writing

$$\epsilon = \sqrt{\frac{e-1}{e+1}}, \quad \epsilon \tan\frac{v}{2} = x, \quad \epsilon \tan\frac{v'}{2} = x',$$

the rest of the notation being preserved, we obtain, by direct integration and substitution,

$$\frac{s}{2} = \frac{1}{1-\epsilon^2}\left(\frac{\epsilon^2 + x^2}{1-x^2} + \frac{\epsilon^2 + x'^2}{1-x'^2}\right),$$

$$\frac{\Delta}{8} = \frac{1}{(1-\epsilon^2)^2}\frac{(\epsilon^2 + x^2)(\epsilon^2 + x'^2)}{(1-x^2)(1-x'^2)},$$

$$t = \mu^{\frac{1}{2}}\left\{\log\frac{1-x'}{1+x'} - \log\frac{1-x}{1+x} + 2(x-x')(1-xx')\right\}.$$

And we must needs find by actual computation the Jacobian

$$\frac{d(s, \Delta, t)}{d(\epsilon, x, x')} = 0.$$

Making $\epsilon = 0$, and giving x, x' their corresponding values in terms of s and Δ, there results

$$\frac{x^2}{1-x^2} = \frac{s+c}{2}; \quad \frac{x'^2}{1-x'^2} = \frac{s-c}{2};$$

and accordingly

$$t = \mu^{\frac{1}{2}}\left\{\begin{matrix}\log\{\sqrt{(s+c+2)} - \sqrt{(s-c)}\} - \log\{\sqrt{(s-c-2)} - \sqrt{(s-c)}\} \\ -\sqrt{\{(s+c+1)^2-1\}} \qquad + \qquad \sqrt{\{(s-c+1)^2-1\}}\end{matrix}\right\}.$$

It is worthy of notice that the effect of making $\epsilon = 0$ or $e = -1$ in this case, like that of making $e = 1$ in the case of elliptic motion, is to reduce the motion to that of a body in a straight line, but with the difference that for the elliptic the reduced motion is that of a body moving between the point of instantaneous rest and the centre of force or point of infinite velocity, whereas for the hyperbola it is that of a body moving on the same side of these two points.

The theorem for the case of the parabola was given by Euler (1744), but reproduced independently by Lambert in the *Insigniores Proprietates*, Sectiones 1, 2, in 1761.

I think the idea of the general theorem may not unlikely have originated in an observation of the accordance in form of the result for parabolic motion with that for motion in a straight line, an accordance easily verified to extend to motion in a circle. Such coincidence, to a mind open to the impressions of analogy, could hardly fail to suggest the necessity of the existence of a deeper-seated law, of which these extreme cases must represent the out-croppings. Euler's theorem is of course included as a particular case in Lambert's, and may be derived from it by making a infinite in the expression for t as a function of s, c, a; but it may also be obtained independently as follows. Calling $4m$ the latus rectum, retaining the rest of the notation, and writing $\tan\frac{v}{2}=q$, $\tan\frac{v'}{2}=q'$, we easily find

$$\tfrac{1}{2}\sqrt{\Delta}=m(1+qq'),$$

$$s=m(2+q^2+q'^2),$$

$$\frac{t}{\sqrt{2}}=m^{\frac{3}{2}}\{(q-q')+\tfrac{1}{3}(q^3-q'^3)\}.$$

Hence the Jacobian

$$\frac{d(\frac{1}{2}\sqrt{\Delta},\, s,\, \sqrt{2}\,.\,t)}{d(m,\, q,\, q')}$$

becomes a multiple of the determinant

$$\begin{vmatrix} 1+qq', & q', & q \\ 2+q^2+q'^2, & 2q, & 2q' \\ 3(q-q')+q^3-q'^3, & 2+2q^2, & -2-2q^2 \end{vmatrix}.$$

Calling this

$$\begin{vmatrix} A, & B, & C \\ D, & E, & F \\ G, & H, & K \end{vmatrix},$$

it will be found that

$$(A,B,C)-\frac{H}{2A-F}(D,E,F)+\frac{E}{2A-F}(G,H,K)=0,\,B,\,-B,$$

$$(A,B,C)+\frac{K}{2A+E}(D,E,F)-\frac{F}{2A+E}(G,H,K)=0,\,-C,\,C;$$

and consequently the Jacobian in question, as before, takes the form

$$\begin{vmatrix} A, & B, & C \\ 0, & B, & -B \\ 0, & -C, & C \end{vmatrix},$$

which is identically zero; so that t is a function only of s, c when a is given, and one solution is left free between m, q, q'.

Making $q = \infty$, we have

$$m(q - q') = \sqrt{(s - \sqrt{\Delta})} = \frac{\sqrt{(s+c)} - \sqrt{(s-c)}}{2},$$

$$m(q + q') = \sqrt{(s + \sqrt{\Delta})} = \frac{\sqrt{(s+c)} + \sqrt{(s-c)}}{2},$$

$$mq = \frac{\sqrt{(s+c)}}{2}, \quad mq' = \frac{\sqrt{(s-c)}}{2};$$

and thus $$t = \tfrac{1}{6}\{(s+c)^{\frac{3}{2}} - (s-c)^{\frac{3}{2}}\}.$$

There is a sort of pendant to Lambert's theorem which is worthy of notice. If we call $\rho - \rho' = v$ and $c^2 - \sigma^2 = D$, writing

$$ae(\sin u' - \sin u) = \Omega,$$

we have also

$$(1 - e^2)a^2\{1 - \cos(u' - u)\} = D,$$

$$ea(\cos u - \cos u') = \sigma,$$

from which we easily obtain

$$\Omega = \sqrt{\frac{e^2c^2 - \sigma^2}{1 - e^2}};$$

so that Ω is a function only of c, e, σ, as by Lambert's theorem it is a function only of c, a, s. Moreover, since

$$\sin\left(\frac{u' - u}{2}\right) = \sqrt{\frac{D}{2(1 - e^2)}},$$

it is apparent that the change in the mean anomaly is a complete function of the two variables $\frac{\sqrt{D}}{b}$, $\frac{c}{a}$, as by Lambert's theorem it is of the two $\frac{\sqrt{\Delta}}{a}$, $\frac{c}{a}$. Comparing the value of Ω given immediately above with that which is contained in Lambert's theorem, the solution of a linear equation leads immediately, after certain simple reductions, to the equation

$$1 - e^2 = \frac{2(c^2 - \sigma^2)}{ss' + c^2 \pm \sqrt{\{(s^2 - c^2)(s'^2 - c^2)\}}},$$

where $s' + s = 4a$. And as there is nothing to determine the signs of ρ or ρ', the above, by interchanging severally and independently ρ, ρ' with $-\rho, -\rho'$, represents eight values of e:—four corresponding to the change of ρ into $-\rho$, and ρ' into $-\rho'$, contained in the expression immediately above written, combined with the equation $s' + s = \pm 4a$; and four in the conjugate expression

$$1 - e^2 = \frac{2(c^2 - s^2)}{\sigma\sigma' + c^2 \pm \sqrt{\{(c^2 - \sigma^2)(c^2 - \sigma'^2)\}}},$$

where $\sigma' + \sigma = \pm 4a$.

Since we have also (calling i the angle between c and a) $\cos i = \frac{\sigma}{ec}$ in the first case, and $\cos i = \frac{s}{ec}$ in the second case, the problem of determining the conic, of which one focus, the major axis, and two points are given, is thus completely solved. This of course comprehends the analytical solution of the problem of determining the magnitude and position of the orbit of a planet from the periodic time, two heliocentric distances, and the included angle, of which no mention is to be found in any of the ordinary books of astronomy, or even in the *Theoria Motus*, where one would naturally expect to find it.

There are thus eight values of e^2, and the solution is eightfold. The sign of $\cos i$ being left ambiguous does not raise the number to 16; for this ambiguity depends upon the fact of the direction of c being incapable of analytical representation; only one of these values of $\cos i$ will appertain to any stated case. If F be the given focus, P, Q the two given points, and G the second focus, by rotating the figure about the line FG, P, Q come into the positions P', Q'; c, s, σ remain unaltered; but the angles between $Q'P'$, FG, and between QP, FG, become supplementary. If we chose to effect a direct solution of the problem of determining the orbit without the aid of the eccentric anomalies, we should have to eliminate θ, θ' between the equations

$$\pm \rho = \frac{a(1-e^2)}{1 - e\cos\theta}, \quad \pm \rho' = \frac{a(1-e^2)}{1 - e\cos\theta'}, \quad c^2 = \rho^2 + \rho'^2 - 2\rho\rho'\cos(\theta - \theta').$$

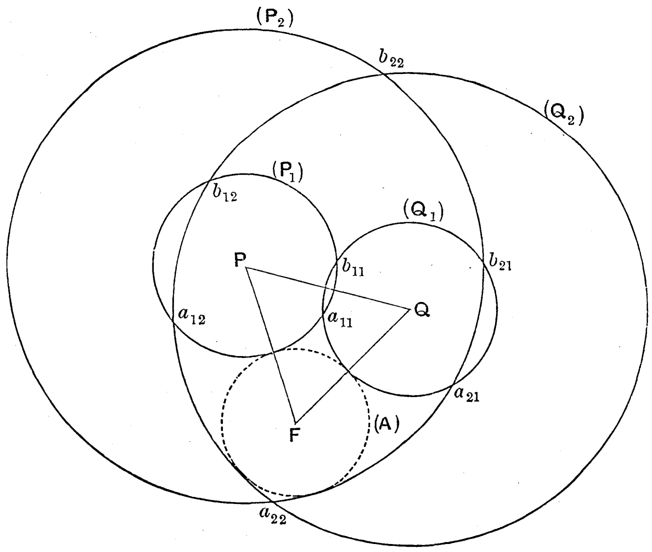
This elimination will be found to lead to a quadratic equation in e^2, the coefficients of e^6 and e^8 vanishing; and we thus obtain an eightfold solution as before, but in a more involved form. Or, again, we might write

$$\{a(1-e^2) + ex\}^2 = \rho^2,$$
$$\{a(1-e^2) + ex'\}^2 = \rho'^2,$$
$$xx' + yy' = c^2,$$
$$x^2 + y^2 = \rho^2, \quad x'^2 + y'^2 = \rho'^2,$$

and between these five equations eliminate x, x', y, y'. By the general theory of elimination, e^2 should rise to the sixteenth degree in the resultant; but in fact it will rise only to the eighth. The following obvious geometrical construction will perfectly account *à priori* for the existence of the excluded infinite values of e^2.

Since $$FP \pm GP = \pm 2a \text{ and } FQ \pm GQ = \pm 2a,$$
G will be any point in the intersection of either of two circles C_1, C_2 with either of the two K_1, K_2, where the squared radii of C_1, C_2 are $(2a + FP)^2$, and of K_1, K_2 $(2a + FQ)^2$, $(2a - FQ)^2$ respectively. Consequently there are eight real or imaginary positions of G at a finite, and eight at an infinite distance.

It is obvious that, if we restrict the orbit to being elliptical, there can never be more than two admissible solutions; but treating the question more generally, any even number of solutions whatever may exist from 0 to 8, both inclusive. I am indebted to my able friend, Dr Hirst, for the following figure, illustrating the interesting case where all eight solutions are real and hyperbolæ.



In this figure $\quad \rho\,(=FP) > 2a, \quad \rho'\,(=FQ) > 2a,$

and likewise

$$\rho + \rho' - 4a > c,$$

$$4a - \rho + \rho' < c,$$

$$4a + \rho - \rho' < c.$$

Supposing FP, FQ to be each greater than $2a$, there will be no difficulty in verifying the following statement.

One pair of hyperbolæ, in which P, Q lie in the branch containing F, will be always real; a second pair, in which they lie in the opposite branch, will be real or imaginary according as s is greater or less than $c + 4a$, that is, according as $2a$ is less or greater than $\frac{s-c}{2}$. A third pair, in which the two given points are distributed between the two branches, will be real or imaginary respectively according as $2a$ is less or greater than $\frac{c+\sigma}{2}$; and a fourth pair, where the same distribution occurs, will be real or imaginary according as $2a$ is greater or less than $\frac{c-\sigma}{2}$.

It is of course only with the first kind of hyperbolæ, that in which the given points lie in the branch concave to the given focus, with which the problem, regarded as an astronomical one, is concerned. But in all cases the formulæ given for the determination of e and i admit of immediate adaptation to logarithmic computation. Thus, for example, if we take the one which meets the case of distribution between the two branches of an hyperbola, namely,

$$e^2 - 1 = \frac{2(s^2 - c^2)}{(\sigma\sigma' + c^2) \pm \sqrt{\{(c^2 - \sigma^2)(c^2 - \sigma'^2)\}}},$$

writing $\quad e = \sec\phi, \quad s = c\sec\lambda, \quad \sigma = c\cos\mu, \quad \sigma' = c\cos\mu',$

we obtain $$\tan\phi = \tan\lambda\sec\frac{\mu \pm \mu'}{2}$$

$$\pm\cos i = \cos\lambda\cos\phi.$$

Viewed as a question of analytical geometry, the investigation as to the reality of the curve would have to be treated in a more general manner; that is, without assuming, as I have done, the necessity of the inequalities $s < \rho_1 + \rho_2$, $\sigma > \rho_1 - \rho_2$, where ρ_1, ρ_2 represent the two given focal distances; for it is a very important, although hitherto strangely neglected geometrical principle, that every real conic is at one and the same time an ellipse and hyperbola; namely, either an actual ellipse accompanied by an ideal hyperbola, or an actual hyperbola accompanied by an ideal ellipse. This may immediately be made manifest by considering how the ordinary rectangular-coordinate equation to a conic, with its origin transferred to a focus, is connected with the property of the conic in respect to its two foci. Calling ρ, ρ' the two focal distances of any point, the equation to rectangular coordinates is obtainable by equating to zero the norm of the quantity $2a \pm \rho \pm \rho'$, where ρ represents $\sqrt{(x^2 + y^2)}$, and ρ' represents $\sqrt{\{(2ae + x)^2 + y^2\}}$, which norm will only be of the *second degree* in x, y, although a product of four factors each of the first degree in x, y, owing to the vanishing of the coefficients of the terms that ought to rise to the fourth degee in the variables. Calling, then, this norm N, we see that the quadratic equation $N = 0$ is satisfied alike by the equations $\rho + \rho' = 2a$ and $\rho - \rho' = 2a$, the difference being that one of these will belong to the apex of an actual, and the other to that of an ideal triangle, according to the sign of $e - 1$.

It may not be quite out of place here to show how immediately the knowledge of the existence of a third focus to the Cartesian ovals, that remarkable discovery of our illustrious Royal Society *Laureate* of the year, flows from a similar process to the one above. For taking the norm of the expression

$$a\sqrt{(x^2 + y^2)} + \sqrt{(b^2x^2 + b^2y^2 + 2bcx + c^2)} + d^2,$$

the equation to any such curve becomes

$$\begin{aligned}&\{(b^2-a^2)(x^2+y^2)+2bcx+c^2\}^2\\ &-2d^2\{(b^2+a^2)(x^2+y^2)+2bcx+c^2\}\\ &+d^4=0,\end{aligned}$$

that is,

$$\begin{aligned}&(b^2-a^2)^2(x^2+y^2)^2+4bc(b^2-a^2)x(x^2+y^2)\\ &\quad+\{2(c^2-d^2)b^2-2(c^2+d^2)a^2\}(x^2+y^2)\\ &\quad+4b^2c^2x^2+4(c^2-d^2)bcx+(c^2-d^2)^2=0;\end{aligned}$$

in which equation a^2, b^2, c^2, d^2 may obviously be replaced by α^2, β^2, γ^2, δ^2, provided

$$\left.\begin{aligned}\beta^2-\alpha^2&=b^2-a^2,\\ \beta\gamma&=bc,\\ \alpha^2\delta^2&=a^2d^2,\\ \gamma^2-\delta^2&=c^2-d^2\end{aligned}\right\}.$$

Writing for a^2, b^2, c^2, d^2, &c. a_1, b_1, c_1, d_1, and squaring the second equation, we obtain a symmetrical system of equations, namely,

$$\left.\begin{aligned}\beta_1-\alpha_1&=b_1-a_1, &\quad \gamma_1-\delta_1&=c_1-d_1,\\ \beta_1\gamma_1&=b_1c_1, & \alpha_1\delta_1&=a_1d_1,\end{aligned}\right\}$$

for determining α_1, β_1, γ_1, δ_1. Throwing out the solution $\alpha_1=a_1$, $\beta_1=b_1$, $\gamma_1=c_1$, $\delta_1=d_1$, only one other solution will be found to exist, which, restoring the original variables, becomes

$$\alpha^2=\frac{b^2-a^2}{c^2-d^2}d^2,\quad \beta^2=\frac{b^2-a^2}{c^2-d^2}c^2,$$

$$\gamma^2=\frac{d^2-c^2}{a^2-b^2}b^2,\quad \delta^2=\frac{d^2-c^2}{a^2-b^2}a^2,$$

with the condition that $\beta\gamma=bc$.

The complete arithmetical determination of the signs to be given to the several quantities α, β, γ, δ requires a distinct and detailed examination, which it would be irrelevant to enter upon in this place; it is enough to see that a second focus G at the distance $\frac{c}{b}$ from a given one F may be moved along the line FG to a new focus H at the distance $\frac{\gamma}{\beta}$ from F, the *modulus* $\frac{b}{a}$ becoming simultaneously replaced by $\frac{\beta}{\alpha}$, and the constant $\frac{d}{a}$ by the constant $\frac{\delta}{\alpha}$. I am not aware that M. Chasles has ever disclosed the *aperçu* which led him to this unlooked-for discovery. It is to be hoped that he will not allow future ages to labour under the same doubt as to the source from which he drew it as we must, it is to be feared, ever remain under with regard to the

origin of Newton's rule, recently demonstrated, or Lambert's theorem, the motive to this paper. In this age of the world *euristic* is even more important to the promotion of science, and its secrets less likely to be recovered than those of mere *apodictic*.

Since a focus may be regarded as the intersection of two tangents from the circular points of infinity, we may generalize the problem of constructing the orbit by considering it as a particular case of constructing the conic which passes through two given points, touches two given straight lines, and has a principal axis of given length.

Taking the two given lines supposed to be inclined to each other at the angle α as the axes of coordinates, the equation to the curve may be written under the form

$$(Ax + Cy + 1)^2 = 2B^2xy,$$

which, writing

$$x = \xi - \frac{C}{2AC - B^2}, \quad y = \eta - \frac{A}{2AC - B^2},$$

becomes

$$A^2\xi^2 + 2(AC - B^2)\,\xi\eta + C\eta^2 = \frac{2B^2}{2AC - B^2}.$$

Adding $\lambda(\xi^2 - 2\cos\alpha\xi\eta + \eta^2)$ to the left-hand side of the equation, the discriminant of that side so augmented becomes

$$(\sin\alpha)^2\lambda^2 + (A^2 + 2\cos\alpha AC + C^2 - 2\cos\alpha B^2)\,\lambda + B^2(B^2 - 2AC).$$

Hence, calling the squared reciprocal of the given principal semiaxis q, and writing $\lambda = \dfrac{2B^2}{2AC - B}q$, we obtain

$$4\sin\alpha^2q^2B^2 + 2(2AC - B)^2(A^2 + 2\cos\alpha\,.\,AC + C^2 - 2\cos\alpha\,.\,B^2)\,q \\ + (B^2 - 2AC)^3 = 0\,;$$

combining which with any of the four couples of linear equations,

$$pA \pm \surd(2pq)\,B + qC + 1 = 0, \quad p'A \pm \surd(2p'q')\,B + q'C + 1 = 0,$$

obtained by substituting for x, y the coordinates of the two given points, we obtain six sets of quadruple solutions, making twenty-four *finite* solutions in all. This result is in perfect accordance with that which applies to the case of the tangents meeting at the focus; for when $\dfrac{1}{q}$ is the square of the principal semiaxis in which the focus lies, we have already found eight solutions; and when $\dfrac{1}{q}$ refers to the other semiaxis, we have

$$\frac{8}{q} = 8a(1 - e^2) = \frac{(s + s')^2(c^2 - \sigma^2)}{ss' + c^2 \pm \surd\{(s^2 - c^2)(s'^2 - c^2)\}}\,;$$

which, considering c, s, σ given, leads to a biquadratic in s' which serves to fix the curve; and as there are four systems of values of s, σ arising from

the permutations of the signs of ρ, ρ', we thus have four times four, or sixteen solutions over and above the previous eight, making twenty-four in all, as in the general case.

We might generalize the problem otherwise by supposing given, not the magnitude of a principal axis, but that of a diameter through the intersection of the two given tangents; or, again, in quite a different direction by supposing three points P, Q, R to be given in a *Cartesian oval* defined by the equation $k\rho - \rho' = 2a$, ρ referring to a given focus F, and ρ' to a second focus G to be determined, a also being given, but k being to be determined. It is easy to see that in this case also the position of G may be obtained by the intersections of circles; for the ratios $PG : QG : RG$ will be known; there will thus be eight pairs of solutions arising from the permutations of the signs of ρ_1, ρ_2, ρ_3 which measure FP, FQ, FR; and calling $\frac{FG}{2a} e$, it would be an interesting analytical question to express the eight systems of k and e in terms of ρ_1, ρ_2, ρ_3, and c_1, c_2, c_3 the three chords joining P, Q, R,—these six quantities, of course, being not independent but connected by the well-known equation between the mutual squared distances of any four points from one another on a plane.

Touching the Cartesian ovals, Mr Crofton has well remarked that a circle may be regarded as one of a very peculiar kind. For if we take any two points electrical images of one another, inverses, in Dr Hirst's language, or, as I prefer to call them, reciprocals or *harmonics* in respect to a given circle, the distances ρ, ρ' of any point in the circle from them will be connected by the equation $-k\rho + \rho' = 0$; so that any pair of harmonics whatever of a circle may be regarded as foci of such curves. The third focus correlated to each pair will evidently be the centre; for, calling its distance from any point in the circle ρ'', we have $0 . \rho + \rho'' = c$; in the first equation the modulus k is finite and the constant zero; in the second the modulus is zero and the constant finite. Consequently a circle is a Cartesian oval, not only as a particular case of a conic, but *proprio Marte* and porismatically in quite another sort of way*. Now it is well known that a conic may be described

* Thus it will be seen that, besides its derivation through the ellipse, the circle descends by a short cut immediately from the Cartesian oval; recalling to mind the singular condition of consanguinity of the ill-fated descendants of Laius, at once children and grand-children to their mother, sons and brothers to their father. Viewed as sprung from the ellipse, there should be but two coincident Cartesian foci to the circle; it is the fraternal aspect of the relationship which brings into view the existence of an infinite number of such foci in the circle; every point in fact being a focus. This is explained by considering the circle so descended, not (like a conic) as a Cartesian oval with a branch at an infinite distance, but without such branch, and as doubled upon itself; thus the circular points at infinity become each double points, and, as well remarked by Mr Cayley, every line through either such double point is analytically a tangent to the curve, and thus every point in the plane of the circle, being the intersection of two such tangents, ought to be, as it is, a *focus*.

by two forces (varying as the inverse square of the distance, and tending to its two foci). This led me to inquire whether some analogous theorem did not hold of a circle in respect to any of its pairs of foci, that is of harmonics; and I find such is the case, as the annexed simple investigation will make manifest.

Call the radius of the circle *unity*; c, $\frac{1}{c}$ the distances of two harmonics from its centre; $\frac{\mu}{\rho^n}$, $\frac{\mu'}{\rho'^n}$ two forces tending to these points respectively; then by duly assigning the initial velocity, we are at liberty to suppose the constant zero in the equation for *vis viva*, so as to be able to write

$$v^2 = \frac{2\mu}{(n-1)\,\rho^{n-1}} + \frac{2\mu'}{(n-1)\,\rho'^{n-1}};$$

we have also

$$\frac{\rho'}{\rho} = \frac{\frac{1}{c}-1}{1-c} = \frac{1}{c}.$$

In order that the circle may be described under the circumstances above supposed, it is necessary and sufficient that

$$v^2 = \frac{\mu}{\rho^n}\cos i + \frac{\mu'}{\rho'^n}\cos i',$$

i, i' being the angles which ρ, ρ' respectively make with the normal; that is

$$v^2 = \frac{\mu}{\rho^n}\left(\frac{1-c^2+\rho^2}{2\rho}\right) + \frac{\mu'}{\rho'^n}\left(\frac{1-\left(\frac{1}{c}\right)^2+\rho'^2}{2\rho'}\right)$$

$$= \tfrac{1}{2}\left(\frac{\mu(1-c^2)}{\rho^{n+1}} + \frac{\mu'\left(1-\frac{1}{c^2}\right)}{\rho'^{n+1}}\right) + \tfrac{1}{2}\left(\frac{\mu}{\rho^{n-1}} + \frac{\mu'}{\rho'^{n-1}}\right).$$

Hence, in order to satisfy this identity, we must have

$$\mu\,\frac{1-c^2}{\rho^{n+1}} + \mu'\,\frac{1-\frac{1}{c^2}}{\rho'^{n+1}} = 0,$$

or

$$\mu = \frac{1}{c^2}\left(\frac{\rho}{\rho'}\right)^{n+1}\mu' = c^{n-1}\,.\,\mu';$$

so that

$$\frac{\mu'}{\rho'^{n-1}} = \frac{\mu}{\rho^{n-1}}.$$

And accordingly the required identity will be completely satisfied if we further make

$$\frac{\mu}{2} = \frac{2\mu}{n-1}, \text{ or } n = 5,$$

which implies

$$\mu = c^4\mu', \text{ or } \frac{\mu}{c^2} = \frac{\mu'}{\gamma^2},$$

c, γ being the distances of the respective centres of force from the centre of the orbit.

The *vis viva* consists of two equal parts, $\frac{\mu}{2\rho^4}$, $\frac{\mu'}{2\rho'^4}$, each centre contributing, as it were, equally to its production. To find the time, calling u the angle which the orbit swept out subtends at the centre, we have

$$\left(\frac{du}{dt}\right)^2 = \frac{\mu}{\rho^4},$$

or

$$t = \int \frac{du\rho^2}{\mu^{\frac{1}{2}}} = \frac{1}{\mu} \int du\,(1 + c^2 - 2c\cos u);$$

and P, the periodic time, will be $\frac{2F}{\mu^{\frac{1}{2}}}(1+c^2)$, or, restoring the value of a to the radius, the period becomes $\frac{2\pi}{\mu^{\frac{1}{2}}}a(a^2+c^2)$, which of course is the same as

$$\frac{2\mu}{\mu'^{\frac{1}{2}}}a(a^2+\gamma^2).$$

If we now suppose the two absolute forces μ, μ', and δ the distance between them, to be given, the problem of determining the magnitude and position of the orbit and the periodic time may be easily effected; for we have only to find c, γ and a from the equations

$$c\gamma = a^2,$$

$$\frac{c^2}{\gamma^2} = \frac{\mu}{\mu'},$$

$$\gamma - c = \delta,$$

from which results

$$a = \frac{(\mu\mu')^{\frac{1}{4}}}{\mu'^{\frac{1}{2}} - \mu^{\frac{1}{2}}}\delta,$$

$$c = \frac{\mu}{\mu'^{\frac{1}{2}} - \mu^{\frac{1}{2}}}, \qquad \gamma = \frac{\mu'^{\frac{1}{2}}}{\mu'^{\frac{1}{2}} - \mu^{\frac{1}{2}}},$$

$$P = 2\pi\frac{\mu^{\frac{1}{4}}.\mu'^{\frac{3}{4}} + \mu^{\frac{3}{4}}.\mu'^{\frac{1}{4}}}{(\mu'^{\frac{1}{2}} - \mu^{\frac{1}{2}})^3}\delta^3.$$

Also the velocity at either apse is given by the formula $v = \frac{\mu}{(a \mp c)^2}$, which gives $\frac{(\mu'^{\frac{1}{4}} \pm \mu^{\frac{1}{4}})^2}{\delta}$ for the two limiting velocities.

Again, the general expression for the time is

$$t = \frac{(a^2 + c^2)}{\mu^{\frac{1}{2}}} \left\{u - \frac{2ac}{a^2 + c^2} \sin u\right\}.$$

Suppose, then, a planet to be describing an ellipse under the attraction of the sun, and a fictitious body moving in a circle described about its axis major to leave an apse simultaneously with the planet, and that its velocity parallel to the axis major always remains equal to that of the planet in the same direction. Then the arc swept out by such body subtends at the centre the angle which measures the eccentric anomaly of the planet, and may be termed its eccentric follower. The motion of this eccentric follower may be physically produced by supposing it to be attracted to two centres of force of proper absolute magnitudes and duly placed in the major axis, attracting according to the inverse fifth power of the distance; this is an immediate consequence from the preceding expression for t.

If we call M the absolute force of the sun, it will readily be seen that we must have

$$\mu = \frac{(a^2 + c^2)^2}{a} M,$$

$$\mu' = \frac{(a^2 + \gamma^2)}{a} M;$$

where c, γ are the distances of the two new centres of force from the centre of the planetary orbit, and satisfy the equation

$$\frac{2ac}{a^2 + c^2} = e,$$

or

$$c^2 - \frac{2a}{e} c + a^2 = 0,$$

which gives

$$c = \frac{a - b}{e}, \quad \gamma = \frac{a + b}{e};$$

b representing the semi-minor axis. c being equal to

$$\frac{a\{1 - \sqrt{(1 - e^2)}\}}{e}, \quad c - ae = a\frac{\sqrt{(1 - e^2)}}{e}\{\sqrt{(1 - e^2)} - 1\},$$

and is always negative, so that the interior centre of force always lies between the centre of the orbit and the sun; when e is small it lies about midway between these two points, but nearer to the latter than the former: for example, if we were to suppose $e = \frac{3}{5}$, we should have $ae = \frac{3a}{5}$, $\frac{a\{1 - \sqrt{(1 - e^2)}\}}{e} = \frac{a}{3}$, which differs not very much from $\frac{3a}{10}$.

It is perhaps remarkable—at all events I am not aware whether any one has remarked, that the motion of the *eccentric follower* of a planet may also be brought about by a single force placed at the sun itself, attracting according to the law which is consistent with the body describing a circle. This is immediately apparent; for if we call S the position of the centre of force, C the centre of the circle, c the distance of S from C, a the radius of the circle, P any point in it, calling i the angle SPC, u the angle PCS, we have

$$v = \frac{h}{p} = \frac{h}{\rho \cos i} = \frac{h}{a - c \cos u};$$

so that $\dfrac{dt}{du} = \dfrac{a^2}{h}\left\{1 - \dfrac{c}{a}\cos u\right\}$, which proves the point in question*.

The force f for this case has been given by Newton in the third section of the *Principia*; it can be obtained instantaneously from the equation

$$v^2 = af \cos i = -\frac{a}{2}\frac{dv^2}{d\rho}\cdot\frac{\rho^2 + a^2 - c^2}{2\rho};$$

so that

$$\frac{dv^2}{d\rho \,.\, v^2} = \frac{-4a\rho}{\rho^2 + a^2 - c^2},$$

or

$$v^2 = \frac{C}{(\rho^2 + a^2 - c^2)^2};\ f = -\frac{C\rho}{(\rho^2 + a^2 - c^2)^3}.$$

Calling ρ' the remainder of the chord R of which ρ is a part,

$$\rho^2 + (a^2 - c^2) = \rho^2 + \rho\rho' = \rho R;$$

so that f varies as

$$\frac{1}{\rho^2 R^3},$$

as given in the *Principia*, and of course, if the force-centre is at the extremity of a diameter, f varies as $\dfrac{1}{\rho^5}$, which is the case in which our two reciprocal foci come together. When one of them is at the centre, the other goes off to infinity, and the actual amount of force exerted by it, $\dfrac{\mu'}{r'^5}$, or $\dfrac{\mu}{r^4}\cdot\dfrac{r'^4}{r'^5}$, becomes zero when $\dfrac{\mu}{r^5}$ is finite; so that this case returns to that of a single force at the centre of the circle. If we wished to find the general law of the respective forces f, f' at the two reciprocal foci suitable to produce motion in the circle we might proceed as follows:—Calling i, i' the angles

* Hence follows the *statical* proposition that the force which tending to any centre retains a point in a circular orbit may be resolved into two forces tending to two fixed centres, each varying as the inverse fifth power of the distance: this proposition will be generalized subsequently in the text.

between the radii vectores drawn to these points from any point in the circle and the radius at that point, and writing

$$V = 2\int dr\,.f, \quad V' = 2\int dr'.f',$$

we have to satisfy the equation

$$\frac{1}{2}\frac{dV}{dr}\cos i + \frac{1}{2}\frac{dV'}{dr'}\cos i' + V + V' = 0.$$

Writing $z = \frac{r}{\sqrt{(c)}} = \frac{r'}{\sqrt{(r)}}$, and taking $\psi(z)$, any arbitrary function of z, we may write

$$\frac{dV}{dr} + \frac{4r}{a^2 - c^2 + r^2} V = \frac{4r}{a^2 - c^2 + r^2}\psi(z),$$

or

$$\frac{dV}{dz} + \frac{4cz}{a^2 - c^2 + cz^2} V = \frac{4cz}{a^2 - c^2 + cz^2}\psi(z);$$

and then

$$\frac{dV'}{dz} + \frac{4\gamma z}{a^2 - \gamma^2 + \gamma z^2} V' = \frac{-4\gamma z}{a^2 - \gamma^2 + \gamma z^2}\psi(z).$$

Integrating these equations, we find

$$V = \frac{4c}{(a^2 - c^2 + cz^2)^2}\int dz\,[z(a^2 - c^2 + cz^2)]\,\psi z,$$

$$V' = \frac{-4\gamma}{(a^2 - \gamma^2 + \gamma z^2)^2}\int dz\,[z(a^2 - \gamma^2 + \gamma z^2)]\,\psi z;$$

also

$$f = \frac{\sqrt{(c)}\,z}{a^2 - c^2 + cz^2}(\psi z - V).$$

Hence, making

$$\psi z = \frac{c^{\frac{3}{2}}\phi' z}{2z(a^2 - c^2 + cz^2)},$$

we have

$$f = \frac{\phi'(z)}{(\gamma - c + z^2)^2} - \frac{4z\phi z}{(\gamma - c + z^2)^3},$$

$$f' = -\frac{\phi' z}{(c - \gamma + z^2)^2} + \frac{4z\phi z}{(c - \gamma + z^2)^3},$$

ϕ being any arbitrary function, and z representing $\frac{r}{\sqrt{(c)}}$ or $\frac{r'}{\sqrt{(r)}}$. A similar method will apply to the determination of the forces at the foci whereby any conic may be described*.

* Employing the equation

$$V + V' + \frac{1}{2}\left(\frac{dV}{dr} + \frac{dV'}{dr'}\right)\rho\cos i = 0,$$

replacing $\rho\cos i$ by its equivalent $\frac{rr'}{a}$, writing $r - a = a - r' = z$, and decomposing the equation above written into

$$V + \frac{1}{2}\frac{dV}{dz}(a^2 - z^2) = \phi z; \qquad V' + \frac{1}{2}\frac{dV'}{dz}(a^2 - z^2) = \phi z,$$

It may be worth while pointing out a somewhat singular consequence of the laws that have been above established for the motion of a body in a circle about two reciprocal points as centres of force. It is an immediate and now well known, although for a time singularly overlooked, consequence* of the linear form of the equation $(\Sigma f \cos i)\,\rho = C - 2\Sigma\int dr\,(f)$ [where f is any central force, and i the angle which it makes with ρ, the radius of curvature at any point], which equation† exhibits the sole necessary and sufficient condition for any determinate orbit being described, that, if several sets

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integrating the two equations, and making suitable substitutions, thence results

$$\left.\begin{aligned} f &= \frac{r'}{r}\psi'(r-a) - 2\mu a\frac{\psi(r-a)}{r^2} \\ f' &= \frac{r}{r'}\psi'(a-r') - 2\mu' a\frac{\psi(a-r')}{r'^2} \end{aligned}\right\}.$$

* See an article by M. Ossian Bonnet among the valuable notes of M. Bertrand's edition of the *Mécanique Analytique*. The principle referred to must be taken with *analytical* latitude, or the range of its application will be unduly restricted. For instance, it is well known and easily demonstrable that a body starting from rest in a position where it is equally drawn by two forces converging to centres attracting according to the law of nature, will oscillate in the arc of an hyperbola. Here the principle seems inapplicable; for the hyperbola will be concave to one focus of attraction and convex to the other, but a curve actually described about either focus would be concave towards it. But in fact the principle does apply; for, analytically speaking, *any conic whatever* may be described about an attractive centre of force varying as the inverse square; only if it be convex to the centre of attraction, its *vis viva* will be a negative quantity, and the motion imaginary. In the case above supposed, the *vis viva* due to each centre of force acting singly will be equal, but with contrary signs, so that the body in such position must be supposed to be at rest; then, by virtue of the principle enunciated, it will for ever continue to move in the hyperbola, in which it would move *really* under the influence of one centre, *imaginarily* under that of the other,—the imaginary motion blended with the real continuous one changing the character of the latter into a reciprocating movement, which is in no way contradictory to M. Ossian Bonnet's theorem, which only determines the *locus*, but not the *direction* of the movement at any point.

† I am informed by the highest authority, the author of Reports on Mechanics, which have become classic, that he has never seen this equation anywhere before employed. It is of course an obvious generalization of Newton's rule, connecting the velocity with that due to a single central force acting through one-fourth of the chord of curvature. As it springs from a combination of the law of *vis viva* with that for centrifugal force, I propose to call it the *Equation of Radial Work*. By aid of it, it is easy to establish the following theorem, giving the most general binary system of forces acting to *two centres*, which will make a body describe *any* given orbit. Call V, V' the respective *force-functions* $\left(\text{so that } \frac{dV}{dr}, \frac{dV'}{dr'} \text{ are the two central forces}\right)$. Call P, P' the squared perpendiculars on the tangent from the two centres respectively, ϕ an *arbitrary* function of any *affection* of the position of the revolving body (for example, of the length of arc or radius of curvature at any point), then

$$V = \frac{\int \phi\, dP}{P}, \quad V' = \frac{-\int \phi\, dP'}{P'},$$

will be the general system in question. When the stored-up work for each point in the orbit is known, the *radial equation* gives the central forces without integration. Thus, for example, if a body move in an ellipse with uniform velocity acted on by forces towards the foci, the equation in question shows that they are equal, and vary as the inverse square of the conjugate diameter.

of forces taken separately can make a body describe a certain path, then all the sets acting collectively will make it describe the same path, provided the *vis viva* at the starting-point, on the latter supposition, is the sum of the *vires vivæ* on the former one.

Suppose, now, a zone of matter bounded by two arbitrary contours P, Q to lie anywhere within a circle C, and another zone bounded by two contours P', Q', the geometrical inverses (or reciprocals) of P, Q to lie outside the same. Then these two zones may be divided into corresponding rectangular elements by transversals drawn through the centre of the circle, points being taken all along every radius of one zone, and corresponding points along the radii of the other. If r, r' be the distances of the centres of any two corresponding elements thus obtained from the centre of the circle, $d\theta$ the angle between the two transversals which pass through both pairs of points in both figures, E, E', the areas of the respective elements, will be

$$d\theta \,.\, dr \,.\, r\,; \;\; d\theta\,(-dr')\,r', \text{ that is, } -d\theta d\left(\frac{a^2}{r}\right)\frac{a^2}{r}\,;$$

so that

$$\frac{E}{E'} = \frac{r^4}{a^4} = \frac{r^2}{r'^2}.$$

Hence if the densities of E, E' be the same, and they *attract* with forces varying as the inverse fifth power of the distance, they will serve to make a body describe the circle in question, E, E' taking the place of μ, μ' and r, r' of c, γ in our previous formulæ; and as this is true of each pair of elements, it will be true of the two entire zones which they compose, the law of density being perfectly arbitrary, except that it must be the same for *corresponding points* in the interior and exterior zones. The contour Q may be made to coincide with Q' at the circumference of C if we please; and then, as a particular case of the proposition above, we may suppose the united zones to consist of homogeneous matter, or, if we please, of matter whose density at any point is only a function of the angular position of the line joining the point to the centre of the circle. Thus, if we suppose a plate of matter of uniform density and of indefinite extent, and attracting according to the law of the inverse fifth power, a point anywhere placed upon it may be made to move in any desired circle under the influence of the plate's attraction, if we cut away a portion of the plate surrounding the centre of such circle, and leave a proper margin exterior to the circle—the rule being that the intrados of the figure so obtained may be of any form whatever, provided the extrados be its electrical image or inverse. The initial velocity to be communicated to the moving point will of course be determined by the form of either of these bounding curves.

It is hardly necessary to add that instead of a zone we may take a patch of matter bounded by a contour of any form within the circle C, and then,

finding the inverse of this contour so as to obtain a corresponding external patch, the two together, by the combined attractions of their particles according to the inverse fifth power of the distance, will serve to make a body describe the circle C; and conversely, since any two circles may be made reciprocals (inverses) to each other by duly determining the centre and radius of the circle of reference, it follows that any two circles of matter attracting according to the above law, will serve to keep a body moving in a certain third circle.

By calculating the attractions of these two circular images, and replacing them by forces tending to their centres, we shall be able to transform and generalize the results previously obtained. But first it will be expedient to recall attention to the form of the single central force which serves to make a body describe a circle. We have found that such force, when the centre lies within the orbit, is of the form $\frac{\mu\rho}{(\rho^2+k)^3}$; and it is easy to see that when external thereto, it takes the form $\frac{\mu\rho}{(\rho^2-k)^3}$, in either case k being the product of the two distances of the force-centre from the extremities of the diameter drawn through it; when the force is external, this product is the square of the tangent drawn to the circle from the centre. At the points of contact the force and velocity both become infinite, and the latter changes its sign.

In a physical sense, only the concave part of the circle will be described by virtue of attraction to the centre, the revolving body going off in a straight line towards the centre* when any point of contact is reached, and in like manner only the convex part by virtue of the repulsive force from the centre, the body going off in a straight line towards infinity on reaching such point; but inasmuch as in either case an infinitesimal deviation from the tangential direction will cause the remainder of the orbit to be described, we may consider, in an *analytical* sense, that the revolving body under the influences of such force describes the entire orbit. We may give the name of *clyclogenous* force to any central force of the form $\frac{-\mu\rho}{(\rho^2\pm k)^3}$, and, if we care to draw the distinction, call it internally cyclogenous or endocyclogenous when the k is positive, and externally cyclogenous or exocyclogenous when k is negative. If we call the cyclogenous-force-function V, so that $\frac{dV}{d\rho}$ is the cyclogenous force itself, we have, by integration, $V=\frac{1}{4}\cdot\frac{\mu}{(\rho^2\pm k)^2}$.

Let us now proceed to calculate the attraction of a circular plate (of radius r) of uniform density, whose particles attract according to the

[* Cf. p. 547, below.]

law of the inverse fifth power of the distance, upon an external particle at the distance ρ from the centre. If we call this $g\dfrac{dP}{d\rho}$, we have

$$P=\frac{g}{4}\int_0^r dr\int_0^{2\pi}\frac{rdr\,.\,d\theta}{(r^2+\rho^2-2r\rho\cos\theta)^2}.$$

By comparison of $\displaystyle\int_0^{2\pi}\frac{d\theta}{(r^2+\rho^2-2r\rho\cos\theta)^2}$ with the integral which represents twice the area of an ellipse of excentricity, $\dfrac{2r\rho}{r^2+\rho^2}$, we find instantaneously

$$P=\frac{\pi g}{4}\int_0^r\frac{2(\rho^2+r^2)\,rdr}{(\rho^2-r^2)^3}=\frac{\pi}{4}\frac{gr^2}{(\rho^2-r^2)^2}.$$

Thus P is of the form of the cyclogenous-force-function, so that the *force of attraction to the centre of a circular plate attracting according to the inverse fifth power of the distance, upon an exterior point, is an external cyclogenous force.* From this we may easily draw the conclusion that any circular orbit cutting orthogonally a circular plate whose particles attract according to the inverse fifth power of the distance may be described (or, at all events, the concave part of it be described) by virtue of such force of attraction.

Let us now consider the joint effect of two such circular plates, *images* of one another, lying one entirely within, the other entirely without a given circle. The centres of two such circles, it will be borne in mind, are *not* images of one another. Let r, r' be the radii of the two images, a of the image-making circle; call the distances of the centres of the images from that of the image-making circle c, c' respectively. The points of contact of the images with a common *exterior* tangent will be corresponding points, and this tangent will pass through the centre of the circle of reference; whence we easily derive $(c^2-r^2)(c'^2-r'^2)=a^4$, and by similar triangles

$$\frac{c}{c'}=\frac{r}{r'}.$$

Hence

$$(c^2-r^2)^2=\frac{c^2}{c'^2}a^4.$$

Whence, remembering that r must be less than c, we have

$$c^2-r^2=\frac{c}{c'}a^2,\quad c^2-r'^2=\frac{c'}{c}a^2;$$

so that

$$r^2=c^2\left(1-\frac{a^2}{cc'}\right);\quad r'^2=c'^2\left(1-\frac{a^2}{cc'}\right)*.$$

* Calling F, G the two centres, F', G' the images of F, G respectively, O the centre of the image-making circle, it is easily seen that $r^2=FO\,.\,FG'$, $r'^2=GO\,.\,GF'$.

Consequently, calling $1 - \frac{a^2}{cc'} = \pm q$, if F, G, two points in the diameter of the image circle, be distant c, c' respectively from its centre, and two cyclogenous forces $\frac{\lambda c^2 \rho}{(\rho^2 \mp qc^2)^3}$, $\frac{\lambda c'^2 \rho'}{(\rho'^2 \mp qc^2)^3}$ tend to F and G, two such forces will serve to make a body describe a circle, and, as we shall see, will be statically equivalent to a single cyclogenous force tending to a fixed point, presently to be determined*.

It follows from what has been shown of any two corresponding elements in the two figures, that the total *vis viva* contributed by each at any moment of time, to the entire amount of stand-up work in the revolving body is the same; consequently, confining our attention to one of the image circles, we see that $v^2 \propto \frac{1}{(\rho^2 \pm qc^2)^2}$. Hence using u to denote, as before, the angle at the centre, we have

$$\frac{du}{dt} \propto a^2 + c^2 \pm qc^2 - 2ac \cos u,$$

which is of the *form* which gives the motion of a planet in eccentric anomaly; consequently, by a proper adjustment of the constants, the motion due to the cyclogenous centres F, G may be made identical with the motion in a circle of radius a with centre at O, about the single cyclogenous-force centre at S. Call ae the distance of S from O, M the absolute force at S, then, comparing the *vis viva* on the two suppositions at the two apsidal points, and again availing ourselves of the law of equal production of *vis viva* from the two force-centres F, G, we obtain

$$\frac{2\lambda c^2}{[(a \mp c)^2 + qc^2]^2} = \frac{M}{[(a \mp ac)^2 + (a^2 - a^2 e^2)]^2}.$$

Hence

$$\frac{(a-c)^2 + qc^2}{(a+c)^2 + qc^2} = \frac{a - ae}{a + ae}$$

or

$$(1+q)\, c^2 - \frac{2a}{e} c + a^2 = 0.$$

Calling c, c' the two roots of this equation, we have

$$\frac{1}{c} + \frac{1}{c'} = \frac{1}{2ae};$$

or, which is the same thing, the points O, F, S, G form a system of four points in harmonic relation.

Hence, if we take a system of points, $F, F', F'' \ldots G, G', G'' \ldots$ in *involution*, the double points of the system being at S and O, the cyclogenous force at S

* The proof of this through the medium of the two circular images requires $-q$ to be employed; but the laws of *analytical* continuity allow q to be made to change its sign.

will be statically equivalent to two cyclogenous forces directed to any two corresponding points F, G.

It is possible that this theorem may be modified so as to admit of further generalization, and be made to extend to an arbitrary system of points in involution, without regard to the condition of O being one of the double points; but I have not had time to consider this point.

In the particular case where F, G become images, in respect to the circular orbit annexed to the force at S, the cyclogenous centres F, G become centres of attraction, following the law of the simple inverse fifth power, as already found. Since in all cases the absolute forces at F, G are proportional to the squares of their distances from O, if we make $cc' = a'^2$, and draw the circle whose centre is at O and radius is a', and take two figures, images of one another in respect to this circle, by the same reasoning as applied to the case of $a' = a$ it may be proved that, provided the densities at corresponding points of such images be the same, and the particles attract according to a certain fixed cyclogenous law, their joint action will support a body in a circular orbit whose radius is a and centre at O. We might again assume two such images to be circular, calculate the law of attraction towards the centres according to the supposed law, and so return to a new system of conjugate points replacing F and G; but I have not had time to ascertain whether such transformation would or would not lead to a new theorem, or merely, as is possible, to a repetition, with a new set of constants, of the one already obtained.

It is hardly necessary to point out how strongly the analogies established in the preceding investigations point to the existence of some simple dynamical theory of the Cartesian ovals under the attraction of forces directed to their foci. The investigation of such theory cannot but tend materially to the elucidation of the essential properties of these most interesting and as yet little-understood curves, the natural parents of the conic sections viewed as focal curves.

In conclusion it may be observed that, in the foregoing paper, it has been seen how a single orbital force passing through a fixed centre may be resolved into others of a more simple form. This suggests a more general subject of investigation, where the force to be so resolved, instead of passing through a fixed point, is tangential, or, better, normal to a fixed curve or surface.

Such an inquiry by no means belongs to ideal mechanics; for it would correspond to the case of the motion under the earth's attraction of a body near the earth's surface, considered as a surface of fluid equilibrium.

88.

ON AN IMPROVED FORM OF STATEMENT OF THE NEW RULE FOR THE SEPARATION OF THE ROOTS OF AN ALGEBRAICAL EQUATION, WITH A POSTSCRIPT CONTAINING A NEW THEOREM.

[*Philosophical Magazine*, XXXI. (1866), pp. 214—218.]

MY new rule (of which the demonstration will be found in a paper by the late lamented Mr Purkiss in the last Number of the *Cambridge, Oxford, and Dublin Mathematical Messenger*) for separating the roots of an algebraical equation, I mean the rule which bears to Newton's rule generalized the same relation as Fourier's to Descartes's, is susceptible of a certain slight improvement as regards the mode of statement, which appears to me deserving of notice.

If we suppose $fx=0$ to be the equation in the theorem as originally stated, I have employed the double progression

$$\begin{array}{llll} fx, & f_1x, & f_2x, \dots & f_nx, \\ Gx, & G_1x, & G_2x, \dots & G_nx, \end{array}$$

where f_rx means $\left(\dfrac{d}{dx}\right)^r fx$, and G_rx means $(f_rx)^2 - \gamma_r f_{r-1}x \,.\, f_{r+1}x$, γ_r being a known function of r involving an arbitrary parameter, confined between limits of which one is dependent on n.

In applying the theorem, it becomes necessary to count the number of compound successions for which, on writing a given value α for x, $f_r \,.\, f_{r+1}$ and $G_r \,.\, G_{r+1}$ are both simultaneously positive, and also the number of the same for which $f_r \,.\, f_{r+1}$, and $G_r \,.\, G_{r+1}$ are simultaneously negative and positive respectively, the succession $\boxed{\begin{array}{c|c} f_r & f_{r+1} \\ G_r & G_{r+1} \end{array}}$ in the first case constituting what I have called a double permanence, and in the other case a variation-

permanence. This latter is of course to be distinguished from a permanence-variation, which corresponds to the supposition of f_r, f_{r+1} bearing like, and G_r, G_{r+1} unlike signs—there being in fact four kinds of succession, namely, double permanences, variation-permanences, permanence-variations, and double variations.

If the enunciation of the theorem can be made to refer to double variations and double permanences exclusively, it is evident that something will have been gained in point of simplicity of statement*; and this can easily be effected in the manner following.

Let
$$H_r x = (f_r x)^3 - \gamma_r . f_{r-1}x . f_r x . f_{r+1}x,$$
so that
$$H_r x = f_r x . G_r x, \quad H_{r+1}x = f_{r+1}x . G_{r+1}x;$$
then, when $f_r x$, $f_{r+1}x$ have the same sign, the nature of the succession $H_r x$, $H_{r+1}x$ will evidently be the same as that of $G_r x$, $G_{r+1}x$; but when $f_r x$, $f_{r+1}x$ have unlike signs, the nature of the succession $H_r x$, $H_{r+1}x$ will be contrary to that of $G_r x$, $G_{r+1}x$.

Accordingly when $\begin{array}{|c|c|} f_r & f_{r+1} \\ G_r & G_{r+1} \end{array}$ constitutes a double permanence, $\begin{array}{|c|c|} f_r & f_{r+1} \\ H_r & H_{r+1} \end{array}$ will also constitute a double permanence; but when $\begin{array}{|c|c|} f_r & f_{r+1} \\ G_r & G_{r+1} \end{array}$ constitutes a variation-permanence $\begin{array}{|c|c|} f_r & f_{r+1} \\ H_r & H_{r+1} \end{array}$ will constitute a variation-variation, that is, a double variation.

If, then, we take for our double progression
$$\begin{Bmatrix} fx, & f_1x, & f_2x, \dots & f_nx, \\ Hx, & H_1x, & H_2x, \dots & H_nx, \end{Bmatrix}$$
the rule, or rather the independent pair of rules referred to, will take the following simplified form.

Supposing a, b to be any two real quantities in ascending order of magnitude, on substituting for x first a and then b, in the simultaneous progressions above written, double permanences (in passing from a to b) may be gained, but cannot be lost: double variations may be lost, but cannot be gained. And the number of real roots included between a and b will either be equal

* Moreover, so stated the theorem becomes more closely analogous to Fourier's. It may not be unreasonable to imagine that a third progression may remain to be invented such that the number of *triple* permanences and *triple* variations of sign in the three combined may afford a new superior limit, and so on *ad infinitum*; but this of course is at present a matter of pure conjecture.

or inferior to the number of double permanences so gained, and also equal or inferior to the number of double variations so lost—the difference, if there be any in either case, being some even number. The value of γ_r is $\frac{\nu+r-1}{\nu+r}$, where ν is limited not to fall within the limits 0 and $-n$. By ascertaining the gain of double permanences and the loss of double variations consequent on the replacement of a by b, we are furnished with two *independent* superior limits to the number of real roots included between a and b.

Postscript.

It often happens that the pursuit of the beautiful and appropriate, or, as it may be otherwise expressed, the endeavour after the perfect, is rewarded with a new insight into the true. So it is in the present instance; for the substitution of the H for the G series, devised solely for the purpose of giving greater clearness to the enunciation of a known theorem, leads to a supplemental theorem which combines with and lends additional completeness and harmony to the original one.

At present the theory stands thus: a superior limit to the number of real roots between two limits a and b is afforded by counting, as x ascends from the one to the other, the loss of changes or gain of permanences (these two numbers are identical) in the f or Fourierian progression, and also by counting the loss of double changes, or gain of double permanences, in the f and H progressions combined: these two are distinct. We have thus the choice of three superior limits. I shall show that a fourth independent one is afforded by considering the loss of changes or gain of permanences in the single H progression, and combining it with such loss or gain in the single f progression.

We have

$$H_r x = f_r x \,.\, G_r x,$$

where

$$G_r x = (f_r x)^2 - \gamma_r (f_{r-1} x)(f_{r+1} x),$$

γ_r being essentially positive for all values of γ.

It has been proved [p. 502, above] (see Mr Purkiss's paper above referred to) that, when $G_r x = 0$,

$$G_r(x+\epsilon) = \frac{\epsilon}{\gamma_{r+1}} \cdot \frac{f_r x}{f_{r+1} x} G_{r+1} x,$$

ϵ being an infinitesimal.

Now suppose $H_r x = 0$. This may happen in two distinct ways, namely, either when $G_r x = 0$, or when $f_r x = 0$.

1. Let $G_r x = 0$, then

$$\frac{d}{dx} H_r x = f_r x \frac{d}{dx} G_r x.$$

Hence $$H_r(x+\epsilon) = \frac{\epsilon}{\gamma_{r+1}} \cdot \frac{(f_r x)^2}{f_{r+1}x} \cdot G_{r+1}x$$
$$= \frac{\epsilon}{\gamma_{r+1}} \left(\frac{f_r x}{f_{r+1}x}\right)^2 H_{r+1}x.$$

2. Let $f_r x = 0$, then
$$\frac{dH_r x}{dx} = G_r x \,.\, f_{r+1}x\,;$$
also $$G_{r-1}x = (f_{r-1}x)^2\,;\; G_r x = -\gamma_r \,.\, f_{r-1}x \,.\, f_{r+1}x\,;\; G_{r+1}x = (f_{r+1}x)^2.$$
Thus $H_{r-1}x$, $H_r(x+\epsilon)$, $H_{r+1}x$ are conformable in signs to
$$f_{r-1}x, \quad -f_{r-1}x \,.\, \epsilon, \quad f_{r+1}x \qquad (\alpha)$$
in this case, and in the case preceding to
$$H_{r-1}x, \quad H_{r+1}x \,.\, \epsilon, \quad H_{r+1}x. \qquad (\beta)$$

The above cases have reference to any *intermediate* H becoming zero; the final H is $(fx)^3$, and the last but one is
$$(f'x)^3 - \gamma_1 (f''x \,.\, f'x \,.\, fx)\,;$$
and accordingly when $fx = 0$, $H_1 x$, $H(x+\epsilon)$ become of the same signs as
$$f'x\,;\; \epsilon f'x. \qquad (\gamma)$$

By combining the results (α), (β), (γ), and denoting by ν the number of real roots included between (a) and (b), it is easy to infer the equation
$$\nu = P - 2\phi + 2\eta,$$
where P is the number of permanences gained in passing up x from a to b in the H progression, ϕ is the collective number of times that any intermediate G vanishes at a moment when the preceding and subsequent H's have like signs, and η is the collective number of times that any intermediate f vanishes at a moment when the two adjacent f's have like signs. But if p is the number of permanences gained from the f (Fourier's series) by passing up x from a to b, we have $\nu = p - 2\eta$, where η represents the same quantity as above.

Hence $$2\nu = P + p - 2\phi\,;$$
and accordingly there emerges a new superior limit to ν, namely, $\frac{P+p}{2}$, an unlooked-for and striking conclusion.

Thus, for example, if $p = P + 2$, ν cannot be greater than $P + 1$, and therefore not greater than P, because it must differ from p or P (whose sum is necessarily even) by an even number*. To make the preceding demonstra-

* And so in general, when $p - P$ is positive and not divisible by $4p$, the superior limit given by Fourier's theorem may be replaced by $\frac{p+P}{2} - 1$.

tion absolutely rigorous, it would be necessary to consider the singular cases when several consecutive terms of the H or G series vanish simultaneously, either with or without the corresponding terms of the f series vanishing too: this inquiry, which is necessarily tedious, and the result of which it is easy to anticipate, must be adjourned to a more suitable occasion.

If we call Λ the new superior limit, we have found

$$\Lambda - \nu = \phi,$$

where ϕ is the collective number of values of x included between a and b for which any function $G_r x$ vanishes, whilst $H_{r-1}x$ and $H_{r+1}x$ have like signs; but since, when $G_r x = 0$, $f_{r-1}x$ and $f_{r+1}x$ must have like signs, ϕ may be defined more simply as the number of values of x between the given limits for which simultaneously, and for any value of r, $G_r x$ vanishes whilst $G_{r-1}x$ and $G_{r+1}x$ have like signs.

This quantity ϕ, the difference between the limit and the number of roots limited, may be odd or even, and not necessarily the latter, as is the case in all existing theorems of a similar nature.

Since $\Lambda = \frac{p+P}{2}$, it follows that when $P=0$, that is, whenever the passage from a to b leaves the number of permanences in the H series unaltered, the limit p given by Fourier's theorem may be replaced by $\frac{p}{2}$ or $\frac{p}{2} - 1$, according as p is or is not divisible by 4.

89.

NOTE ON THE PERIODICAL CHANGES OF ORBIT, UNDER CERTAIN CIRCUMSTANCES, OF A PARTICLE ACTED ON BY A CENTRAL FORCE, AND ON VECTORIAL COORDINATES, ETC., TOGETHER WITH A NEW THEORY OF THE ANALOGUES TO THE CARTESIAN OVALS IN SPACE, BEING A SEQUEL TO "ASTRONOMICAL PROLUSIONS."

[*Philosophical Magazine*, XXXI. (1866), pp. 287—300.]

A VERY singular and previously unnoticed species of discontinuity arises when, according to the equations of motion interpreted in the ordinary manner, a particle solicited by a continuous central force *would seem* as if it ought to describe an orbit external to the force-centre. An instance of this kind, probably for the first time, presented itself in a question incidentally brought forward by myself in the paper inserted in the January number of this *Magazine*, where* I alluded, in passing, to the case of a body acted on by a central force capable of making it move in a circle exterior to the force-centre, and fell into the not unnatural error, which has since been pointed out to me, and which is obvious on a moment's reflection, of stating that on arriving at a point where the motion points to the force-centre, that is, at the point where the tangent to the circle passes through this centre, the particle would go off in a straight line on account of the motion and the force coinciding in direction. But it is clear that since the instantaneous area $\frac{1}{2}\rho^2\frac{d\theta}{dt}$ remains finite at such point, it cannot abruptly become zero; the radial velocity becoming infinite, does not entitle us to reject the transverse part which remains finite; thus the radius vector ρ will continue to revolve in the same direction as before it reached the tangential point; it will therefore swing off to another curve, so that the true orbit will possess an inflexion at that point. The new curve, it may easily be proved, will be a circle equal to the former, and related to it in the manner following: let us suppose O to be the force-centre and two tangents drawn from O to meet the original circle in A and B, so that the line AB divides the circle into two unequal

[* p. 538, above.]

segments, and that the particle has been travelling, say in the upper segment, from A to B; draw the angle BOC equal to the angle AOB, and in it place a circle equal to the former, touching the part OB, OC in B and C; then the particle will describe the *lower* segment of this new circle; and so in like manner, after reaching C, will undergo a new inflexion at that point and pass on to a new circle touching OC and OD, the latter inclined to the former at the same angle as OC to OB and OB to OA. Thus, if we repeat the angular sector AOB indefinitely, and in each such sector place equal circles touching the rays of the sector, and call their upper and lower segments P, Q respectively, the particle will describe the successive arcs $P_1, Q_2, P_2, Q_3, P_3, \ldots$ *ad infinitum*. If the sectorial angle be an even aliquot part of 360°, the complete orbit will be a single anautotomic broken curve returning into itself, as, for instance, if the sector be 90° the orbit will be $P_1, Q_2, P_2, Q_2, P_3, Q_3, P_4, Q_4, P_1, Q_2, \ldots$. If the angle be an odd aliquot part of the same, the orbit will be a line returning into but crossing itself as many times as there are circles, so that in fact the whole of each circle will be described in a complete period, namely, the upper and lower segments alternately in the first half period, and the lower and upper in the second half thereof, the *period* being double the time of a revolution if the latter is defined as the interval between the body leaving and returning to any the same point. Thus, for example, if the angular sector be 72°, the orbit will be

$$P_1 Q_2 P_2 Q_3 P_3 Q_4 P_4 Q_5 P_5 Q_1 P_2 Q_2 P_3 Q_3 P_4 Q_4 P_5 Q_5 P_1 \ \&\text{c.}$$

In like manner, if the angle included between the tangents be any commensurable part of 360°, as $\frac{m}{n}$ 360°, where m and n are integers prime to one another, the orbit will be a closed one containing mn alternate segments, or mn entire circles, according as n is even or odd. By taking n even and giving m any arbitrary odd value, a waving line will be produced forming an original and, I think, elegant pattern for a *circular lace border*. For this purpose $\frac{m}{n}$ should not be too small, in order that the disproportion between the alternate circular segments and the ratio of the border to the interior may not be so great as to offend the eye; and m not too great, in order that the traces of the pattern may not become too complicated.

I conjecture that $[m=3;\ n=64]$ and $[m=5;\ n=128]$, producing respectively 3 and 5 twists, and 5 or 6 and 6 or 7 flexures within a quadrant of each twist, would be eligible systems for the purpose. In general n ought to be even, and m a large moderate odd integer.

If the angle between the tangents to the circle from the force-centre be not an aliquot or commensurable part of 360°, the orbit will be a non-reentrant curve intersecting itself an infinite number of times. Similar or

analogous conclusions are of course applicable to every case where the orbit, *seemingly* indicated by the equations of motion, is an oval, or, more generally, any curve to which tangents admit of being drawn from the force-centre,—a self-evident (now that it is stated) but none the less a very surprising feature in the *mathematical* theory of central forces. I say *mathematical*; for it ought in fairness to be observed that since it is impossible to conceive a force of infinite magnitude resulting from the attraction of a finite mass, the question involves not so much a discussion of any real phenomenon, as of the principles of interpretation applicable to an extreme case, valueless as to the establishment of a distinct independent conclusion, although not without latent importance as a safeguard against errors which might flow from the adoption of an erroneous mode of interpretation*.

It may readily be found that the velocity at any point of the orbit must be that due to infinity (otherwise a different and much more complicated curve would result), and then with the usual notation the differential polar equation to the curve becomes

$$\left(\frac{d\theta}{dr}\right)^2 = C\frac{(r^2-k^2)^2}{\sqrt{\{r^4-r^2(r^2-k^2)^2\}}},$$

which is easily seen to be true of any arc of a circle. The phenomenon to be noticed is, that when $\frac{d\theta}{dr}=0$, since t is the real independent variable, θ does not attain a maximum or minimum, for it is $\frac{dr}{dt}$ becoming infinite, not $\frac{d\theta}{dt}$ becoming zero, which accounts for $\frac{d\theta}{dr}$ vanishing; accordingly $\frac{d\theta}{dr}$ in passing through zero must be taken with a change of sign, which accounts for the discontinuity of the orbit regarded as a geometrical curve. This change of sign in the radical is very analogous to what happens when we calculate the potential of a spherical shell, and trace its value as the attracted point continuously receding from the centre passes from within to without the shell.

As connected with this subject of motion in a circle, I may mention that Mr Crofton has pointed out to me that my theorem concerning a homogeneous circular plate whose molecules attract according to the inverse fifth power of the distance, namely that its resultant attraction is capable of

* If we accept the very reasonable axiom that no law of force is admissible which would involve the consequence of a *finite* mass exerting an infinite attraction at a finite distance, we can find an *à priori* limit to the negative exponent of the power of the distance which can *possibly* express any law of force in nature. If my memory serves me aright, a distinguished rising French analyst, in contravention of this axiom, has assumed, for the purpose of explaining certain optical phenomena, a law of force according to some very high inverse power of the distance transcending such limit. It will be seen below that the inverse fifth power is inadmissible on this ground, and is capable of leading to irreconcileable contradictions.

making a particle move in any circle cutting the plate orthogonally, admits of being established upon my own principles without calculating, as I have done, the law of the attraction (Astronomical Prolusions*, *Phil. Mag.* Jan. 1866, p. 73); for the whole plate may be shown to be its own inverse in respect to any such orthogonal dividing circle; that is, the two parts into which it is divided by the plate will be inverses to each other in respect to the orthogonal circle, and consequently conjointly will serve to make a particle move in a segment of such circle exterior to the plate†.

Mr Crofton has also made a partial extension of the theorem to the case of a plate of the form of either one of a conjugate pair of Cartesian ovals, in a remarkable paper on the theory of these curves, lately read before the London Mathematical Society. In the "Prolusions" I raised the question of determining the force at a focus required to make a body move in such oval. This may easily be solved by aid of vectorial coordinates; and as it seems desirable to place on record the tangential affections of a curve expressed in terms of such coordinates, which I am not aware has hitherto been done, I subjoin the investigation for the purpose. The results will be seen to be of great use in simplifying the solution of the important problem of determining the most general motion of a body attracted to two or more fixed centres, a problem to which I purpose hereafter to return.

If F, G be two foci, c their distance from one another, f, g their distances from any point in a curve, ds the element of the arc at the point (f, g), θ, η the angles which ds makes with f and g, we have

$$\cos\theta = \frac{df}{ds}, \quad \cos\eta = \frac{dg}{ds}.$$

Call $g + f = u$, $g - f = v$, and let ω be the angle between f and g. Then

$$(\cos\theta)^2 + (\cos\eta)^2 + (\cos\omega)^2 - 2\cos\omega \, . \cos\theta \, . \cos\eta - 1 = 0‡.$$

Hence
$$(ds)^2 = \frac{df^2 + dg^2 - 2\cos\omega \, . \, df \, . \, dg}{(\sin\omega)^2}.$$

[* p. 539, above.]

† And equally it follows that a homogeneous plate whose molecules exert a *repulsive* force following the inverse fifth power of the distance, would serve to make a particle move in the *interior* segment of an orthogonal circle. Quære as to how the motion must be conceived to take place when the attracted or repelled particle enters or quits the plate? To fix the ideas, suppose the plate attractive. The orbit described within the plate must touch the radius, for the force becomes infinite in the direction of the radius, and must tend towards the centre without becoming convex to it, on account of the force being attractive. I do not see how these conditions can be reconciled, except by supposing the remainder of the motion to take place along the radius itself, which involves the supposition of the transverse velocity at immergence becoming instantaneously destroyed, and the same at emergence when the force is repulsive.

‡ The left-hand side of this equation, calling the directions of f, g, ds, A, B, C, is

$$\begin{vmatrix} 0, & \cos AB, & \cos AC, & 1 \\ \cos BA, & 0, & \cos BC, & 1 \\ \cos CA, & \cos CB, & 0, & 1 \\ 1, & 1 & 1, & 0 \end{vmatrix}$$

And, by trigonometry,

$$1+\cos\omega=\frac{u^2-c^2}{2fg}\,; \quad 1-\cos\omega=\frac{c^2-v^2}{2fg}\,.$$

Hence
$$(ds)^2=\frac{(du)^2(1-\cos\omega)-(dv)^2(1+\cos\omega)}{(\sin\omega)^2}$$

$$=\frac{(c^2-v^2)\,du^2+(c^2-u^2)\,dv^2}{(u^2-v^2)}\,;$$

and again,

$$\sin\theta=\sqrt{\left(\frac{(ds)^2-(df)^2}{(ds)^2}\right)}=\frac{df-dg\,.\cos\omega}{\sin\omega ds}=\frac{du(1-\cos\omega)-dv(1+\cos\omega)}{2\sin\omega ds}$$

$$=\frac{(c^2-v^2)\,du-(u^2-c^2)\,dv}{4fg\sin\omega\,ds}$$

$$=\frac{(c^2-v^2)\,du+(u^2-c^2)\,dv}{4fg\sin\omega ds}=\frac{(c^2-v^2)\,du-(c^2-u^2)\,dv}{\sqrt{[(u^2-v^2)\{(c^2-v^2)du^2+(c^2-u^2)\,dv^2\}]}},$$

and similarly,

$$\sin\eta=\frac{(c^2-v^2)\,du+(c^2-u^2)\,dv}{\sqrt{[(u^2-v^2)\{(c^2-v^2)\,du^2+(c^2-u^2)\,dv^2\}]}}.$$

It is worthy of passing observation that the above expressions lead immediately to the integral of the fundamental equation in the addition of elliptic functions; for if we call p, q the two perpendiculars from the foci upon ds, we have

$$\frac{(c^2-v^2)^2(du)^2-(u^2-c^2)^2\,dv^2}{(c^2-v^2)\,du^2-(u^2-c^2)\,dv^2}=4fg\sin\theta\,.\sin\eta=4pq.$$

Suppose now
$$4pq=c^2-a^2.$$

Then
$$(c^2-v^2)^2\,du^2-(u^2-c^2)^2\,dv^2=(c^2-a^2)(c^2-v^2)\,du^2-(c^2-a^2)(u^2-c^2)\,dv^2,$$

or
$$\frac{du^2}{(u^2-a^2)(u^2-c^2)}-\frac{dv^2}{(v^2-a^2)(v^2-c^2)}=0.$$

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and in like manner for four lines in space A, B, C, D in spaces, the determinant

$$\begin{vmatrix} 0, & \cos AB, & \cos AC, & \cos AD, & 1 \\ \cos BA, & 0, & \cos BC, & \cos BD, & 1 \\ \cos CA, & \cos CB, & 0, & \cos CD, & 1 \\ \cos DA, & \cos DB, & \cos DC, & 0, & 1 \\ 1, & 1, & 1, & 1, & 1 \end{vmatrix}=0.$$

This important equation is nowhere *explicitly* given in treatises on trigonometry or determinants, but is virtually included in a theorem which is to be found in Balzer, and probably elsewhere, as affirmed concerning the four sides of a wry quadrilateral; for *any* four lines in space which meet in a point being given, a wry quadrilateral may be formed with sides parallel respectively to the same. The above equation enables us to express the element of a curve in space in terms of vectorial coordinates and their differentials.

The integral, therefore, of this equation must express the fact that u and v are, or may be regarded as, the sum and difference of the distances of two fixed points distant c apart from any point in a fixed straight line, the product of whose distances from those points is $c^2 - a^2$, or also, if we please, as the sum and difference of the distances of two fixed points distant a apart from any point in a fixed straight line the product of whose distances from the points is $a^2 - c^2$.

Thus, parting from the first construction, if we write $y + \lambda x = L$ as the equation to the straight line, the origin being taken midway between the two points, and the axis of x coincident with the line joining them, we obtain

$$c^2 - a^2 = \frac{L^2 - \lambda^2 \dfrac{c^2}{4}}{1 + \lambda^2},$$

or

$$L^2 = \frac{c^2}{4}\lambda^2 + (c^2 - a^2)(1 + \lambda^2);$$

we have also $\quad u^2 = y^2 + \dfrac{c^2}{4} - cx + x^2; \quad v^2 = y^2 + \dfrac{c^2}{4} + cx + x^2;$

so that $\quad x = \dfrac{v^2 - u^2}{2c}, \quad y^2 = u^2 - v^2 - \dfrac{c^2}{2} - \dfrac{(v^2 - u^2)^2}{2c^2},$

and that the required integral will be

$$\sqrt{\left((u^2 - v^2) - \frac{c^2}{2} - \frac{(v^2 - u^2)^2}{2c^2}\right)} + \lambda \frac{v^2 - u^2}{2c}$$
$$+ \sqrt{\left(\frac{c^2}{4}\lambda^2 + (c^2 - a^2)(1 + \lambda^2)\right)} = 0,$$

which, completely rationalized, will lead to an equation of the eighth degree in u, v, and quadratic in λ^2.

A similar rational equation in u, v, μ^2 can be obtained by interchanging a and c with one another, and λ with μ; and as each equation represents the complete integral, μ^2 will necessarily be a *linear function* of λ^2 when each is regarded as a function of u, v. This linear relation we can establish *à priori*; for we have

$$y + \lambda x = \sqrt{\left(\frac{c^2}{4}\lambda^2 + (c^2 - a^2)(1 + \lambda^2)\right)},$$

$$y + \mu x = \sqrt{\left(\frac{a^2}{4}\mu^2 + (a^2 - c^2)(1 + \mu^2)\right)}.$$

Hence making $x = 0$, we have

$$(5a^2 - 4c^2)\mu^2 - (5c^2 - 4a^2)\lambda^2 + 8(a^2 - c^2) = 0.$$

If we are content to leave the integral irrational in λ or μ respectively, then it presents itself under the form of a biquadratic rational equation in u and v.

Combining the above construction of the integral with the well-known one through spherical triangles, we obtain an interesting geometrical theorem, namely, that if from a given spherical lune two arcs be cut off by an arc of constant length, their *sines* may always be represented by the sum and difference of the distances of two fixed points from a variable point in a fixed straight line; and moreover there will be two systems of such line and associated points.

Besides the general integral, we have also the singular ones given by

$$u = a \text{ or } v = a, \text{ or } u = c \text{ or } v = c,$$

indicating the familiar proposition that the product of the focal distances from the tangents of an ellipse or hyperbola are constant; $u = a$ and $v = c$ will correspond to an ellipse and hyperbola, of which the foci in the one are the vertices of the other, and *vice versâ*. If from any external point we draw a pair of tangents to either of these curves, $\frac{du}{dv}$, that is $\frac{df+dg}{df-dg}$, and therefore $\frac{df}{dg}$, will have the same value at each point of contact; so that if α, α' and β, β' be the angles which the tangents respectively make with the focal distances of the points of contact, we have $\frac{\cos\alpha}{\cos\alpha'} = \frac{\cos\beta}{\cos\beta'}$ and also $\alpha' - \alpha$ the same in absolute magnitude as $\beta' - \beta$, from which it is easy to infer $\alpha = \beta$, $\alpha' = \beta'$, showing that the tangents to an ellipse or hyperbola make equal angles with the focal distances at the points of contact, as is also known from the theory of confocal conics.

In precisely the same manner we may integrate the general equation $F(2p, 2q) = C$, where

$$2p = \sqrt{\left(\frac{u+v}{u-v}\right)}\frac{(c^2-v^2)\,du + (c^2-u^2)\,dv}{\sqrt{\{(c^2-v^2)\,du^2 + (c^2-u^2)\,dv^2\}}},$$

$$2q = \sqrt{\left(\frac{u-v}{u+v}\right)}\frac{(c^2-v^2)\,du - (c^2-u^2)\,dv}{\sqrt{\{(c^2-v^2)\,du^2 + (c^2-u^2)\,dv^2\}}},$$

F being any form of function whatever; the integral will always be

$$\sqrt{\left\{(u^2-v^2) - \frac{c^2}{2} - \left(\frac{v^2-u^2}{2c^2}\right)^2\right\}} + \lambda\frac{v^2-u^2}{2c} + L = 0,$$

where the relation between L and λ depends upon, and may be determined from, the nature of F*.

* By varying the curve to which ds refers, we may obtain innumerable classes of differential equations whose integrals can be determined. Moreover, by taking ds the element of a curve in space referred to three foci, ds can be expressed by aid of the theorem given in a previous footnote as a function of the three focal distances f, g, h and their differentials; and consequently the lengths of the perpendiculars upon it from the three foci can be expressed in like manner, and we may thus obtain integrable forms of simultaneous *binary* systems of differential equations between f, g, h.

As regards the expression for ρ, the radius of curvature in terms of vectorial coordinates, we may employ the well-known formula

$$\frac{1}{\rho}=\frac{ds^2}{\sqrt{\{(d^2x)^2+(d^2y)^2\}}},$$

where
$$x=\frac{f^2+c^2-g^2}{2c}=\frac{uv+c^2}{2c},$$

$$y=\frac{2\sqrt{\left(\frac{u+c}{2}\cdot\frac{u-c}{2}\cdot\frac{c+v}{2}\cdot\frac{c-v}{2}\right)}}{c}=\frac{\sqrt{\{(u^2-c^2)(c^2-v^2)\}}}{2c};$$

so that
$$2\frac{c}{\rho}=\frac{ds^2}{\sqrt{[\{d^2(uv)\}^2-\{d^2\sqrt{[(c^2-u^2)(c^2-v^2)]}\}^2]}}$$
$$=\frac{(c^2-v^2)\,du^2+(c^2-u^2)\,dv^2}{(u^2-v^2)\sqrt{[\{d^2(uv)\}^2-\{d^2\sqrt{[(c^2-u^2)(c^2-v^2)]}\}^2]}},$$

which I have not thought it necessary to reduce further. As regards the original question of determining the central force towards a focus, say F, proper to make a body move in a Cartesian oval, we have

$$-F=\frac{1}{2}\frac{d}{df}v^2=2h^2\frac{d}{df}\cdot\left(\frac{1}{2p}\right)^2,$$

where $\frac{h}{2}$ is the instantaneous area, and, if the equation to the oval be $f-kg=m$,

$$df=kdg;\quad du=(1+k)\,dg;\quad dv=(1-k)\,dg;\quad u=\left(1+\frac{1}{k}\right)f-\frac{m}{k};$$
$$v=\left(1-\frac{1}{k}\right)f+\frac{m}{k};$$

so that
$$\left(\frac{1}{2p}\right)^2=\frac{(u-v)\{(1+k)^2(c^2-v^2)+(1-k)^2(c^2-u^2)\}}{(u+v)\{(1+k)(c^2-v^2)+(1-k)(c^2-u^2)\}^2},$$

from which F may be calculated and expressed under the form $\frac{P}{f^2Q}$, where P and Q are each rational integral functions of the fourth degree in f.

It does not seem to me worth while to work out the actual values of P, Q for the general form of the oval (in algebra, as in common life, there is wisdom in knowing where to stop); but it did appear to me desirable to ascertain the *form* of the expression for the retaining force, which, it is hardly necessary to add, it would have been quite impossible to do had the ordinary system of coordinates been employed. The fact of this force being a rational function of the distance is a result not without interest; and for particular varieties of the curves belonging to the class of Cartesian ovals, it will be easy to obtain its actual value as a function of the distance.

POSTSCRIPT.

On the Curve in Space which is the Analogue to the Cartesian Ovals in plano.

By a Cartesoid we may understand a surface such that a linear relation exists between the distances of any point in it from three fixed points in a plane, and by a twisted Cartesian the intersection of two Cartesoids whose three fixed points of reference are identical. A twisted Cartesian, then, will be a curve in space whose distances from three fixed points (its foci) are connected by two linear relations: from this it is obvious that it may be conceived also as the intersection of two surfaces of revolution generated by the rotation about their lines of foci of two plane Cartesians having one focus in common, so that it will consist of a system of closed rings. If F, G, H, K be any four points in a plane, and if the areas of the triangles GHK, HKF, KFG, FGH be called F_1, G_1, H_1, K_1 respectively, and P be any point in space, it is easy to prove that

$$F_1 . PF^2 - G_1 . PG^2 + H_1 . PH^2 - K_1 . PK^2 = E,$$

where E is a sort of *geometrical invariant* independent of the position of P. Its value may be expressed by the equation

$$-16E^2 = \begin{vmatrix} 0, & FG^2, & FH^2, & FK^2 \\ GF^2, & 0, & GH^2, & GK^2 \\ HF^2, & HG^2, & 0, & HK^2 \\ KF^2, & KG^2, & KH^2, & 0 \end{vmatrix}.$$

By making P coincide with F we find

$$\pm E = FG^2 . HKF + FH^2 . GKF - FK^2 . GFH.$$

Hence, if the position of K be determined by linear coordinates, x, y, and of F, G, H by coordinates of the like kind, it is obvious that E becomes a rational quadratic function of x, y; F_1, G_1, H_1 linear functions of x, y; and K_1 independent of x, y.

Let P be any point in a twisted Cartesian whose foci are F, G, H; ρ, σ, τ the distances of P from these foci. Then we have

$$l\rho + m\sigma + n\tau + p = 0, \qquad (1)$$

$$l'\rho + m'\sigma + n'\tau + p' = 0, \qquad (2)$$

where l, m, n, p; l', m', n', p' are constants.

Let v be the distance of P from K, then

$$F_1\rho^2 - G_1\sigma^2 + H_1\tau^2 - E = -K_1 v^2, \qquad (3)$$

and v will be a linear function of ρ, σ, τ, provided that the values of ρ, σ in terms of τ determined from (1) and (2) make the left-hand side of (3) a perfect square.

The condition that this may happen is

$$\begin{vmatrix} F_1, & 0, & 0, & 0, & l, & l' \\ 0, & -G_1, & 0, & 0, & m, & m' \\ 0, & 0, & H_1, & 0, & n, & n' \\ 0, & 0, & 0, & -E, & p, & p' \\ l, & m, & n, & p, & 0, & 0 \\ l', & m', & n', & p', & 0, & 0 \end{vmatrix} = 0. \qquad (4)$$

It is easy to see that the determinant above written consists exclusively of terms in which only *binary* combinations of F_1, G_1, H_1, E appear. Consequently equation (4) is an equation of the third degree in x, y. When this equation is satisfied, K is a focus just like F, G, H. Hence we may conclude that any given twisted Cartesian possesses an infinite number of foci, every point that lies in a certain curve of the third degree being a focus. When three foci are given there are four disposable parameters, and no more, for determining this curve, which therefore cannot be any cubic curve, but is subject to satisfy two conditions. This cubic curve of foci for the twisted Cartesian is the analogue of the three focal points appertaining to the ordinary plane Cartesian*.

We are now in a position to obtain a much simpler mode of genesis of the twisted Cartesian. If F, G, H be any three points in a right line whose distances from each of a group of points in a plane more than *two* in number are subject to two linear relations, it is easy to prove that these latter will lie in a Cartesian oval, of which F, G, H are the three foci. If then we draw any transversal in the plane of the focal cubic cutting it in three points F, G, H, and make a plane revolve about this line, each group of points in which the twisted curve is cut by this revolving plane being subject to the same two linear conditions of distance from F, G, H, they and therefore the entire twisted curve will lie in a surface generated by the revolution of a certain Cartesian oval about F, G, H. By drawing F, G, H parallel to an asymptote†, one of the points, say H, goes off to infinity, and F, G become the foci of a

* It is due to Mr Crofton to state that the idea which has led to the discovery of this property of the twisted Cartesian was suggested by the method employed by that excellent geometer for establishing the existence of the third focus for the plane ovals, as described by him in a remarkable paper on the theory of these curves read before the London Mathematical Society on the 19th instant. It is important to notice that, since the distances of the points in the twisted curve from any one of the original foci are linearly related to those from any other point L, and also from any other point M in the focal cubic, the distances from L and M are themselves linearly related.

† It will presently appear that there is but one real asymptote to the focal cubic.

conic; and as we may draw any other transversal parallel to the former cutting the cubic in two other points F', G', we learn that the twisted Cartesian is always expressible as the intersection of two surfaces of revolution of the second degree whose axes are parallel, and is thus a curve of only the fourth order. It follows, moreover, that the focal cubic is the locus of the foci of a family of conics in involution whose axes are parallel.

But we may still further simplify the conception of these remarkable analogues to the ovals of Descartes. One of the system of parallels last described will be the asymptote itself meeting the cubic in only one point, so that the revolving conic becomes a parabola; and again, if we draw another transversal parallel to the asymptote and touching the cubic, the two foci come together, and the conic becomes a circle. Hence *every twisted Cartesian is the intersection of a sphere and a paraboloid of revolution**.

We are now in a position to turn back upon the focal cubic itself and make it disclose its true nature; for it will be no other than one of the two curves of foci of the system of conics passing through four points which lie in a circle. The axes of such a system always retain their parallelism; and consequently there will be two separately determinable curves of foci—those, namely, which lie in one set of parallel axes, and those which lie in the other. By a general theorem of M. Chasles, the complete curve of foci is of the sixth order, and consequently each of the two in question ought to be, as we learn from the preceding theory it is, a curve of only the third degree†.

The equation of either may easily be found, and is of the form

$$x(x^2 + y^2 + A) + Bx^2 + Cxy + Dy^2 = 0,$$

to which there is only one real asymptote, namely, $x + D = 0$. This, then, is the general equation to the focal cubic to a twisted Cartesian, and shows it to belong to the class of circular cubics.

The focal cubic is or may be determined by a circle involving three constants and four points arbitrarily chosen in the circle, which, together with the three constants for fixing the plane of the circle, give ten parameters in all.

It passes through the intersections of the three pairs of opposite sides of the quadrilateral inscribed in the circle, the centre of the circle, and the two circular points at infinity; the special relations of the three intersections to the cubic await further investigation. The twisted cubic with which it is associated may be determined by means of two right cones, each involving

* Or, as is evident from the text, the intersection of two (and therefore also of *three*) right cones with parallel axes whose plane will contain the focal cubic.

† Every focal cubic to a given twisted Cartesian has thus its conjugate corresponding to another twisted Cartesian, which may be regarded as the conjugate of the first; and the mutual relations of such curves seem to *invite* further investigation.

six constants; but as the axes must be coplanar and parallel, the number of parameters is reduced from twelve to ten, thus showing that, when the focal curve is given, the associated ovals are determined (in this respect differing from the plane ovals, in which one parameter remains indeterminate when the trifocal system of points—the analogue of the focal cubic—is given). It will probably be found that when five points in the focal curve are given, thus leaving two parameters disposable, the twisted ovals drawn through any given point will cut each other orthogonally, as Mr Crofton has shown to be the case for the plane curves in his beautiful paper on the Cartesian ovals. I find that when the focal cubic is defined by means of the circle $x^2+y^2-c^2=0$, and of its intersection with the parabola $Ax^2+2ex+2fy+g=0$, its equation becomes $Aex(x^2+y^2+c^2)+(A^2-Ag)x^2-(ey-fz)^2=0$.

I have already implicitly alluded in a preceding footnote, but think it well again to call express attention, to the remarkable property of the new ovals, of giving *circular* perspective projections on the same plane for three different positions of the eye, the lines joining the eye with the centre of each projection being all three parallel to one another and perpendicular to the plane of the picture. This fact involves the truth of the elegant and probably well-known elementary geometrical proposition, that if the opposite sides of a quadrilateral inscribed in a circle be produced, the lines which bisect the acute angles thus formed will be perpendicular to one another, and respectively parallel to the two bisectors of the angles formed by the diagonals at their intersection. I must now leave to professed geometers (among whose glorious ranks I do not claim to be numbered) the further study of those wonderful twin beings, twisted Cartesians as I have called them, but which those who so think fit may of course designate more simply as ovals with the name of their originator prefixed. By supposing the vertices of the three containing cones to be brought indefinitely near to the plane of the picture, my ovals ought to revert to the Cartesian form.

90.

SUPPLEMENTAL NOTE ON THE ANALOGUES IN SPACE TO THE CARTESIAN OVALS *IN PLANO*.

[*Philosophical Magazine*, XXXI. (1866), pp. 380—385.]

To complete the theory given in the last Number of the *Magazine*, concerning the new ovals in space, I ought to notice that the focal cubic there spoken of is the circular cubic of which the four points in the circle, by means of which it is determined, are the foci. This will become evident from comparison with Dr Salmon's *Higher Plane Curves*, p. 175. My focal cubic is the locus of one set of foci of a system of conics whose axes are parallel, which pass therefore through four points lying in a circle. The axis in which the foci are taken, and which is parallel to the real asymptote, in general meets the focal curve in two points. Whenever these points come together, this parallel to the asymptote becomes a tangent; and the foci do come together for the circle itself and for the three pairs of lines which can be drawn through the four points in question. Hence the focal cubic not only passes through the centre of the circle and through the intersections of the three pairs of lines just spoken of, but at each of these four points is parallel to the real asymptote, that is, to the line bisecting one of the angles in which the diagonals cross. It has also two circular points at infinity. All these conditions are fulfilled by one of Dr Salmon's pair of circular cubics, of which the four points in question are the foci. These curves are therefore identical; or, to express the same idea more fully, the *two* conjugate circular cubics, of which four points in a circle are the foci, together constitute the *complete* locus of the foci of the system of conics which can be drawn through those four points*. It is interesting, moreover, to notice that the spherical curve

* Hence, as shown by Dr Salmon, the focal cubic consists of an oval and a serpentine branch. The two associated focal cubics, the same eminent author has shown, may be regarded as the locus of the intersections of similar conics having for their respective pairs of foci the two pairs of points which make up the given set of four foci; but their simpler geometrical definition, as the complete locus of the foci of the conics drawn through the four given points, appears to have escaped observation.

which is the intersection of any two right cones with parallel axes, and which is necessarily contained also in a third right cone fulfilling the same condition, may be regarded as the *inverse* of any plane section of the spindle or *tore* formed by the revolution of a circle about an axis cutting it, in respect to either point of intersection of the spindle with its axis. The spherical curve in question is of course no other than the so-called pair of twisted Cartesian ovals; and its focal curve may be any of Dr Salmon's circular cubics of the first kind, that is, one whose four real foci lie in a circle. Finally, if a double-curvature (that is, twisted) Cartesian is given, we may define its focal curve very simply as *one of the two circular cubics of which the points in which it is intersected by the plane passing through the axes of its containing right cones are the four real foci.* The Cartesian itself is contained in a sphere, in a paraboloid of revolution, in three right cones with parallel axes*, and also in three surfaces of revolution produced by the rotation of cardioids (with their triple foci lying respectively at the points of inflexion of the focal curve) about the stationary tangents†.

If the two parabolas drawn through the four given points lying in a circle which serve to determine the focal curve be

$$x^2 + 2ex + 2fy = 0, \quad y^2 + 2gx + 2hy = 0,$$

I find that the equation to the focal curve which is the locus of the foci lying in the y axis of the conics drawn through the four foci is

$$h(x-e)(x^2+y^2) + (fh - ke + kx)^2 - 2(fh - ke + kx)hy - h^2x^2 = 0.$$

* So remarkable is this property of the three cones, that, at the risk of tedious reiteration, I think it desirable to present it under the same vivid form in which it strikes my own mind. If any two indefinite straight lines cross, they may be regarded as representing a couple of right cones generated respectively by the revolution of the lines about the two bisectors of the angle which they form. Imagine now a quadrangle inscribed in a circle; its diagonals and pairs of opposite sides produced indefinitely will represent three couples of right cones. This triad of couples may be resolved into a couple of triads, the cones of each triad having their axes parallel *inter se* and perpendicular to those of the other triad; the three cones of each triad respectively will have a *common* intersection, the two intersections being consociated twisted Cartesians whose focal cubics are respectively the two consociated circular cubics of which the angles of the quadrangle are the common foci. Moreover each such twisted Cartesian is a spherical curve lying in the sphere of which the circle circumscribed about the quadrangle is a great circle. The verification of these laws of intersection might be used to form the subject of a new and instructive *plate* for students of ordinary *descriptive geometry*.

† It is interesting to trace the change of form in the general double-curvature Cartesians in regard to the real and imaginary. For this purpose conceive a sphere penetrated by a conical bodkin of indefinite length; when the point of the bodkin just pricks the sphere externally, the curve consists of a single point; as the bodkin is pushed in, the curve becomes a single oval; when the point of the bodkin again meets the sphere internally, the curve will consist of an oval and a conjugate point; then two ovals are formed; then when the bodkin and sphere touch, of an oval and a conjugate point; then of a single oval; and after the bodkin again touches the sphere, in the other side, of a single point; and finally the curve returns wholly into the *limbo* of the imaginary, whence it originally issued. There is apparently nothing analogous to this in the geometrical genesis of the plane Cartesian ovals.

When $x^2+y^2=0$, this gives $(fh-ke+kx-hy)^2=0$, showing that $x=0$, $y=0$ is a *focus*, which demonstrates the focal character of each of the four fixed points.

Departing from the theory of the quasi-Cartesian ovals, if in general we take *any* four fixed points lying at the intersections of the two conics,

$$U = ax^2 + by^2 + 2hxy + 2gzx + 2fzy,$$

and

$$V = \alpha x^2 + \beta y^2 + 2\eta xy + 2\gamma zx + 2\phi zy,$$

the foci of the conic $U+\lambda V$ will be given by the equality

$$\begin{vmatrix} a+\lambda\alpha, & h+\lambda\eta, & g+\lambda\gamma, & 1 \\ h+\lambda\eta, & b+\lambda\beta, & f+\lambda\phi, & i \\ g+\lambda\gamma, & f+\lambda\phi, & 0, & -(x+iy) \\ 1, & i, & -(x+iy), & 0 \end{vmatrix} = 0,$$

which, by equating real and imaginary parts, gives two equations between x, y, λ.

By aid of these equations λ may be expressed as a rational integral function of x, y, which we know *à priori* must be of the second degree only, since otherwise, on substituting for its value in either of the equations between x, y, λ, we should obtain an equation above the sixth degree in x, y, contrary to Chasles's theorem (we shall also see that this is the case, without having recourse to this theorem, by the reasoning below).

Let $x^2+y^2=0$, then the above equality becomes

$$(f-ig+(\phi-i\gamma)\lambda)^2=0,$$

showing that the origin, that is, any one at will and therefore all of the four fixed points, is a focus. If λ were above the second degree in x, y, the line joining this point with either of the circular points at infinity would be *always* at least a triple tangent to the focal curve; but in the case where the focal curve breaks up into two distinct subloci, we have seen that these tangents to each sublocus are simple, that is, that they are double in regard to the whole locus, wherefore λ must be always a quadratic function only of x, y. Consequently each circular point at infinity must itself be a double point upon the curve.

If now we regard the four given points as syzygetic foci of a curve (a term indispensable to give precision to the theory of foci when interpreted in the Plückerian sense), that is, if we suppose a plane curve defined by the equation

$$\lambda\sqrt{(A)}+\mu\sqrt{(B)}+\nu\sqrt{(C)}+\pi\sqrt{(D)}=0,$$

where A, B, C, D are the characteristics of the infinitesimal circles of which the four given points are the centres; and if in the norm of the linear function above written we make $\lambda\pm\mu\pm\nu\pm\pi=0$, and conjoin with this two other equations between λ, μ, ν, π, which will make the term $(x^2+y^2)^3(Lx+My)$

in the norm vanish identically, that is, the equations $L=0$, $M=0$, we shall obtain a group* of curves of the sixth degree, each possessing precisely the same geometrical characters as have been proved to be satisfied by the curve of foci of the conics $U+\lambda V$ drawn through the four fixed points, namely of having the circular points at infinity for double points, and being doubly touched† by each line joining either of them with any one of the four fixed points; and if we are at liberty to assume (which, however, requires further investigation‡) that the curve containing the foci of $U+\lambda V$ must be identical with one of the group, then this curve of foci will be defined by the equation

$$l\sqrt{(A)}+m\sqrt{(B)}+n\sqrt{(C)}+p\sqrt{(D)}=0\,;$$

whence, calling a, b, c, d the four fixed points, and F, G, H the points of intersection of the opposite sides of the quadrangle $abcd$, the signs of the square roots below written must be capable of being so assumed that the determinant

$$\begin{vmatrix} (aF)^{\frac{1}{2}}, & (bF)^{\frac{1}{2}}, & (cF)^{\frac{1}{2}}, & (dF)^{\frac{1}{2}} \\ (aG)^{\frac{1}{2}}, & (bG)^{\frac{1}{2}}, & (cG)^{\frac{1}{2}}, & (dG)^{\frac{1}{2}} \\ (aH)^{\frac{1}{2}}, & (bH)^{\frac{1}{2}}, & (cH)^{\frac{1}{2}}, & (dH)^{\frac{1}{2}} \\ 1, & 1, & 1, & 1 \end{vmatrix}$$

shall be equal to zero, constituting a remarkable theorem concerning seven points (four quite arbitrary) in a plane. If (as seems probable) the case supposed is what actually obtains, a geometrical rule must exist for determining the proper combination of signs to be employed in the above determinant; and then l, m, n, p will be proportional to the first minors of the three first lines of the matrix above written§.

The above theory, very hastily sketched out under the pressure of other occupations, will serve at all events to manifest in how very imperfect and inchoate a form the theory of foci at present exists, and may serve to raise

* No two of the group can be identical; for in such case one of the focal distances could be eliminated, and the syzygy be reducible by one term, contrary to hypothesis.

† In general, if a curve is defined by a homogeneous linear relation between its distances from r, or a non-homogeneous linear relation between its distances from $r-1$ points, it will easily be seen that the line joining each such point with either circular point at infinity will be a tangent *touching* the curve in 2^{r-2} points.

‡ To the group of syzygetic curves there are only eight parameters, and these eight tangents (all double) at the four foci are common to each member of the group and to the locus of the foci of $U+\lambda V$. To complete the proof of the supposed identity of the latter with one of the former, it is necessary to show that the *number* of curves having the eight tangents in question is precisely equal to the number of systems of solutions of the equations

$$l+m+n+p=0,$$
$$L=0, \quad M=0.$$

§ Calling the determinant D, the equation $D=0$ serves to fix the allowable combinations of the doubtful signs, just as in Cardan's rule a certain equation, which the product of the two associated cube-roots in the solution is bound to satisfy, fixes their allowable combinations of values.

the question whether there is not a family of curves distinguishable by the possession of syzygetic foci, and which may be termed syzygetic or norm curves (including as elementary members of the group conics, Cartesian ovals, circular cubics, &c.), forming a distinct genus, and calling for a special and detailed examination of their focal properties*.

* In an evil hour for the cause of sound nomenclature, and by an over-hasty generalization, a property of foci properly so called was substituted for their true definition as centres of linear relationship. The temptation was great, it must be allowed; for the new definition gave a means of describing each focus *per se* without reference to the associated points: it calls to mind the analogous attempted definition of *man* as a *featherless biped*, and is open to the like kind of objections. The mischief being done, and under cover of the authority of names so great that to root it out seems now hopeless, the best remedy to apply is, as I think, that used in the text, of distinguishing the centres of linear relationship as *syzygetic* foci or foci *proper*. There is room for a grand chapter in the promised and anxiously-expected new edition of Dr Salmon's *Higher Plane Curves*, on a systematic and exhaustive development of the laws of foci proper, and the algebraical philosophy, as it may well be termed, of true focal curves, that is, curves the distances of whose points from one or more sets of fixed points are subject to linear relations. Nothing can be more curious than the study of the way in which, starting from a given set of fixed points, other foci (as in the Cartesian ovals) are found capable of replacing one or more of the given ones, constituting the theory of substitution—and then, again, how, as in the conic sections and in circular cubics, besides this faculty of mutual substitutability of foci of the same set, one set may be entirely replaced by one or more other sets, constituting the theory of plurality or distribution. Algebra cannot but gain largely by these ideas of substitution and distribution being fully worked out.

In answer to my objections to the undue extension of the term *focus*, it has been urged that a focus, as originally presenting itself in the theory of conics, is susceptible of two distinct definitions—first as a member of a syzygetic group, and again as a point whose squared distance from any point in its curve is the square of a linear function of the coordinates, —that it is legitimate to generalize the conception from either of these points of view, and that the latter leads to the definition of a focus as a point whose squared distance from any point in its curve, multiplied by a quantic, gives rise to a second quantic containing a squared linear function as a factor. But I answer to this, that the generalization is carried too far and too fast, two steps in enlargement of the original condition being taken at once to arrive at it; that the first step should be to define a focus as a point such that the squared distance in question, multiplied by a quantic, viewed as a function of the coordinates, shall be a perfect square; and that when this first step is taken, the foci so obtained are foci of a peculiar kind, and probably retain their quality as *foci* proper, or centres of linear relationship. At all events they possess the property of giving, by their junctions with the circular points at infinity, multiple tangents to the curve, according to the law stated in a previous footnote concerning such foci.

If the word *focus* is retained to signify the proper or syzygetic species, some slight modification of the word may be used to denote the genus, namely, foci which satisfy the larger definition of being points of intersection of the simple tangents to the circular points at infinity. I thought of the word *focal* for the purpose; but this is objectionable, for the reason that it would probably be found advisable to retain that word to denote the class of *curves* which possess foci proper. On the whole, the word *subfocus* seems to me best to meet the exigency of the case, and possesses the recommendation of being capable (with dialectic variations) of passing current in each of the five accepted tongues—Latin, German, French, English, and Italian, which happily at the present day may be regarded as the common property and inheritance of mathematical Europe.

91.

NOTE ON A *MEMORIA TECHNICA* FOR DELAMBRE'S, COMMONLY CALLED GAUSS'S, THEOREMS.

[*Philosophical Magazine*, XXXII. (1866), pp. 436—438.]

THE most subtle reagents employed in spherical analysis and transformation are the following four admirable formulæ, "commonly ascribed to Gauss, but in reality due to Delambre*":—

$$\cos\frac{c}{2}\cos\frac{A+B}{2}=\sin\frac{C}{2}\cos\frac{a+b}{2},$$

$$\cos\frac{c}{2}\sin\frac{A+B}{2}=\cos\frac{C}{2}\cos\frac{a-b}{2},$$

$$\sin\frac{c}{2}\cos\frac{A-B}{2}=\sin\frac{C}{2}\sin\frac{a+b}{2},$$

$$\sin\frac{c}{2}\sin\frac{A-B}{2}=\cos\frac{C}{2}\sin\frac{a-b}{2}.$$

Four out of the six binary combinations of these four equations give by simple division Napier's *Analogies*, a term which seems almost equally appropriate to designate Delambre's formulæ. It need hardly be remarked that whilst Napier's analogies may be immediately deduced from Delambre's formulæ, the converse is not true.

If we call the products on the left-hand side of the equations P, Q, R, S, and their polar reciprocals P', Q', R', S', it is worthy of notice that the formulæ become

$$P=-P',\quad Q=R',\quad R=Q',\quad S=-S'.$$

* Todhunter's *Spherical Trigonometry*, p. 27. See also Davies's edition of *Hutton's Course*, Vol. II. p. 37.

The formulæ may be expressed collectively by the easily remembered disjunctive elective equation

$$\begin{matrix}\cos\\ \sin\end{matrix}\frac{c}{2}\begin{matrix}\cos\\ \sin\end{matrix}\frac{A\pm B}{2}=\begin{matrix}\cos\\ \sin\end{matrix}\frac{C}{2}\begin{matrix}\cos\\ \sin\end{matrix}\frac{a\pm b}{2}.$$

The number of products on each side of the equation, if all the combinations of trigonometric affection and algebraical sign are exhausted, is 2^3 or eight. Out of each 8, 4 only are to be preserved and colligated each with each. Thus the number of systems capable of formation is

$$\left(\frac{8.7.6.5}{1.2.3.4}\right)^2(1.2.3.4)=24\times 70^2=117600,$$

of which one only is valid. This accounts *à priori* for the difficulty of recollecting these formulæ, a difficulty often complained of and still oftener felt, and which is one reason of their being comparatively little used by junior students. Two observations easily retained in the memory will serve, I think, in a great degree to remove this difficulty.

Rule 1. On opposite sides of any one equation the trigonometric affections of the angles are contrary, and those of the sides similar.

Rule 2. *The trigonometric affection of the uniliteral factor of each product governs the algebraic sign of the biliteral factor,* in the following manner:—

Comparing products which lie on the *same* side of the equations, *like and unlike* affections go with like and unlike signs; comparing those which lie on *opposite* sides of the equations, *unlike and like* affections go with *like and unlike* signs.

These two rules are not quite sufficient in themselves; for they would be satisfied not only by the four true equations, but also by the four following false ones:—

$$\cos\frac{c}{2}.\cos\frac{A-B}{2}=\sin\frac{C}{2}.\cos\frac{a-b}{2},$$

$$\cos\frac{c}{2}.\sin\frac{A-B}{2}=\cos\frac{C}{2}.\cos\frac{a+b}{2},$$

$$\sin\frac{c}{2}.\cos\frac{A+B}{2}=\sin\frac{C}{2}.\sin\frac{a-b}{2},$$

$$\sin\frac{c}{2}.\sin\frac{A+B}{2}=\cos\frac{C}{2}.\sin\frac{a+b}{2}.$$

To make the system of rules complete so as to exclude *à priori* the construction of the four false deductions, it is necessary and sufficient to bear in mind

that, on the left-hand side of the equation, the *cosine*-affection of the uniliteral term is associated with the *plus* sign in the biliteral one*.

But even without this check the false equations may be put to the question and made severally to disclose their character as such by applying any one of them to the limiting case of a triangle on a sphere continuing of *finite radius*, but in which the angles become respectively 180°, 0, 0, and consequently the side opposite the first equal or capable of being equal to the sum of the other two. Thus writing in the first and third of the last written formulæ $C=180°$, $B=0$, $A=0$, we ought to be able to derive $c=a+b$, but find instead $a=b\pm c$ in the first, and $a=b+c$ in the third. And similarly in the second and fourth, writing $A=180°$, $B=0$, $C=0$, we ought to be able to derive $a=b+c$, but find instead $a=-b\pm c$ in the second, and $a=-b+c$, or $a+b+c=360°$ in the fourth. We might easily deduce other defective criteria from the reciprocal limiting case of a spherical triangle in which one side is zero and the two others each 180°, in which case the angle opposite the first augmented by 180° will equal the sum of the other two. Furthermore, using accents, as before, to denote polar reciprocation, the false system takes the form

$$P-P'=0,\quad Q+R'=0,\quad R+Q'=0,\quad S-S'=0,$$

in lieu of the true form,

$$P+P'=0,\quad Q-R'=0,\quad R-Q'=0,\quad S+S'=0.$$

A direct geometrical proof of these potent formulæ appears to be a *desideratum*.

* Rule 2, with the addition to it, may be easily retained in the memory by aid of the scheme below written,

	left	right
cos	+	−
sin	−	+

but, as subsequently shown in the text, the bordering of the square may be affixed at random, that is, the words *left* and *right* or *cos* and *sin* may be interchanged without leading to any error but of a kind susceptible of immediate detection and remedy.

92.

NOTE ON THE PROPERTIES OF THE TEST OPERATORS WHICH OCCUR IN THE CALCULUS OF INVARIANTS, THEIR DERIVATIVES, ANALOGUES, AND LAWS OF COMBINATION; WITH AN INCIDENTAL APPLICATION TO THE DEVELOPMENT IN A MACLAURINIAN SERIES OF ANY POWER OF THE LOGARITHM OF AN AUGMENTED VARIABLE.

[*Philosophical Magazine*, XXXII. (1866), pp. 461—472.]

SUPPOSE ϕ_1 denotes any algebraical function of the two sets of elements,

$$\left(a, b, c, \ldots, \frac{d}{da}, \frac{d}{db}, \frac{d}{dc}, \ldots\right).$$

Let $\psi *$ in general signify the process of operating with ψ upon all that follows†.

Suppose $\phi_1 * \phi_1 = \phi_2$, where the operating elements $\frac{d}{da}, \frac{d}{db}, \ldots$ of course can only operate upon the operands $a, b, c, \ldots$ in the second ϕ. In like manner, let

$$\phi_1 * \phi_1 * \phi_1 = \phi_1 * \phi_2 = \phi_3,$$

and in general

$$(\phi_1 *)^{n-1} \phi_1 = \phi_n.$$

† The symbol of an operator consists of two parts, the *corpus* or quantity, and the *asterisk* or sign of operation. Thus a simple extensor operator has one of the extensors for its corpus; a compound extensor operator has any algebraical function of any number of extensors for its corpus. The operator which represents the combined effect of two or more operators following each other in any specified order may be termed their resultant; the theorems in the text amount to saying that the resultant of any number of simple or compound extensor operators is independent of the order in which its components occur, and is equivalent to some third compound extensor operator. One great problem to be solved is to determine the corpus of a resultant in terms of the corpora of its two components. This is done in the text for the simple case where each component corpus is a simple power of one of the extensors. To attain clearness of conception, the first condition is *language*, the second *language*, the third *language*—Protean speech—the child and parent of thought.

It will follow from this that

$$\phi_1 * \phi_1 * = (\phi_1^2 + \phi_2) *,$$

$$\phi_1 * \phi_1 * \phi_1 * = (\phi_1^3 + 2\phi_1\phi_2 + \phi_3) * \dagger;$$

and in the general case there will be found no great difficulty in obtaining the following theorem,

$$(\phi_1 *)^i = \Pi i \,.\, \text{coefficient of } t^i \text{ in } e^T,$$

where $$T = \phi_1 t + \phi_2 \frac{t^2}{1.2} + \phi_3 \frac{t^3}{1.2.3} + \ldots, \qquad \text{(A)}$$

a relation which may be expressed by means of the identity ‡

$$e^{t\phi_1 *} = (e^T) * \S, \qquad \text{(A) } bis$$

which important equation has been previously noticed by Professor Cayley under a somewhat less general form.

With the exception of noticing that $(\phi_1 *)^r$ and $(\phi_1 *)^s$ are commutable symbols by virtue of their definition, that is, that

$$(\phi_1 *)^r (\phi_1 *)^s = (\phi_1 *)^s (\phi_1 *)^r,$$

I am not at present aware that this theory of derivation when the form of ϕ is left undetermined presents much that is remarkable. Very different, however, is the case when we proceed to give to ϕ the particular form in which it enters into the calculus of invariants: a most surprising and unexpected system of relations then springs up between the various orders of operators; and a vast and inexhaustible theory opens out before us, of which I want leisure to be able to do more than briefly notice one or two salient features.

† So more generally if ϕ, ψ be any two functions of $a, b, c, \ldots \frac{d}{da}, \frac{d}{db}, \frac{d}{dc}, \ldots$ we have ¶

$$\phi * \psi * = (\phi\psi) * + [\phi * \psi] *,$$

and similarly $$\psi * \phi * = (\psi\phi) * + [\psi * \phi] *.$$

Hence if two operators $\phi *$, $\psi *$ are commutable, so, in respect to the symbol of operation $*$, are the two operants ϕ, ψ.

The force of the bracket explains itself. This wonderful symbol has the faculty of extending itself without ambiguity to every possible development, however new, of mathematical language. It is susceptible only of a metaphysical definition as signifying the exercise, with regard to its content, of that faculty of the human mind whereby a multitude is capable of being regarded as an individual, or a complex as a monad. In a word, it is the symbol of individuality and unification.

[‡ Cf. p. 608, below.]

§ Thus, for example, let ϕ_1 represent $x\frac{d}{dx}$, then $\phi_2, \phi_3, \ldots$ will be all equal to ϕ_1; accordingly $T = (e^t - 1)\phi_1$, and the formula in the text becomes

$$e^{\left(tx\frac{d}{dx}\right)*} = \left(e^{(e^t-1)x\frac{d}{dx}}\right)*,$$

a remarkable formula of expansion.

[¶ Cf. p. 610, below.]

Let $E_1 = a\dfrac{d}{db} + 2b\dfrac{d}{dc} + 3c\dfrac{d}{dd} + \ldots + a'\dfrac{d}{db'} + 2b'\dfrac{d}{dc'} + \ldots + a''\dfrac{d}{db''} + \ldots,$

that is, $$= \Sigma\left(a\frac{d}{db} + 2b\frac{d}{dc} + 3c\frac{d}{dd}\ldots\right).$$

Then if I be any function of the coefficients $a, b, c, \ldots;\ a', b', c', \ldots$ in the algebraic forms $(a, b, c, \ldots)(x, 1)^p$; $(a', b', c', \ldots)(x, 1)^p \ldots$, and I_1 be what I becomes when we substitute for $a, b, c, \ldots;\ a', b', c', \ldots$, the values which these coefficients assume when $x + h$ is written in place of h, it is, or ought to be, well known that

$$I_1 = I + E_1{*}Ih + (E_1{*})^2 I\frac{h^2}{1\,.\,2} + (E_1{*})^3 I\frac{h^3}{1\,.\,2\,.\,3} + \ldots.$$

Here $$E_1{*}E_1 = 2\Sigma\left(a\frac{d}{dc} + 3b\frac{d}{dd} + 6c\frac{d}{de} + \ldots\right),$$

$$E_1{*}E_1{*}E_1 = 2\,.\,3\Sigma\left(a\frac{d}{dd} + 6b\frac{d}{de} + \ldots\right),$$

$$\ldots = \ldots;$$

it will therefore become convenient slightly to depart from the notation applied to the general form ϕ, and to write

$$E_1 = \Sigma\left(a\frac{d}{db} + 2b\frac{d}{dc} + \ldots\right),$$

$$E_2 = \Sigma\left(a\frac{d}{dc} + 3b\frac{d}{dd} + \ldots\right),$$

....

$$E_n = \Sigma\left(a\frac{d}{da_n} + (n+1)\,b\frac{d}{db_n} + \frac{1}{2}(n+1)(n+2)\,c\frac{d}{dc_n} + \ldots\right);$$

where $a_n, b_n, c_n, \ldots$ are used to express the elements n steps more advanced than $a, b, c, \ldots$ respectively; we have then by the general theorem

$$e^{tE_1*} = (e^T){*}, \tag{B}$$

where T now takes the form

$$E_1 t + E_2 t^2 + E_3 t^3 + \ldots$$

I propose to give to the E series of operators the general name of Extensor Operators, or simply Extensors.

The first remarkable, I may say marvellous, property of these extensors is, that they form a sort of closed group; that is, any two algebraical functions whatever of the extensors regarded as algebraic functions of the quantities $a, b, c, \ldots;\ \dfrac{d}{db}, \dfrac{d}{dc}, \ldots$ being used as new operators and applied in succession to the same operand, the result is the same as if *some single third algebraical function* of the extensors had operated alone on this operand. The second

great fact is, that the order in which the above-described operations take place is indifferent, that is, that the two operators above described are commutable; in other words, we have always

$$\left.\begin{aligned}&\theta(E_1, E_2, E_3, \ldots)* \psi(E_1, E_2, E_3, \ldots)*\\ &\quad= \Omega(E_1, E_2, E_3, \ldots)*\\ &\quad= \psi(E_1, E_2, E_3, \ldots)*\theta(E_1, E_2, E_3, \ldots)*\end{aligned}\right\}. \tag{C}$$

Thus, for example,

$$E_\mu{}^i * E_\mu{}^j * = F(E_\mu, E_{2\mu})* = E_\mu{}^j * E_\mu{}^i *,$$

where, writing $m = \frac{\Pi(2\mu)}{(\Pi\mu)^2}$, F represents the *quasi*-hypergeometric series,

$$\left.\begin{aligned}&E_\mu{}^{i+j} + i\,.\,j\,.\,mE_\mu{}^{i+j-2}\,.\,E_{2\mu} + \frac{i(i-1)j(j-1)}{1\,.\,2} m^2 E_\mu{}^{i+j-4}\,.\,E_{2\mu}{}^2\\ &\quad + \frac{i(i-1)(i-2)j(j-1)(j-2)}{1\,.\,2\,.\,3} m^3\,.\,E_\mu{}^{i+j-6}\,.\,E_{2\mu}{}^3 + \ldots\end{aligned}\right\}, \tag{D}$$

and $E_\mu{}^i * E_\nu{}^j *$ will be expressible under a form *quam proximè* analogous. My immediate intention in this brief notice being merely to call attention to the surprising properties of these functions, I shall conclude with adding a slight extension of theorem (B) above given, namely,

$$e^{tE_1*} = (e^T)*.$$

This may be regarded as a particular case of a more general theorem which I have discovered, namely,

$$E_1{}^j * e^{tE_1*} = e^{tE_1*}\, E_1{}^j * = \left\{\left(\frac{dT}{dt}\right)^j e^T\right\} *,$$

a theorem which, with a simple change in the coefficients of T, may be extended to the still more general form $E_\omega{}^j * e^{tE_\omega*}$, so as to give a simple solution of the equation

$$X* = (E_\omega *)^i E_\omega{}^J *,$$

where X is a form to be determined as an algebraical function of $E_\omega, E_{2\omega}, E_{3\omega}$, &c. ....

The cardinal problem to be solved in the theory of extensors is the determination of Ω in formula (C), where ψ and θ are any given functional forms.

In the further development of this theory, it will probably be found expedient to suppose the number of the elements, $a, b, c, \ldots j, k, l$, to become finite, which will limit the number of the derived extensors, and to study the mutual reactions of the correlated series of extensors (with their derivatives), which we may characterize respectively as the E and H series, where

$$E_1 = \Sigma\left(a\frac{d}{db} + 2b\frac{d}{dc} + \ldots\right),$$

$$H_1 = \Sigma\left(l\frac{d}{dk} + 2k\frac{d}{dl} + \ldots\right).$$

Either of the above two primitive forms (as it is the imperishable glory of Professor Cayley to have discovered†) is sufficient in itself for testing the nature of every invariant satisfying the necessary and obvious condition of weight, and for deducing the complete form of a covariant from either of its extreme terms; which latter consideration affords, I think, a sufficient ground for the name (of some kind or another so much needed) *Extensors*, which I propose to give to these too-long-suffered-to-remain anonymous test operators and their derivatives.

Postscript.

Since the above was sent to press it has occurred independently to Professor Cayley, to whom I had communicated a sketch of the theory, and to myself, that the general conclusions contained in the text above would remain valid for a much more general class of operants than those there defined; and there can be little or no doubt that such is the case for all operants *lineo-linear* in a set of elements $a, b, c, \ldots$, and their præ-reciprocals $\frac{d}{da}, \frac{d}{db}, \frac{d}{dc}, \ldots$. Moreover a material improvement in the nomenclature has suggested itself, which I proceed to explain. It is most important in this theory to be able to distinguish between the corpus or root of an operator viewed as a function and the operator itself, and to be in possession of a single name for the former. Accordingly, in conformity with the general terminology of the new algebra, I propose to substitute the name of Protractor for Extensor to signify the *operator*, so as to be able to use the word Protractant to signify the *corpus*. Also I shall give the analogous names of Pertractant‡ and Pertractor—the former to the lineo-linear function above referred to, the latter to this function *energized*, that is, converted into an operator by the addition of the asterisk $*$, the symbol of operative power.

We thus start with a pertractant P_1 which is *energized* into a pertractor, $P_1 *$; with this latter we continue to operate any number of times upon the original pertractant, and obtain a succession of new derived pertractants, into which it appears at present to be convenient, for the sake of uniformity, to introduce the numerical divisors 2, 3, 4, ..., so that we may define P_{n+1}, the nth *derivative* pertractant, as equal to $\frac{(P*)^n P}{\Pi(n+1)}$.

† But this magnificent discovery, whereby the determination of the number of fundamental invariants to a binary quantic of a given degree is reduced to a problem in the partition of numbers, it is but justice to M. Hermite to state, took its rise in that great analyst's discovery of the octodecimal invariant of the binary quintic. So long as the existence of this fourth invariant to that form was unsuspected, it must have remained impossible to conjecture the sufficiency of the single partial differential equation-test.

‡ Thus the "*Universal Mixed Concomitant*" $x\frac{d}{dx} + y\frac{d}{dy} + z\frac{d}{dz} + \ldots$ is of the genus Pertractant.

We thus obtain a series of pertractants, P_1, P_2, P_3, ..., which may be termed the primitive and prime derivative pertractants of the family.

Again, we may form any algebraical function of the primitive and its prime derivatives, and such function may be termed a compound derivative of the family; this in its turn, by the addition of the symbol of operative power, may be energized into a pertractive operator, which, containing only a single asterisk, is to be regarded as a simple or single derived pertractor, although its corpus is a compound derivative.

The first leading proposition of the theory is, that all operators so formed are *commutable*, so that, being subject to the laws of algebraical operation, they may themselves be made the subjects of algebraical functions. The second great proposition is, that any such function of one or more pertractors is reducible to the form of a *single pertractor*, that is, is an energized function of the prime pertractants P_1, P_2, P_3, ...

The theorems that have been stated concerning protractants and protractors will continue to subsist for the much more general class of pertractors and pertractants. Thus, for example, theorem (D) in the text above, when we take $\mu = 1$, becomes

$$E_1{}^i * E_1{}^j * = E_1{}^{i+j} + i\,.\,jE_1{}^{i+j-2}(2E_2) + \frac{i(i-1)j(j-1)}{1\,.\,2} E_1{}^{i+j-4}(2E_2)^2 + \dots.$$

Mr Cayley verifies this theorem when for E_1, the leading protractant, we substitute P_1, a pertractant, as follows. Take only a single element x and its symbolical reciprocal $\dfrac{d}{dx}$, so that $P_1 = x\dfrac{d}{dx}$; then

$$P_2 = \tfrac{1}{2}P_1, \text{ and } P_1{}^i\,.\,P_1{}^j = \left\{x^i\left(\frac{d}{dx}\right)^i\right\} * \left\{x^j\left(\frac{d}{dx}\right)^j\right\},$$

is easily seen to be

$$P_1{}^{i+j} + \frac{i\,.\,j}{1} P_1{}^{i+j-1} + \frac{i(i-1)j(j-1)}{1\,.\,2} P_1{}^{i+j-2} + \dots$$

$$= P_1{}^{i+j} + \frac{i\,.\,j}{1} P_1{}^{i+j-2}(2P_2) + \frac{i(i-1)j(j-1)}{1\,.\,2} P_1{}^{i+j-4}(2P_2)^2 + \dots,$$

as before.

But I find that the theory admits of a still further and most important extension. Thus far we have been dealing with operants and operators derived from a single one of the former. But we may easily form a set of two or more, say k pertractants, that is, functions lineo-linear in $a, b, c, \dots$; $\dfrac{d}{da}, \dfrac{d}{db}, \dfrac{d}{dc}, \dots$ commutable *inter se*†; these being *energized* into operators

† This imports into the subject a beautiful theory of commutable matrices. In the case of two letters we have two types of commutable pertractors, from which all the rest may be derived by the laws of pertraction stated in the text. These two fundamental systems are:—

which are made to act on the functions themselves, will give rise to $\frac{1}{2}r(r+1)$ first derivatives, which, energized in their turn, will be commutable *inter se* and with the original operators: the derivatives of the next order enjoying the same properties will be $\frac{1}{6}r(r+1)(r+2)$, and so on. Thus, as before, we obtain the *prime* pertractive derivants of various orders, with the difference that there are now several of such prime derivants belonging to each order. Any function of these gives rise to a compound-pertractive derivant, the number of which is of course unlimited; these may be energized into operators, subject *inter se* to all the laws of algebraical operation, and any function of one or more of such compound-pertractive derivators will be equivalent to some single derivator belonging to the same family. In a word, the theory may be extended from the case of *Monocephalous* to that of *Polycephalous* pertractive functions and operators and their derivatives.

I will conclude for the second time with the statement of an expansion in a series which, as far as I have been able to ascertain, is new to writers on the differential calculus, to which I was led by applying to the *operand* a^x the symbolical equation previously given in a footnote. The equation in question may be written as follows:

$$e^{\left(ta\frac{d}{da}\right)}* = \left\{\left(e^{(e^t-1)}\right)^{a\frac{d}{da}}\right\} *;$$

from this I have been able to deduce by a mental calculation, the steps of

---

(1) $x\delta_x$; $y\delta_y$.

(2) $\begin{pmatrix} a, & b \\ c, & d \end{pmatrix}(x, y)(\delta_x, \delta_y)$; $x\delta_x + y\delta_y$.

In the case of three letters, the four following types of commutable systems present themselves:—

(1) $x\delta_x$; $y\delta_y$; $z\delta_z$.

(2) $\begin{pmatrix} a, & b \\ c, & d \end{pmatrix}(x, y)(\delta_x, \delta_y)$; $x\delta_x + y\delta_y$; $z\delta_z$.

(3) $ax\delta_y + by\delta_z + cz\delta_x$; $\frac{1}{a}y\delta_x + \frac{1}{b}z\delta_y + \frac{1}{c}x\delta_z$.

(4) $\begin{pmatrix} a, & b, & c \\ d, & e, & f \\ g, & h, & k \end{pmatrix}(x, y, z)(\delta_x, \delta_y, \delta_z)$; $x\delta_x + y\delta_y + z\delta_z$.

Whether the above four systems are independent, and whether they constitute an exhaustive enumeration in the case of the three letters, I have not yet had time to ascertain.

The reader will please to bear in mind that any linear function of the terms in each system, or of them and their derivatives, is commutable with those terms themselves; thus, for example, the last system but one is quite as extensive as if we included in it

$$\lambda ax\delta_y + \lambda by\delta_z + \lambda cz\delta_x + \frac{\mu}{a}y\delta_x + \frac{\mu}{b}z\delta_y + \frac{\mu}{c}x\delta_z + \lambda\mu x\delta_x + \lambda\mu y\delta_y + \lambda\mu z\delta_z,$$

in which it will be noticed that the three last terms may be obtained (to a constant factor *près*) by operating with the sum of the three first upon the sum of the three middle terms, or *vice versâ*.

which I am unable to recall, a development which would be exceedingly difficult to obtain from the method of Maclaurin's theorem. I find

$$\{-\log(1-x)\}^n = x^n + S_{n,1}\frac{x^{n+1}}{n+1} + S_{n+1,2}\frac{x^{n+2}}{(n+1)(n+2)} + S_{n+2,3}\frac{x^{n+3}}{(n+1)(n+2)(n+3)} + \dots,$$

where in general $S_{i,j}$ signifies the sum of the $\frac{i(i-1)\dots(i-j+1)}{1.2\dots j}$ products of the combinations of the numbers 1, 2, 3, ... i, taken j and j together. This development may be easily verified inductively by aid of the identical equation

$$\frac{d}{dx}\{\log(1-x)\}^n = -\frac{n\{\log(1-x)\}^{n-1}}{1-x},$$

combined with the relation

$$\begin{aligned} S_{n+j-1,j} &= S_{n+j-2,j} + jS_{n+j-2,j-1}\dagger \\ &= S_{n+j-2,j} + jS_{n+j-3,j-1} + j(j-1)S_{n+j-4,j-2} \\ &\quad + j(j-1)(j-2)S_{n+j-5,j-3} + \&c. \end{aligned}$$

It is obvious that the coefficients of the powers of x in the above expansion must be all of them integral functions of n, and must also contain n in every term except the first; and when so expressed as integer functions of n, the result obtained on the supposition of n being a positive integer will continue to subsist for all values of n. From the first part of this statement, it follows that $S_{i,j}$ may always be expressed under the form

$$\{(i+1)i(i-1)\dots(i-j+1)\}\,\phi_{j-1}(i),$$

where $\phi_{j-1}(i)$ is a quantic in i of the degree $j-1$.

Furthermore, if we suppose

$$\phi_{j-1}(i) = \frac{\psi_{j-1}(i)}{2^\alpha . 3^\beta . 5^\gamma . 7^\delta . 11^\epsilon \dots p^{\phi(p)}\dots},$$

p being any prime number, and ψ a function of i of the degree $(j-1)$ all whose coefficients are integer, and (consistently with this being the case) as small as they can be made, there is no difficulty in obtaining the value of $\phi(p)$ under the following form,

$$\phi(p) = \Sigma_{\mu=\infty}^{\mu=0} E\frac{j}{(p-1)p^\mu}\ddagger,$$

† The equation in differences $S_{n,j} = S_{n-1,j} + jS_{n-1,j-1}$ gives an easy algorithm for calculating $S_{n,j}$, and shows *à priori* that it is divisible by $(n+1)n\dots(n-j+1)$.

‡ Consequently $\phi(p)$, the exponent of p, is always less than $\frac{pj}{(p-1)^2}$, and *à fortiori* than $\frac{j}{p-2}$.

where, as usual, the symbol E signifies that *only* the integer part is to be preserved of the number upon which it acts. The value of the coefficient of i^{j-1} in $\phi_{j-1}(i)$ is easily ascertained to be $\frac{1}{(1\,.\,2\,.\,3\ldots j)\,2^j}$, and consequently the coefficient of i^{j-1} in ψ is always an odd number, the number of times that 2 is contained in this denominator being

$$\Sigma_{\mu=\infty}^{\mu=0} E\,\frac{j}{2^\mu}.$$

The maximum prime in the denominator of the fraction which expresses $\phi_{j-1}(i)$ enters always as a simple factor, because, as we know by M. Bertrand's theorem, there is always a prime number included between $q+1$ and $2q+2$. Consequently, supposing j to be $2q$ or $2q+1$, since there exists a prime number p greater than $q+1$, and not greater than $2q+1$, this prime number will appear in the denominator of $S_{n,j}$ with the exponent $E\left[\frac{2q \text{ or } 2q+1}{p-1}\right]$, that is, *unity*.

Conversely, if by any means not founded on the above theorem we could ascertain this fact, we should be in possession of an entirely new proof of that celebrated theorem. It is perhaps also worthy of a passing notice, that $(-)^j\,.\,\phi_{(j-1)}(j-1)$ may easily be proved to be equal to the coefficient of t^j in $\log\log(1+t)-\log t$†.

I have calculated the values of $S_{i,1}$, $S_{i,2}$, $S_{i,3}$, $S_{i,4}$, which are as follows:

$$\frac{(i+1)\,i}{2},\quad \frac{(i+1)\,i\,(i-1)}{2^3\,.\,3}(3i+2),\quad \frac{(i+1)\,i\,(i-1)\,(i-2)}{2^4\,.\,3}\,i\,(i+1),$$

$$\frac{(i+1)\,i\,(i-1)\,(i-2)\,(i-3)}{2^4\,.\,3^2\,.\,5}(15i^3-15i^2+10i-8)\,‡.$$

† And more generally if

$$S_{n+j-1,j}=\{(n+j)\,(n+j-1)\ldots n\}\,(C_j n^{j-1}+C_{j-1}n^{j-2}+\ldots+C_1),$$

$$(-)^j\,C_\omega=\frac{C_1}{\Pi\omega}\,.\text{ coefficient of } t^j \text{ in } \left(\log\frac{\{\log(1+t)\}}{t}\right)^\omega.$$

‡ In his great and most useful work on the *Calculus* (p. 264), Professor De Morgan has applied Arbogast's method to the expansion of $\{\log(1+x)\}^n$, and worked out his results completely as far as the coefficients of x^4 inclusive. His $\frac{C}{n}$, $\frac{E}{n}$, $\frac{F}{n}$, when $i-1$, $i-2$, $i-3$ are substituted in these quotients for n, become identical with the *non-trivial*, or so to say outstanding factors in my $S_{i,2}$; $S_{i,3}$; $S_{i,4}$ respectively.

I have since calculated the same factors for $S_{i,5}$, $S_{i,6}$ corresponding to Professor De Morgan's $\frac{G}{n}$, $\frac{H}{n}$, when n is replaced by $i-4$, $i-5$ respectively. The calculations are rather laborious, extending in the latter case to 8 places of digits; but comparatively very small numbers appear in the final expressions. For $S_{i,5}$ I find the outstanding factor takes the exceedingly simple form

$$\frac{i\,(i+1)(3i^2-i-6)}{2^8\,.\,3^2\,.\,5},$$

and for $S_{i,6}$ the form

$$\frac{63i^5-315i^3+224i^2+140i-96}{2^{10}\,.\,3^3\,.\,5\,.\,7}.$$

I think there can be little doubt that the outstanding factor in $S_{i,j}$ becomes more liable to

The following observation from Professor Cayley will be found interesting:—

"In the case of two variables, if

$$P_1 = (ax + by)\frac{d}{dx} + (cx + dy)\frac{d}{dy},$$

then in the notation of matrices,

$$P_1 = \begin{Bmatrix} a, & b \\ c, & d \end{Bmatrix} (x, y)\left(\frac{d}{dx}, \frac{d}{dy}\right),$$

$$P_2 = \tfrac{1}{2}\begin{Bmatrix} a, & b \\ c, & d \end{Bmatrix}^2 (x, y)\left(\frac{d}{dx}, \frac{d}{dy}\right),$$

$$P_3 = \tfrac{1}{6}\begin{Bmatrix} a, & b \\ c, & d \end{Bmatrix}^3 (x, y)\left(\frac{d}{dx}, \frac{d}{dy}\right);$$

whence also $P_1 * P_2 = P_2 * P_1 = \tfrac{1}{2}\begin{Bmatrix} a, & b \\ c, & d \end{Bmatrix}^3 (x, y)\left(\frac{d}{dx}, \frac{d}{dy}\right) = 3P_3$,

which accords with your theorem,

$$E_1 * E_2 * = E_2 * E_1 * = E_1 E_2 * + 3E_3 *."$$

I have taken the liberty of writing in the above $\frac{d}{dx}, \frac{d}{dy}$ for δ_x, δ_y, and P_1 for δ in the original. It will be useful to bear in mind that in any operator such as $E_1 *$ or $E_2 *$, *the asterisk forms an integral part* of the symbol†. Thus $E_1 * E_2 *$, if we choose, may be written under the form of $E_1 *$ multiplied by $E_2 *$, that is, $(E_1 *) \times (E_2 *)$, where the cross is the sign of ordinary algebraical multiplication.

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decomposition into algebraical factors in proportion as the number $j+1$ becomes more separable into *numerical* factors, that is, in proportion as $j+1$ contains a smaller number of *distinct* prime factors. For this reason I purpose calculating $S_{i,7}$, $S_{i,8}$ against the appearance of the next Number of the *Magazine*. The nature of the roots, as regards being real or imaginary in the equation $S_{i,j}=0$, is also probably well deserving of study. It is worthy of notice that in each of the irreducible factors of $S_{i,j}$ for the values of j above considered, the coefficients are composed exclusively of the prime factors which enter into $j+1$. It is hardly necessary to observe that the quantities $\frac{S_{n+j-1,j}}{(n+1)(n+2)\ldots(n+j)}$, when expressed in a rational integral form, are the coefficients of the powers of x in the series for $[\log(1+x)]^n$, when n is regarded no longer as a positive integer, but as an arbitrary variable.

† The operant, sign of operation, and operand form a triad somewhat analogous to the subject, copula, and predicate of the logicians; and as in the admirable new school of philosophical grammar the copula is for certain purposes incorporated with the predicate, so *ex converso* in this system the sign of operation is taken up by the operant; but, herein advantageously differing from the practice of the grammarians alluded to, the combination assumes a distinct name from its leading element and is styled an operator.

I ought to mention that my information in this matter is derived from the statements which have appeared in the public prints, and not from a direct study of that wonderful manual of the quintessence of grammar so unpretendingly ushered into the world as a *primer*, but which, whatever name it goes by, can hardly fail to bring about a philosophical revival of the intellect of the rising generation of Englishmen. I wait for a favourable opportunity of leisure to address the full energies of my mind to the invigorating and congenial task of mastering its subtle differentiations and profound arduous abstractions.

93.

ON THE MOTION OF A RIGID BODY ACTED ON BY NO EXTERNAL FORCES.

[*Philosophical Transactions of the Royal Society of London*, CLVI. (1866), pp. 757—780.]

[Cf. p. 602.]

As conveying an image of the motion of a rigid body acted on by no forces, Poinsot's well-known method of representation, whether by a rolling ellipsoid or a shifting cone, labours under an obvious imperfection; the *time* is not put in evidence by it. Thus when the ellipsoid, with which alone I intend here to deal, is employed, it is true that the proportional value of the velocity of rotation about the instantaneous axis is geometrically measured by the radius vector drawn from the fixed point to the invariable tangent plane, and so by a process of summation the time of passing from one position to another may be considered as inferentially determined; but there is nothing to convey to the senses, or to the mind's eye, a notion of the effect of this summation, and thus the relation of the most important element—the time—to the position of a free revolving body remains unexpressed. I shall begin with showing how by a slight addition to Poinsot's ideal kinematical apparatus this defect may be completely removed, and the time between successive positions conceived to register itself mechanically. As the property upon which this depends readily lends itself to a geometrical form of proof, I shall, in the first instance, follow that mode of investigation, as being the more germane to the matter in hand, reserving to a later point in the memoir the analytical demonstration; that is to say, assuming Poinsot's ellipsoid, and the law which connects the velocity with the position of the body, I shall show how the time may be, as it were, mechanically extracted and summed.

It will be well, then, in the first instance to recall some simple properties of confocal ellipsoids which I shall have occasion to employ. If parallel tangent planes be drawn to a system of confocal ellipsoids, it is well known (see Dr Salmon's great work on *Surfaces*, Art. 202, 1st edition, or Art. 184, 2nd edition) that the points of contact lie in a plane curve, and that this

curve is an equilateral hyperbola. Since a concentric sphere with an infinite radius belongs to the system of confocal ellipsoids supposed, it follows that the point of intersection of the perpendicular from the centre of the ellipsoid upon the tangent planes with the plane at infinity, is a point in this curve, or, in other words, such perpendicular is contained in the plane of the hyperbola, and is an asymptote to the latter. The above is all that is required to establish the dynamical theorems necessary for my immediate purpose.

The revolving body being assumed to have moments of inertia A, B, C about the principal axes, the ellipsoid

$$Ax^2 + By^2 + Cz^2 = 1,$$

rigidly connected with the body, and which may be termed its kinematical exponent, is supposed to have its centre fixed, and to turn with a purely rolling motion upon a plane in contact with it which contains the constant impulsive couple L, capable at each moment of time in any position into which the body has turned, of communicating to it from rest the motion which it then actually possesses. If we suppose that the angular velocity of rotation is always equal to LRP, where P is the length of the perpendicular distance of the fixed centre from the tangent plane, and R is the length of the radius vector drawn from it to the point of contact, the path and velocity of the motion of the body in rigid connexion with the ellipsoid is completely represented; this is Poinsot's theorem stated in its complete form.

To fix the ideas, let us consider the invariable plane to be horizontal; if we were to apply a second plane parallel to the former fixed one, and also touching the ellipsoid, this would in no respect affect the motion—the ellipsoid might be made to roll between the two planes instead of rolling upon the under one alone; but if we were arbitrarily to alter the form of the upper part of the surface, the motion of rolling would in general be no longer possible; the only motion that could take place would be that of swinging round the vertical axis perpendicular to the two planes. In order that the ellipsoid may be able to roll as well as to swing, a certain geometrical condition must be satisfied, namely, the plane passing through the radius vector from the centre O to R, the point of contact with the given plane, and through the vertical perpendicular in question POp, must contain the point of contact r of the upper surface with the upper plane; for then, and then only, the rotation about OR may be resolved into two rotations about Or, Op respectively, and the ellipsoid whilst it rolls about OR, will be swinging round Op [or it may obviously at the same time be rolling and swinging (the latter in unequal degrees) upon each of the parallel tangent planes]; if this condition were not fulfilled, the ellipsoid, in the act of rolling upon the lower plane according to the direction of its motion, would either quit the upper one or tend to force it upwards; but as the upper, like the lower plane

is supposed to be at a fixed distance from the centre, this tendency would be resisted, and thus the supposed motion of rolling upon the lower plane without quitting the contact with the upper one could not be realized.

The condition that OR, POp, Or shall lie on one plane, we have seen will be fulfilled if the upper surface be a portion of an ellipsoid confocal with the lower one, and in that case the body may remain continually in contact with both planes whilst it rolls on the lower one; and we have thus a complete solution of the kinematical problem of determining what form must be given to the upper part of a body, the lower portion of whose surface is ellipsoidal, in order that it may be able to roll as well as swing between, and in contact with, two parallel fixed planes.

Call, then, the squared semi-axes of the lower surface a^2, b^2, c^2, and those of the upper one $a^2-\lambda$, $b^2-\lambda$, $c^2-\lambda$, and let us proceed to calculate the respective values of the two rotations about Op, Or equivalent to the single rotation LPR about OR.

In PO, RO produced set off OP_1, OR_1 equal to OP, OR, and draw R_1r' parallel to Op, and rp perpendicular to Op, and make $Or=r$, $Op=p$; then by virtue of what has been remarked above, r, R_1 lie in a hyperbola, of which OpP_1 is an asymptote, and the rotation about the instantaneous axis OR is represented by $L.P.OR_1$, and may be resolved into $L.P.Or'$ about Or' and $L.P.r'R_1$ about Op.

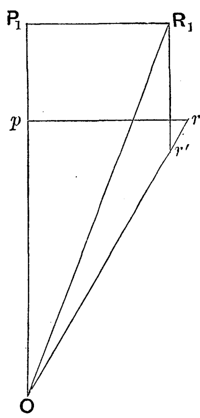

Fig. 1.

But $$L.P.Or' = L.P.r.\frac{Or'}{Or} = L.r.P.\frac{P_1R_1}{pr}$$
$$= L.r.P.\frac{Op}{OP_1} = L.r.p,$$

and $$L.P.R_1r' = L.P\,(OP_1 - P_1R_1 \cot r'Op)$$
$$= L.P\left(OP_1 - P_1R_1\frac{Op}{pr}\right)$$
$$= L.P\left(P - p\frac{p}{P}\right)$$
$$= L\,(P^2-p^2) = L\lambda;$$

for if α, β, γ be the angles which OP, Op make with the axes of the ellipsoid,
$$P^2 = a^2(\cos\alpha)^2 + b^2(\cos\beta)^2 + c^2(\cos\gamma)^2,$$
$$p^2 = (a^2-\lambda)(\cos\alpha)^2 + (b^2-\lambda)(\cos\beta)^2 + (c^2-\lambda)(\cos\gamma)^2,$$
$$P^2-p^2 = \lambda\,\{(\cos\alpha)^2 + (\cos\beta)^2 + (\cos\gamma)^2\} = \lambda.$$

Observing, then, that the motion has been resolved into a variable rotation Lpr about Or, and a uniform rotation $L\lambda$ about Op, and that accordingly the motion of a free body whose moments of inertia are as $\frac{1}{a^2}$, $\frac{1}{b^2}$, $\frac{1}{c^2}$

differs only by the uniform rotation $L\lambda$ from that of another one whose moments of inertia are as $\frac{1}{a^2-\lambda}$, $\frac{1}{b^2-\lambda}$, $\frac{1}{c^2-\lambda}$; we derive the following theorem:—

If the reciprocals of each of the moments of inertia of any number of rigid bodies B, B_1, B_2, B_3, ... differ from one another by constant quantities, say those of the second, third, fourth, &c., from those of the first by λ_1, λ_2, λ_3, ..., and these bodies be arranged with their corresponding principal axes parallel and be set in motion by an impulsive couple L given in magnitude and direction, then, after the lapse of any interval of time t, the principal axes of all the bodies will remain equally inclined to the axis of the given couple, and moreover the parallelism of the axes may be restored by turning B_1, B_2, B_3, ... about the axis of the couple through angles proportional to the time, namely, $L\lambda_1 t$, $L\lambda_2 t$, $L\lambda_3 t$, ... respectively.

It may be further noticed that if, at any moment of time, ω, ω_1 are the angular velocities of B, B_1 about their respective instantaneous axes,

$$\begin{aligned}\omega^2-\omega_1{}^2 &= L^2(P^2 . R^2 - p^2 . r^2)\\ &= L^2\{P^2(R^2-P^2)-p^2(r^2-p^2)\}+L^2(P^4-p^4)\\ &= L^2\lambda(P^2+p^2),\end{aligned}$$

that is, the difference between the squared velocities of any two bodies of the set is constant throughout the motion.

The above is a theory of rigid bodies whose kinematical exponents are confocal ellipsoids, and it has been shown that the motion of the whole set of bodies thus related, both as regards position and velocity, is completely determined when we know the motion of any one of them. It will hereafter appear from the analytical treatment of the subject that an analogous theorem applies to bodies whose kinematical exponents, instead of being confocal, are what may be termed contrafocal ellipsoids; ellipsoids, that is to say, the sums instead of the differences of whose squared axes are the same in all three directions.

By turning an ellipse through 90° round its centre we obtain a contrafocal ellipse; and contrafocal ellipsoids will be those all of whose principal sections are contrafocal.

To every infinite series of confocal ellipsoids there will correspond another such series, each ellipsoid of one series being contrafocal to each of the other, and it may very easily be seen that no two ellipsoids taken respectively out of the two opposite series can be obtained from each other by a mere change of place, as is the case with contrafocal ellipses; so in the instance of binary covariants and contravariants, any such can be converted into each other by the simple interchange of x, y with y, $-x$, but no such or similar commutability exists between covariants and contravariants of the ternary species.

It may be here convenient to notice that the kinematical exponent (or momental ellipsoid) of a given uniform ellipsoid is not the ellipsoid itself, but the *reciprocal* of the contrafocal ellipsoid whose squared semi-axes are $\lambda - a^2$, $\lambda - b^2$, $\lambda - c^2$, where $\lambda = a^2 + b^2 + c^2$.

It is now clear how the time of passage from one position to another is susceptible of mechanical measurement. Let the upper part of Poinsot's ellipsoid, whose semi-axes are a, b, c, be pared away until it assumes the form of a segment of an ellipsoid whose squared semi-axes are $a^2 - \lambda$, $b^2 - \lambda$, $c^2 - \lambda$; let the lower surface be in contact with a rough plane absolutely fixed, whilst its upper surface is so with a parallel *plate* not absolutely fixed, but capable of turning round an axis perpendicular to the two planes, and which if produced would pass through the centre of the ellipsoid. Then, when by the hand or any mechanical contrivance the body is made to spin like a sort of top upon the lower plane, it will also spin upon the *plate* above, and at the same time by the friction drive it round the vertical axis; the angle of rotation round this axis will give the exact measure of the time which the *free* body ideally associated with the ellipsoid would occupy in passing from one position to another. If this angle (which of course may be made to register itself by the motion of a hand upon a fixed dial-plate immediately over the rotating one which carries the index) be called ϕ, the time in question will be $\frac{\phi}{L\lambda}$, where it is particularly deserving of notice that the denominator $L\lambda$ is independent of the initial position of the body; hence by supposing the plane and rotating-plate to be capable by a preliminary adjustment of being shifted to any required distance from one another, the ellipsoid may be started from any position we please, and the value of the divisions of the dial-plate which register the time will remain invariable.

The greater the value of λ which measures the degree of divergency of the two juxtaposed surfaces, the larger will be the divisions representing a given quantity of time; and there is no impediment to λ receiving its maximum value, which is the square of the least semi-axis (say c). The upper confocal surface then degenerates into a curve or hoop resting upon and driving before it the rotating-plate. This gives precision to the form to be assigned to the upper surface. Again, as regards the lower surface, whose form involves two parameters, namely, the ratios of the three axes, it will hereafter appear that we may without any loss of generality reduce it to depend upon a single parameter by assuming the reciprocal of the square of one of its axes equal to the sum of the reciprocals of the squares of the remaining two.

Hence with a single series of ellipsoids every possible kind of motion of a free rigid body may be completely represented both as regards time and place. Each ellipsoid with its confocal hoop may be regarded as complete in

form, the former being imagined to consist of segments capable of being separated at will, so as to expose in succession each part as it is wanted of the interior hoop; and by an apparatus mechanically executable the motion may be followed without any break throughout the whole of one or any number of periods of revolution of the instantaneous axis.

Thus, then, the time of rotation of a free body may be kinematically determined. It may also, and even more simply, be measured off by direct observation of the time which a uniform ellipsoid spinning with its centre fixed upon an indefinitely rough plane occupies in passing from one position to another. To establish this somewhat remarkable law, let us consider the general case when the moments of inertia of the rolling ellipsoid have any values A, B, C. The resultant of the pressure and friction which coerce the ellipsoid to follow its actual path is a force always meeting the axis of instantaneous rotation, and giving rise therefore to an impressed couple whose axis is perpendicular to the former one. This being the case, and the ellipsoid subject to no other external force, its *vis viva* will be constant for just the same reason as the *vis viva* is so in the case of a system of particles connected in any manner, as by strings, whether elastic or inelastic, dragging each other along one or more surfaces, and acted on by no other forces except the reactions exerted by such surface or surfaces.

To render this perfectly clear, let v_1, v_2, v_3 denote the angular velocities of the rotating body about its principal axes; λ, μ, ν the angles between these axes and the instantaneous axis; J the magnitude of the couple produced by a force meeting the axis of rotation; then, by Euler's equations, we have

$$A\frac{dv_1}{dt} - (B-C)\,v_2v_3 = J\cos\lambda,$$

$$B\frac{dv_2}{dt} - (C-A)\,v_1v_3 = J\cos\mu,$$

$$C\frac{dv_3}{dt} - (A-B)\,v_1v_2 = J\cos\nu;$$

also $$v_1\cos\lambda + v_2\cos\mu + v_3\cos\nu = 0.$$

Hence $$Av_1dv_1 + Bv_2dv_2 + Cv_3dv_3 = 0,$$

and $$Av_1^2 + Bv_2^2 + Cv_3^2 = K,$$

a constant, as was to be proved.

In the case actually under consideration, if ω_1, ω_2, ω_3 are the angular velocities of the associated free body, and τ the time corresponding to t, so that dt, $d\tau$ are the intervals of time of the rolling and the free body undergoing the same infinitesimal angular displacement of position, we have

$$v_1 = \rho\omega_1, \quad v_2 = \rho\omega_2, \quad v_3 = \rho\omega_3,$$

and $$dt = \frac{d\tau}{\rho}.$$

Hence $$\rho^2 = \frac{K}{A\omega_1^2 + B\omega_2^2 + C\omega_3^2};$$

so that using the notation in ordinary use for the motion of a free body,

$$dt = \frac{d\tau}{\rho} = \frac{\omega d\omega \sqrt{(A\omega_1^2 + B\omega_2^2 + C\omega_3^2)}}{\sqrt{\{(\omega^2 - e_1)(\omega^2 - e_2)(\omega^2 - e_3)\}}},$$

and thus the time t of the rolling ellipsoid is known as an elliptic function in terms of ω^2.

Furthermore, by the well-known equations of *vis viva* and conservation of areas applied to the free body whose kinematical exponent is the ellipsoid with semi-axes a, b, c, that is, whose moments of inertia may be denoted by $\frac{1}{a^2}, \frac{1}{b^2}, \frac{1}{c^2}$, we have

$$\frac{\omega_1^2}{a^2} + \frac{\omega_2^2}{b^2} + \frac{\omega_3^2}{c^2} = M,$$

$$\frac{\omega_1^2}{a^4} + \frac{\omega_2^2}{b^4} + \frac{\omega_3^2}{c^4} = L^2.$$

Consequently if A, B, C are respectively representable by

$$\frac{\lambda}{a^4} + \frac{\mu}{a^2},\quad \frac{\lambda}{b^4} + \frac{\mu}{b^2},\quad \frac{\lambda}{c^4} + \frac{\mu}{c^2},$$

the multiplicator of $\omega d\omega$ in the numerator of the expression above given for dt, becomes a constant, namely, $\lambda L^2 + \mu M$. But this is the case when the density of the ellipsoid is uniform; for then

$$A : B : C :: b^2 + c^2 : c^2 + a^2 : a^2 + b^2,$$

and the determinant

$$\begin{vmatrix} \frac{1}{a^2}, & \frac{1}{b^2}, & \frac{1}{c^2} \\ \frac{1}{a^4}, & \frac{1}{b^4}, & \frac{1}{c^4} \\ b^2 + c^2, & c^2 + a^2, & a^2 + b^2 \end{vmatrix}, \text{ that is } \frac{1}{a^4 b^4 c^4} \Sigma (a^2 - b^2)(a^2 + b^2),$$

vanishes; in fact it is easily seen that

$$b^2 + c^2 = -\frac{a^2 b^2 c^2}{a^4} + \frac{b^2 c^2 + c^2 a^2 + a^2 b^2}{a^2},$$

$$c^2 + a^2 = -\frac{a^2 b^2 c^2}{b^4} + \frac{b^2 c^2 + c^2 a^2 + a^2 b^2}{b^2},$$

$$a^2 + b^2 = -\frac{a^2 b^2 c^2}{c^4} + \frac{b^2 c^2 + c^2 a^2 + a^2 b^2}{c^2}.$$

Hence any uniform ellipsoid, with its centre fixed, compelled by friction to roll on a rough horizontal plane will move precisely like a free body with properly assigned moments of inertia acted on by no external forces, as was to be proved. We see from what has been shown above that a uniform ellipsoid whose semi-axes are a, b, c, and which rolls on a rough horizontal plane, will keep pace with the motion of a uniform free ellipsoid, provided that the moments of inertia of the latter are in the ratios of $\frac{1}{a^2} : \frac{1}{b^2} : \frac{1}{c^2}$, that is, provided its axes are in the proportions of

$$\sqrt{\left(\frac{1}{b^2}+\frac{1}{c^2}-\frac{1}{a^2}\right)} : \sqrt{\left(\frac{1}{c^2}+\frac{1}{a^2}-\frac{1}{b^2}\right)} : \sqrt{\left(\frac{1}{a^2}+\frac{1}{b^2}-\frac{1}{c^2}\right)},$$

and thus the relative rate of motion of the rolling ellipsoid will not be affected if an interior ellipsoid whose axes are in the proportions above written is entirely removed or its density altered in any ratio. The internal ellipsoid will in fact move precisely as if it were free and detached from the surrounding crust, and might be annihilated without affecting the motion of the latter, in analogy with the well-known fact that any weight at the centre of oscillation of a compound pendulum may be abstracted without affecting its motion.

The theories of the free body and of the ellipsoid constrained by pressure and friction to follow its path, and which has been proved above to keep

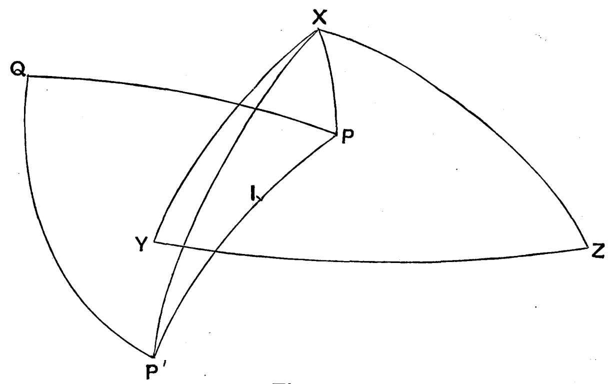

Fig. 2.

exact pace with it, are so interwoven that it would be unsatisfactory to leave the theory of the latter incomplete in any point, and I shall therefore proceed to calculate the value of the pressure and friction corresponding to any position of the rolling body. On a sphere described about the fixed point, let P and I denote the position of the instantaneous axis of rotation, and the perpendicular to the fixed plane respectively. The pole of the friction couple will be denoted by a point P' in the plane of PI distant by a quadrant from P, for its plane passes through P and through Q the pole of PI, and the pole

of the pressure couple will obviously lie at Q itself. Let X, Y, Z mark in the sphere the positions of the principal axes.

Then XPP' being a quadrantal triangle,

$$\cos XP' = \sin XP \cos XPI = \frac{1}{\sin PI}(\cos XI - \cos XP \cos PI)$$

$$= \frac{1}{\sin PI}\left(\frac{\omega_1}{a^2 L} - \frac{\omega_1}{\omega}\frac{M}{L\omega}\right) = \frac{\omega_1}{L \sin PI}\left(\frac{1}{a^2} - \frac{M}{\omega^2}\right),$$

where $$\omega^2 = \omega_1^2 + \omega_2^2 + \omega_3^2.$$

Again, for greater simplicity, making $\rho = 1$, that is, considering the motions of the rolling body and the free nucleus to absolutely coincide in time, we have from the Eulerian equations,

$$J \cos \lambda = (b^2 + c^2)\frac{d\omega_1}{d\tau} + (b^2 - c^2)\,\omega_2\omega_3$$

$$= \frac{c^2 - b^2}{b^2 c^2}(a^2 b^2 + a^2 c^2 - b^2 c^2)\,\omega_2\omega_3.$$

Hence if $[F]$ be the couple due to F the friction force,

$$[F] = \Sigma\,(J \cos \lambda \cos XP')$$

$$= \frac{\omega_1\omega_2\omega_3}{La^2b^2c^2 \sin PI\,\omega^2}\,\Sigma\,(c^2 - b^2)(a^2b^2 + a^2c^2 - b^2c^2)(\omega^2 - Ma^2)$$

$$= \frac{2(c^2 - b^2)(b^2 - a^2)(a^2 - c^2)\,\omega_1\omega_2\omega_3}{La^2b^2c^2 \sin PI}$$

$$= \frac{2(c^2 - b^2)(b^2 - a^2)(a^2 - c^2)\,\omega_1\omega_2\omega_3\,\omega}{a^2b^2c^2\sqrt{(L^2\omega^2 - M^2)}}.$$

And as the arm at which the friction acts, that is, the distance of the fixed centre from the point of contact between the ellipsoid and the fixed plane is $\frac{\sqrt{M}}{L}\sec PI$, that is, $\frac{\omega}{\sqrt{M}}$, we have

$$F = 2\,\frac{(c^2 - b^2)(b^2 - a^2)(a^2 - c^2)\,\omega_1\omega_2\omega_3}{a^2b^2c^2}\sqrt{\left(\frac{M}{L^2\omega^2 - M^2}\right)},$$

the mass of the ellipsoid throughout being treated as unity.

We might, in like manner, through the algorithm of spherical triangles, proceed to calculate the value of the pressure couple $[P]$ which is equal to the sum of the components $J\cos\lambda$, $J\cos\mu$, $J\cos\nu$ multiplied respectively by the sines of the perpendicular arcs dropped upon PI from X, Y, Z. But it will be obtained more expeditiously in its simplest form by first calculating J itself, the value of the entire couple, and then using the equation

$$[P]^2 = J^2 - [F]^2.$$

For brevity, in place of a^2, b^2, c^2, ω^2, L^2 write f, g, h, Ω, Λ respectively, and let $f+g+h=p$, $fg+gh+hf=q$, $fgh=r$. Then

$$\frac{\Omega_1}{f^2}+\frac{\Omega_2}{g^2}+\frac{\Omega_3}{h^2}=\Lambda,$$

$$\frac{\Omega_1}{f}+\frac{\Omega_2}{g}+\frac{\Omega_3}{h}=M,$$

$$\Omega_1+\Omega_2+\Omega_3=\Omega.$$

So that if $(f-g)(g-h)(h-f)=\zeta$, and $\kappa=\dfrac{fgh}{\zeta}$,

$$\Omega_1=\kappa\frac{h-g}{h^2g^2}\{\Omega-(g+h)M+gh\Lambda\},$$

$$\Omega_2=\kappa\frac{g-f}{f^2h^2}\{\Omega-(h+f)M+hf\Lambda\},$$

$$\Omega_3=\kappa\frac{h-g}{g^2f^2}\{\Omega-(f+g)M+fg\Lambda\}.$$

Hence $$\begin{aligned}(\Lambda\Omega-M^2)[F]^2=&-4\Omega^4+(8pM-4q\Lambda)\Omega^3\\&+\{(4p^2+4q)M^2-(4pq+12r)M\Lambda+4pr\Lambda^2\}\Omega^2\\&+\{(4pq-4r)M^3-(4q^2+4pr)M^2\Lambda+8qrM\Lambda^2-4r^2\Lambda^3\}\Omega.\end{aligned}$$

Also from the Eulerian equations,

$$\begin{aligned}J^2&=\Sigma\frac{(g-h)^2}{g^2h^2}(fg+fh-gh)^2\,\Omega_2\Omega_3\\&=\frac{1}{\zeta}\Sigma\{(g-h)(fg+fh-gh)^2N_1\},\end{aligned}$$

where $$\begin{aligned}N_1=\Omega^2-(2f+g+h)M\Omega+(fg+fh)\Lambda\Omega+(f^2+fg+fh+gh)M^2\\-\{f^2(g+h)+2fgh\}M\Lambda+f^2gh\Lambda^2.\end{aligned}$$

But

$$\Sigma(g-h)(fg+fh-gh)^2=4\Sigma(g^3h^2-g^2h^3)-4(fg+fh+gh)\Sigma(g^2h-h^2g)=0,$$

$$\begin{aligned}\Sigma(g-h)(fg+fh-gh)^2(2f+g+h)&=\Sigma(fg-fh)(fg+fh-gh)^2\\&=4\Sigma(fg-fh)g^2h^2=-4fgh\zeta,\end{aligned}$$

$$\begin{aligned}\Sigma(g-h)(fg+fh-gh)^2(fg+fh)&=-\Sigma gh(g-h)(fg+fh-gh)^2\\&=-(f^2g^2+f^2h^2+g^2h^2)\Sigma gh(g-h)+2fgh\Sigma gh(g^2-h^2)\\&=\{(f^2g^2+f^2h^2+g^2h^2)-2fgh(f+g+h)\}\zeta,\end{aligned}$$

$$\begin{aligned}\Sigma(g-h)(fg+fh-gh)^2(f^2+fg+fh+gh)&=\Sigma f^2(g-h)(fg+fh-gh)^2\\&=\Sigma(fg+fh+gh)^2f^2(g-h)=-(fg+fh+gh)^2\zeta,\end{aligned}$$

$$\begin{aligned}\Sigma(g-h)(fg+fh+gh)^2\{f^2(g+h)+2fgh\}&=\Sigma(f^2g^2-f^2h^2)(fg+fh-gh)^2\\&=-4(fg+fh)\Sigma gh(f^2g^2-f^2h^2)=-4fgh(fg+fh+gh)\zeta,\end{aligned}$$

$$\Sigma(g-h)(fg+fh-gh+gh)^2f^2gh=fgh\Sigma(fg-fh)(fg+fh-gh)^2=-4f^2g^2h^2\zeta.$$

Hence $$J^2 = \{4rM + (q^2 - 4pr)\Lambda\}\Omega - (qM - 2r\Lambda)^2,$$

and $$(\Lambda\Omega - M^2)J^2 = \{4rM\Lambda + (q^2 - 4pr)\Lambda^2\}\Omega^2$$
$$- \{4rM^3 + (2q^2 - 4pr)M^2\Lambda - 4qrM\Lambda^2 + 4r^2\Lambda^3\}\Omega + M^2(qM - 2r\Lambda)^2.$$

Hence $$(\Lambda\Omega - M^2)[P]^2 = (\Lambda\Omega - M^2)J^2 - (\Lambda\Omega - M^2)[F]^2$$
$$= 4\Omega^4 - (8pM^2 - 4q\Lambda)\Omega^3 + \{(4p^2 + 4q)M^2 - (4pq + 8r)M\Lambda + q^2\Lambda^2\}\Omega^2$$
$$- \{4pqM^3 - (2q^2 - 8pr)M^2\Lambda + 4qrM\Lambda^2\}\Omega + M^2(qM - 2r\Lambda)^2.$$

Hence $$[P] = \frac{2\Omega^2 - (2pM - q\Lambda)\Omega + M(qM - 2r\Lambda)}{\sqrt{(\Lambda\Omega - M^2)}};$$

and as the arm at which this couple acts is

$$\sqrt{\left(\frac{\omega^2}{M} - \frac{M}{L^2}\right)}, \text{ that is } \sqrt{\left(\frac{\Lambda\Omega - M^2}{M\Lambda}\right)},$$

the pressure $$P = \frac{2\Omega^2 - (2pM - q\Lambda)\Omega + (qM^2 - 2rM\Lambda)}{\Lambda\Omega - M^2}.$$

If we call the constant perpendicular from the centre and the radius vector to the point of contact h and l respectively, and substitute for $\frac{\Omega}{\Lambda}$, $\frac{M}{\Lambda}$ their respective values h^2l^2, h^2, we may express $\frac{F}{P}$ as a function of h, l, and making this a maximum in respect to l, the least sufficient value of the coefficient of friction necessary to ensure rolling may be deduced in terms of the quantities $\frac{a}{h}$, $\frac{b}{h}$, $\frac{c}{h}$.

Also if θ denote the angle between the axis of the couple J and the pole of the plane PI, we have

$$(\cos\theta)^2 = \frac{[P]^2}{J^2} = \frac{\{2h^4l^4 - (2ph^2 - q)h^2l^2 + (qh^4 - 2rh^2)\}^2}{(h^2l^2 - h^4)\{(4rh^2 + q^2 - 4pr)h^2l^2 - (qh^2 - 2r)^2\}},$$

or $$\cos\theta = \frac{h\{2h^2l^4 - (2ph^2 - q)l^2 + (qh^2 - 2r)\}}{\sqrt{(l^2 - h^2)}\sqrt{\{(4rh^2 + q^2 - 4pr)h^2l^2 - (qh^2 - 2r)^2\}}}.$$

It has been already seen how, by the method of confocal ellipsoids, the number of constants entering into the question of the rotation of a rigid body about its centre of gravity has virtually been reduced by a unit; to render this important theory complete, and to give it the fullest extension of which it is capable, a corresponding dynamical theory of contrafocal ellipsoids remains to be developed, and might undoubtedly be discussed by analogous geometrical methods; but it will be found more expedient to take up the subject afresh from a purely analytical point of view, and then the theory will present itself in all its completeness under a single aspect.

Calling α, β, γ the angles which the invariable axis makes with the principal axes of the rotating body, we have the well-known equations

$$\cos\alpha = \frac{A\omega_1}{L}, \quad \cos\beta = \frac{B\omega_2}{L}, \quad \cos\gamma = \frac{C\omega_3}{L},$$

(immediate deductions from the self-obvious principle of the constancy of the couple competent at any instant to communicate to the rotating body the motion it is then actually endued with, conjoined with the geometrical property of the principal axes that the moment in respect to any one of them of the momenta of the particles of the body due to rotation about either of the other two is zero).

Consequently from the principle of *vis viva*, that is, from the equation

$$A\omega_1^2 + B\omega_2^2 + C\omega_3^2 = M,$$

in addition to the equation

$$(\cos\alpha)^2 + (\cos\beta)^2 + (\cos\gamma)^2 = 1, \tag{1}$$

we have the equation

$$\frac{(\cos\alpha)^2}{A} + \frac{(\cos\beta)^2}{B} + \frac{(\cos\gamma)^2}{C} = \frac{M}{L^2}, \tag{2}$$

and the Eulerian system of equations*,

$$A\frac{d\omega_1}{dt} - (B-C)\,\omega_2\omega_3 = 0, \quad B\frac{d\omega_2}{dt} - (C-A)\,\omega_3\omega_1 = 0, \quad C\frac{d\omega_3}{dt} - (A-B)\,\omega_1\omega_2 = 0,$$

* To make this paper complete within itself so as to come within the comprehension of those who have no previous knowledge of the special problem which it treats, it seems desirable to indicate an elementary method of obtaining these ofttimes herein quoted equations.

1. Suppose no external forces in operation. Consider the effects of the three partial velocities ω_1, ω_2, ω_3 in succession as if the others were non-existent.

Referring to fig. 3, ω_1 tends to produce no motion about OY or OZ in the time dt, because the moments of the centrifugal forces about these axes, quantitatively represented by Σmzx, Σmxy respectively, are each zero by virtue of the geometrical definition of the principal axes.

Thus to each partial velocity in the time dt is due only a motion of rotation about its own axis. Hence if $d\gamma$ is the variation in γ due to ω_1,

$$d\gamma = ZZ' \cos YZI = \omega_1 dt \frac{\cos\beta}{\sin\gamma},$$

or

$$d\cos\gamma = \omega_1 \cos\beta\, dt.$$

Similarly as regards the variation of $\cos\gamma$ due to ω_2,

$$d\cos\gamma = -\omega_2 \cos\alpha\, dt.$$

Hence the total variation $d\cos\gamma = (\omega_1\cos\beta - \omega_2\cos\alpha)\,dt$,

that is

$$\frac{C}{L}d\omega_3 = \left(\frac{B\omega_2\omega_1}{L} - \frac{A\omega_1\omega_2}{L}\right)dt,$$

or

$$d\omega_3 = \frac{B-A}{C}\omega_2\omega_1 dt,$$

with analogous equations for $d\omega_2$, $d\omega_1$.

becomes
$$\left.\begin{aligned}\frac{d\cos\alpha}{dt}-L\left(\frac{1}{C}-\frac{1}{B}\right)\cos\beta\cos\gamma=0,\\ \frac{d\cos\beta}{dt}-L\left(\frac{1}{A}-\frac{1}{C}\right)\cos\gamma\cos\alpha=0,\\ \frac{d\cos\gamma}{dt}-L\left(\frac{1}{B}-\frac{1}{A}\right)\cos\alpha\cos\beta=0.\end{aligned}\right\} \quad (3)$$

The above equations suffice to express the relations of the angles which the invariable line in space makes with fixed lines in the moving body to one another and to the time: to complete the solution it will be sufficient to express in terms of the time, or of any quantity dependent on the time, the position of any of the planes drawn through a principal axis and the invariable line.

The letters X, Y, Z, I retaining their previous signification, let ZZ' represent the infinitesimal angular displacement of Z due to the rotation ω_1 about X in the time dt.

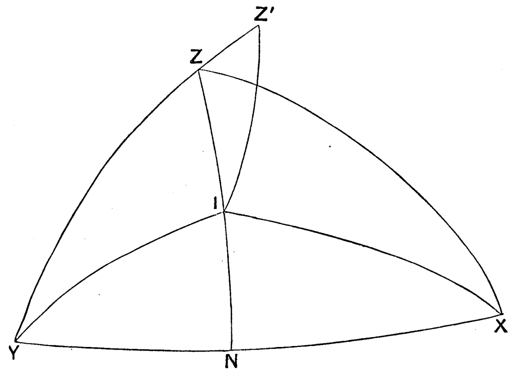

Fig. 3.

When the impressed couples about OX, OY, OZ respectively are L, M, N the variations in the angular velocities due to them being
$$\frac{Ldt}{A},\ \frac{Mdt}{B},\ \frac{Ndt}{C},$$
these quantities must be added to the values of $d\omega_1$, $d\omega_2$, $d\omega_3$ indicated above. We have thus the equations in question.

It may be as well also here to indicate in the fewest words the rationale of the ellipsoidal representation of the motion.

A, B, C being the principal moments of inertia, and $Ax^2+By^2+Cz^2=1$ the equation to the ellipsoid, the relation
$$\omega_1 : \omega_2 : \omega_3 :: A\cos\alpha : B\cos\beta : C\cos\gamma$$
shows that the invariable line coincides in direction with the pedal to the radius vector drawn in the direction of the instantaneous axis.

2. Consequently the length of such pedal being
$$\frac{(\cos\alpha)^2}{A}+\frac{(\cos\beta)^2}{B}+\frac{(\cos\gamma)^2}{C},$$
which is constant, a plane drawn at that constant length perpendicular to the invariable line

Then $$ZIZ' = ZZ'\frac{\sin IZZ'}{\sin IZ'} = ZZ'\frac{\cos NX}{\sin IZ}.$$

But $$\frac{\cos NX}{\cos NY} = \frac{\cos IX}{\cos IY} = \frac{\cos \alpha}{\cos \beta},$$

or $$\cos NX = \frac{\cos \alpha}{\sqrt{\{(\cos \alpha)^2 + (\cos \beta)^2\}}},$$

and $$\sin IZ = \sqrt{\{1 - (\cos \gamma)^2\}} = \sqrt{\{(\cos \alpha)^2 + (\cos \beta)^2\}}.$$

Hence $$ZIZ' = L\frac{\dfrac{(\cos \alpha)^2}{A}dt}{(\cos \alpha)^2 + (\cos \beta)^2}.$$

Similarly, if ZIZ_1 be the angular displacement of the plane ZI measured in the same direction as before,

$$ZIZ_1 = L\frac{\dfrac{(\cos \beta)^2}{B}dt}{(\cos \alpha)^2 + (\cos \beta)^2},$$

and the rotation about Z causes no displacement of the plane in question. Hence if the horary angles, as they may be called, which measure the angular deviations of the planes XI, YI, ZI from a fixed meridian plane through I be called ξ, η, ζ, we have

$$\frac{d\xi}{dt} = L\frac{\dfrac{(\cos \beta)^2}{B} + \dfrac{(\cos \gamma)^2}{C}}{(\cos \beta)^2 + (\cos \gamma)^2},\quad \frac{d\eta}{dt} = L\frac{\dfrac{(\cos \gamma)^2}{C} + \dfrac{(\cos \alpha)^2}{A}}{(\cos \gamma)^2 + (\cos \alpha)^2},\quad \frac{d\zeta}{dt} = L\frac{\dfrac{(\cos \alpha)^2}{A} + \dfrac{(\cos \beta)^2}{B}}{(\cos \alpha)^2 + (\cos \beta)^2}* \quad (4).$$

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touches the ellipsoid in every position into which it turns, and therefore the ellipsoid with its centre fixed rolls on such plane. This proves the identity of the two motions *quâ* space.

3. The moment of inertia in respect to the instantaneous axis being represented by the inverse squared length of the radius vector of the ellipsoid in the direction of that axis, the square root of the *vis viva* (a constant) is proportional to the angular velocity divided by the radius vector drawn to the point of contact, so that the former is proportional to the latter; this completes the representation by expressing through means of the ellipsoid the relation of the motion of the associated free body to time, or at all events it gives the law from which that relation may be extracted.

The above contains the whole sum, pith, and substance of Poinsot's ellipsoidal mode of representation.

* By combining this with the system of equations previously found, both η and ζ may readily be obtained under the form of elliptic functions of the third kind in terms of $\cos\gamma$, but $\eta - \zeta$ or the angle I in the quadrantal spherical triangle XIY of fig. 3 will also be expressible as a function of α, β, and therefore of γ. The comparison of the forms of $\eta - \zeta$ given by the two methods respectively, leads therefore to a theorem in elliptic functions; Professor Cayley has worked this out, and finds that it is the well-known theorem which expresses the dependence between two elliptic functions of the third order, the product of whose parameters is equal to the square of the modulus. I subjoin an extract from his letter, in which I have only introduced some slight changes in the lettering:—

If, now, preserving L constant we replace $\frac{1}{A}, \frac{1}{B}, \frac{1}{C}, M$ by

$$\frac{1}{A}-\lambda;\ \frac{1}{B}-\lambda;\ \frac{1}{C}-\lambda;\ M-\lambda L^2,$$

the equations (1), (2), (3) remain unaltered, and the right-hand sides of equations (4) become each of them simply altered by the addition of the term $-L\lambda$, which may be expressed by saying that the difference between

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"Writing
$$Ap^2+Bq^2+Cr^2=M,$$
$$A^2p^2+B^2q^2+C^2r^2=L^2,$$
your theorem is
$$\int \frac{M-Ap^2}{L^2-A^2p^2}\,dt - \int \frac{M-Bq^2}{L^2-B^2q^2}\,dt = \cos^{-1}(\ldots\ldots)$$
$$=\tan^{-1}(\ldots\ldots),$$
where
$$dt=-\frac{Cdr}{(A-B)\,pq}.$$
Whence expressing everything in terms of r, this is
$$\int \frac{F+Gr^2}{(1+nr^2)\sqrt{(R)}}\,dr - \int \frac{F_1+G_1r^2}{(1+n_1r^2)\sqrt{(R)}}\,dr = \tan^{-1}(\ldots\ldots);$$
write for shortness,
$$AM-L^2=a,\quad BM-L^2=b,$$
$$B-C=\alpha,\quad C-A=\beta,\quad A-B=\gamma.$$
Then we have
$$B\gamma q^2=a+C\beta r^2;\quad -A\gamma p^2=b-C\alpha r^2;$$
or if $r^2=-\frac{a}{C\beta}\theta^2$,
$$B\gamma q^2=a\,(1-\theta^2);\quad -A\gamma p^2=b\left(1+\frac{a\alpha}{b\beta}\theta^2\right);$$
so that using θ instead of r, the radical is
$$\sqrt{\{(1-\theta^2)(1-\kappa^2\theta^2)\}},\quad \kappa^2=-\frac{a\alpha}{b\beta},$$
$$L^2-A^2p^2=L^2+\frac{A}{\gamma}(b-C\alpha r^2)=\frac{1}{\gamma}(Ba-AC\alpha r^2)$$
$$=\frac{Ba}{\gamma}\left(1+\frac{AC\alpha}{Ba}\cdot\frac{a}{C\beta}\theta^2\right)$$
$$=\frac{Ba}{\gamma}\left(1+\frac{A\alpha}{B\beta}\theta^2\right),$$
$$L^2-B^2q^2=L^2-\frac{B}{\gamma}(a+C\beta r^2)=\frac{1}{\gamma}(-Ab-BC\beta r^2)$$
$$=-\frac{Ab}{\gamma}\left(1-\frac{BC\beta}{Ab}\cdot\frac{a}{C\beta}\theta^2\right)$$
$$=-\frac{AB}{\gamma}\left(1-\frac{Ba}{Ab}\theta^2\right).$$
So that the form is
$$\int d\theta\,\frac{F+G\theta^2}{(1+n\theta^2)\sqrt{(\Theta)}} - \int d\theta\,\frac{F_1+G_1\theta^2}{(1+n_1\theta^2)\sqrt{(\Theta)}}=\ldots\ldots,$$
where
$$n=\frac{A\alpha}{B\beta},\quad n_1=-\frac{Ba}{Ab},\quad \kappa^2=-\frac{a\alpha}{b\beta},$$
and thus
$$nn_1=-\frac{a\alpha}{b\beta}=\kappa^2,$$
so that the relation is the known one between the two forms
$$\int\frac{d\theta}{(1+n\theta^2)\sqrt{(\Theta)}}\ \text{and}\ \int\frac{d\theta}{\left(1+\frac{\kappa^2}{n}\theta^2\right)\sqrt{(\Theta)}},$$
with reciprocal parameters."

the displacements at any moment of time of two bodies whose kinematical exponents are confocal ellipsoids, is equivalent to a displacement round the invariable line proportional to the time elapsed since the positions were coincident or parallel, as previously found by geometrical reasoning.

Again, if we replace $\frac{1}{A}, \frac{1}{B}, \frac{1}{C}, M$ by

$$\lambda - \frac{1}{A}; \;\; \lambda - \frac{1}{B}; \;\; \lambda - \frac{1}{C}; \;\; \lambda L^2 - M,$$

the equations (1), (2), (3) will remain unaltered, provided we write $180 - \alpha$; $180 - \beta$; $180 - \gamma$ in place of α, β, γ, and the equations (4) will receive an augmentation of $L\lambda$ on their right-hand sides, but remain otherwise unaltered, provided we substitute $-\xi, -\eta, -\zeta$ for ξ, η, ζ. Or again, we may state the same result without substituting for the angles of inclinations their supplements, but leaving them unaltered if we change the sign of L; showing that if two bodies whose kinematical exponents or momental ellipsoids are contrafocal, be set in a parallel position at rest, and are acted on by two equal and coaxial but contrary impulsive couples, their principal axes will continue throughout the motion to make equal but contrary angles with the invariable line, and will admit of being brought back to a position of parallelism by means of a rotatory displacement about the invariable line proportional to the time. Thus, leaving out of consideration this displacement, *correlated* solid bodies (as those may be termed whose kinematical exponents are confocal ellipsoids) may be made to move equally and similarly, and *contrarelated* ones (as we may term those whose kinematical exponents are contrafocal ellipsoids) equally and contrarily without the action of any external force. It will eventually be seen that there is a practical advantage in considering L as retaining the same sign in both cases, and throwing the contrariety of motion in the second case upon the change of the inclinations α, β, γ into their supplements.

Thus the motion of a body is arithmetically given when that of any other of the series of those to whose kinematical exponents its own is either confocal or contrafocal has been determined.

Alike for the two cases of con- and contra-focalism it will be convenient to disregard this uniform motion of rotation, treating it in the light merely of a correction*, so that the motions of all the bodies contained in either one series may be considered in regard to themselves as *coincident*, and as *supplemental* (in a sense that explains itself) in regard to the motions of the bodies belonging to the other series. I shall now show as a corollary from the

* The apparent motions of any two correlated or contrarelated bodies to two spectators standing respectively on the invariable plane of each may be made identical or similar, provided a certain uniform angular velocity be imparted to one of these planes.

above proposition that, with the above understanding, the motion of any rigid body may (subject to an unimportant exception that will be stated in its proper place) be made identical with that of one real indefinitely flattened disk, and supplemental to that of another. The case of a disk, it will be noticed, is that in which one of the principal moments of inertia becomes equal to the sum of the other two; in general these moments of inertia must not only be positive, but each must be not greater than the sum of the other two, as is the case with the lengths of the sides of a triangle; in the extreme case, when the body is reduced to but two dimensions, the greatest becomes equal to the sum of the other two, and conversely, when this is so, the body can only be of the form of a flat disk; the above is obvious when it is remembered that the moments of inertia are the sums of the three intrinsically positive quantities Σmx^2, Σmy^2, Σmz^2 taken two and two together. So also it is well to notice that the modular quantity $\frac{M}{L^2}$ in equation (2) is not absolutely arbitrary, but besides being essentially positive, is conditioned to lie between the least and greatest of the quantities $\frac{1}{A}$, $\frac{1}{B}$, $\frac{1}{C}$, since otherwise the quantities $(\cos\alpha)^2$, $(\cos\beta)^2$, $(\cos\gamma)^2$ in equations (1) and (2) could not all remain positive, and consequently such equations would not correspond to any real case of motion.

Let A, B, C be arranged in order of magnitude, and suppose

$$\frac{1}{A_1}=\frac{1}{A}-\frac{1}{\mu},\quad \frac{1}{B_1}=\frac{1}{B}-\frac{1}{\mu},\quad \frac{1}{C_1}=\frac{1}{C}-\frac{1}{\mu},\quad \frac{M_1}{L^2}=\frac{M}{L^2}-\frac{1}{\mu},$$

and let μ be so determined as to make one of the quantities A_1, B_1, C_1 equal to the sum of the other two. Then

(1) Any imaginary value of μ must be neglected.

(2) Any value of μ which makes A_1, B_1, C_1 of different algebraical signs must be neglected.

(3) If μ, being real, makes A_1, B_1, C_1 all positive, these quantities will correspond to the moments of a real disk whose representative ellipsoid is confocal to that of the body whose moments of inertia A, B, C are given.

(4) If μ, being real, makes A_1, B_1, C_1 all negative, by taking $-A_1$, $-B_1$, $-C_1$, that is, the reciprocals of $\frac{1}{\mu}-\frac{1}{A}$, $\frac{1}{\mu}-\frac{1}{B}$, $\frac{1}{\mu}-\frac{1}{C}$ as the new moments of inertia, we evidently shall have obtained a reduction to a disk of the supplemental or contrafocal kind.

In case (3) $M-\frac{L^2}{\mu}$, and in case (4) $\frac{L^2}{\mu}-M$ is to be substituted for M, so that the necessary condition of $\frac{L^2}{M}$ being intermediate between the greatest

and least of the quantities A, B, C will continue to be fulfilled in the disk by $\frac{L^2}{M_1}$ remaining intermediate between the greatest and least of the quantities A_1, B_1, C_1.

Suppose $A_1 + B_1 = C_1$, then

$$\frac{A}{A-\mu} + \frac{B}{B-\mu} = \frac{C}{C-\mu},$$

or

$$(A + B - C)\mu^2 + AB\mu + ABC = 0.$$

The determinant (that is, negative discriminant) of this equation is

$$AB(AB - CA - CB + C^2) \text{ or } AB(A-C)(B-C),$$

so that if C is the least or greatest moment of inertia, μ will have real values, but will be unreal if C is the mean moment of inertia.

Suppose now that $A_1 + B_1 = C_1$ for one value of μ, to find the values A', B', C' corresponding to the conjugate disk, we obtain from the above equation in μ, by substituting A_1, B_1, C_1 for A, B, C,

$$2A_1B_1\mu - A_1B_1C_1 = 0, \text{ or } \mu = -\frac{C_1}{2},$$

and accordingly

$$\frac{1}{A'} : \frac{1}{B'} :: \frac{1}{A_1} - \frac{2}{A_1+B_1} : \frac{1}{B_1} - \frac{2}{A_1+B_1} :: \frac{B_1-A_1}{2A_1} : \frac{A_1-B_1}{2B_1} :: \frac{1}{A_1} : -\frac{1}{B_1}.$$

Hence if A_1, B_1 have the same signs, A', B' have opposite signs, and *vice versâ*, if A_1, B_1 have opposite signs A', B', and therefore A', B', C' have all the same signs for $C' = A' + B'$.

Consequently one and one only of each of the two solutions for disks drawn perpendicular respectively to the extreme principal axes, makes the three moments of inertia all of the same sign, and consequently each such solution leads either to a direct or supplemental reduction to the disk form.

Now, suppose that A, B, C being all of the same signs, A has become equal to $B + C$, so that the equation in μ becomes

$$2B\mu^2 + 2AB\mu + ABC = 0,$$

or

$$\mu^2 - A\mu + \frac{AC}{2} = 0.$$

Let μ_1, μ' be the two values of μ from this equation, so that

$$\frac{1}{A_1} = \frac{1}{A} - \frac{1}{\mu_1},\quad \frac{1}{B_1} = \frac{1}{B} - \frac{1}{\mu_1},\quad \frac{1}{C_1} = \frac{1}{C} - \frac{1}{\mu_1},$$

$$\frac{1}{A'} = \frac{1}{A} - \frac{1}{\mu'},\quad \frac{1}{B'} = \frac{1}{B} - \frac{1}{\mu'},\quad \frac{1}{C'} = \frac{1}{C} - \frac{1}{\mu'},$$

and

$$A_1 + B_1 = C_1,\quad A' + B' = C'.$$

Then $$\frac{1}{A_1}+\frac{1}{A'}=\frac{2}{A}-\left(\frac{1}{\mu_1}+\frac{1}{\mu'}\right)=\frac{2}{A}-\frac{2}{C}=2\left\{\frac{1}{B+C}-\frac{1}{C}\right\}=-\frac{2}{AC},$$

$$\frac{1}{A_1}\cdot\frac{1}{A'}=\frac{1}{A^2}-\left(\frac{1}{\mu_1}+\frac{1}{\mu'}\right)\frac{1}{A}+\frac{1}{\mu_1\mu'}$$

$$=\frac{1}{A^2}.$$

Hence if A, B, C have the positive sign, A_1 and A' are both negative, and if A, B, C have the negative sign, A_1 and A' are both positive; consequently, on the first supposition, the signs of one of the two systems A_1, B_1, C_1; A', B', C' will be all negative, and on the second supposition all positive. Hence one of the two reductions falls under case (3), that is, is proper or direct, and the other under case (4), and is improper or supplemental. As nothing in nature exists in vain, it will presently be seen that the choice which is always possible between these two modes of reduction leads to an important simplification of the cases which arise in the problem of rotation, and that there need never be any room for doubt as to which of the two sorts of reduction should be employed in any specified problem.

The case of exception to which allusion has been made in anticipation, arises when two of the moments of inertia are equal; for then, supposing A, A, C to be the original moments of inertia, the new moments of inertia will be A_1, A_1, C_1; and since C_1 cannot be zero, we can only suppose $C_1=2A_1$; and making

$$\frac{1}{A_1}=\frac{1}{A}-\frac{1}{\mu},\quad \frac{1}{C_1}=\frac{1}{C}-\frac{1}{\mu},$$

the equation in μ becomes

$$\frac{2A}{A-\mu}=\frac{C}{C-\mu},$$

or $$(2A-C)\,\mu=AC,\quad \mu=\frac{AC}{2A-C},$$

and $$A_1=\frac{A\mu}{\mu-A}=\frac{AC}{2\,(C-A)},\quad C_1=\frac{AC}{C-A};$$

so that the reduction will be proper or improper according as the unequal moment of inertia is greater or less than either of the equal ones.

A relation has been obtained geometrically in the commencement of this memoir between the squared velocities of any two dynamically equivalent bodies represented by confocal ellipsoids. To complete the theory, it is proper to find the exact nature of this relation when a given body has been reduced to a disk, whether by the direct or supplemental method.

First, in the case of direct reduction, using v_1, v_2, v_3 for the angular velocities of the disk, and ω_1, ω_2, ω_3 for those of the associated body in

corresponding positions about the principal axes, and v, ω for the total angular velocities of the disk and body respectively,

$$v_1 = \frac{L}{A_1}\cos\alpha, \quad v_2 = \frac{L}{B_1}\cos\beta, \quad v_3 = \frac{L}{C_1}\cos\gamma,$$

$$\omega_1 = \frac{L}{A}\cos\alpha, \quad \omega_2 = \frac{L}{B}\cos\beta, \quad \omega_3 = \frac{L}{C}\cos\gamma,$$

$$\frac{1}{A_1} = \frac{1}{A} - \lambda, \quad \frac{1}{B_1} = \frac{1}{B} - \lambda, \quad \frac{1}{C_1} = \frac{1}{C} - \lambda.$$

Hence $$\omega^2 = \Sigma\omega_1^2 = \Sigma\left(\frac{L}{A_1} + L\lambda\right)^2 (\cos\alpha)^2 = \Sigma v_1^2 + 2L^2\lambda\Sigma\frac{(\cos\alpha)^2}{A_1} + L^2\lambda^2$$

$$= v^2 + 2L^2\lambda\frac{M}{L^2} + L^2\lambda^2,$$

or $$\omega^2 - v^2 = \lambda L^2\left(\frac{2M}{L^2} + \lambda\right).$$

And again, in the case of supplemental reduction, using v_1, v_2, v_3, v for the partial and total angular velocities of the disk,

$$v_1 = -\frac{L}{A_1}\cos\alpha, \quad v_2 = -\frac{L}{B_1}\cos\beta, \quad v_3 = -\frac{L}{C_1}\cos\gamma,$$

$$\frac{1}{A'} = \lambda - \frac{1}{A}, \quad \frac{1}{B'} = \lambda - \frac{1}{B}, \quad \frac{1}{C'} = \lambda - \frac{1}{C},$$

$$\omega^2 = \Sigma\left(L\lambda - \frac{L}{A_1}\right)^2 (\cos\alpha)^2 = v^2 - 2L^2\lambda\frac{M}{L^2} + L^2\lambda^2,$$

or $$\omega^2 - v^2 = \lambda L^2\left(-\frac{2M}{L^2} + \lambda\right);$$

showing that in both cases alike the difference between the squared velocity of the body and that of either its representative disks is constant throughout the motion, as might also have been predicted *à priori* from the form of the elliptic function connecting the time with the squared velocity. In the case of disk motion there is a distinctive feature which is deserving of notice. In this case we have

$$A\omega_1^2 + B\omega_2^2 + (A + B)\,\omega_3^2 = M,$$

$$A^2\omega_1^2 + B^2\omega_2^2 + (A + B)^2\,\omega_3^2 = L^2.$$

Hence $$AB\,(\omega_1^2 + \omega_2^2) = (A + B)^2 M - L^2,$$

showing that the angular velocity with which the disk turns about a line in its own plane is constant throughout the motion, whilst the velocity about the axis perpendicular to its plane is continually varying, in the first particular agreeing with, and in the second differing from what takes place for a body of three dimensions with two of its principal moments of inertia equal.

It is easy to see how in the general case every conceivable motion of a body of any form may be tabulated and reduced to a table of treble entry, and how greatly the use of such tables may be facilitated, and seemingly distinct cases reduced to identity by aid of the twofold method of reduction above explained. Let us consider the case of a body whose principal moments of inertia are A, B, C, arranged in ascending order of magnitude.

We have seen that the quantity $\frac{M}{L^2}$ must always be intermediate between $\frac{1}{A}$ and $\frac{1}{C}$.

If the direct reduction be employed instead of $\frac{1}{A}, \frac{1}{B}, \frac{1}{C}, \frac{M}{L^2}$, we shall have

$$\frac{1}{A}-\lambda,\ \frac{1}{B}-\lambda,\ \frac{1}{C}-\lambda,\ \frac{M}{L^2}-\lambda, \text{ say } \frac{1}{A_1}, \frac{1}{B_1}, \frac{1}{C_1}, \frac{M_1}{L^2};$$

and if $\frac{M}{L^2}$ is intermediate between $\frac{1}{B}$ and $\frac{1}{C}$, $\frac{M_1}{L^2}$ will be intermediate between $\frac{1}{B_1}$ and $\frac{1}{C_1}$, where $C_1 = A_1 + B_1$.

On the other hand, if the supplemental method be employed,

$$\lambda-\frac{1}{A},\ \lambda-\frac{1}{B},\ \lambda-\frac{1}{C},\ \lambda-\frac{M}{L^2}, \text{ say } \frac{1}{C'}, \frac{1}{B'}, \frac{1}{A'}, \frac{M'}{L^2},$$

where $C' = A' + B'$ will take the place of $\frac{1}{A}, \frac{1}{B}, \frac{1}{C}$; so that if $\frac{M}{L^2}$ is intermediate between $\frac{1}{A}$ and $\frac{1}{B}$, $\frac{M'}{L^2}$ will be intermediate between $\frac{1}{B'}$ and $\frac{1}{C'}$.

Hence by using the direct method of reduction in the case where $\frac{L^2}{M}$ is greater than B, and the supplemental method of reduction where $\frac{L^2}{M}$ is less than B, the original body can be always replaced by a disk of which A_1, B_1, $A_1 + B_1$ are the new principal moments of inertia, L the given initial impulsive couple, M the new *vis viva*, and where the ascending order of the magnitudes is $\frac{1}{A+B}, \frac{M}{L^2}, \frac{1}{B}, \frac{1}{A}$, so that $\frac{BM}{L^2}, \frac{AM}{L^2}$ will be both of them less than unity. This reduction being effected when the motion of the disk is known, that of the associated body is given.

Calling the two parameters $\frac{BM}{L^2}, \frac{AM}{L^2}$, q_2, q_1* respectively, an inspection

* Calling A, B, C the original moments of inertia, it is important to notice that we have seen that no real distinction of motion arises from $\frac{M}{L^2}$ lying between $\frac{1}{A}$ and $\frac{1}{B}$ on the one hand, or between $\frac{1}{B}$ and $\frac{1}{C}$ on the other; the so-called two kinds of polhods and Legendre's primary

of the system of equations (1, 2, 3, 4) at pp. [588—590] will show that the angles $\alpha, \beta, \gamma, \xi, \eta, \zeta, \omega$ are known, and may be registered in a table when $\frac{M}{L}t$, q_1, q_2 are given, the time t being reckoned from some determinate epoch, which must be so fixed as to be identical for the disk and the associated body*. We may assume as such epoch indifferently the moment when the axis of the disk has its maximum, or when it has its minimum inclination to the invariable line, that is, when the quantity $(\cos\gamma)^2$ in the equations

$$\left.\begin{aligned}(\cos\alpha)^2 + (\cos\beta)^2 + (\cos\gamma)^2 = 1,\\ \frac{(\cos\alpha)^2}{q_1} + \frac{(\cos\beta)^2}{q_2} + \frac{(\cos\gamma)^2}{q_1+q_2} = 1,\end{aligned}\right\}*$$

attains its maximum or minimum value; the equations being linear between $(\cos\alpha)^2$, $(\cos\beta)^2$, $(\cos\gamma)^2$, say between x, y, z, the extreme values of z of course correspond to the zero values of x and y respectively.

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distinction of the problem into his cases (1) and (2) turn entirely upon this difference, but the two kinds of motion are convertible into one another (save as to the correction for the uniform displacement round the invariable line) by the theory of contra-relation. The real essential distinction of cases can only arise from particular values being assumed by q_1, q_2.

The quantities $0, q_1, q_2, 1, q_1+q_2$ are written in natural ascending order.

The two *singular* cases are (A) when $q_1=q_2$, which is the case of two equal moments, (B) when $q_2=1$, which is Legendre's *Troisième Cas, Cas très-remarquable*, Arts. 26, 27, corresponding to the instantaneous axis describing the so-called "separating polhod."

Besides these properly called singular cases, there are what may be termed special cases arising from sequences of two or three terms in the above quinary scale becoming approximately equal, or *subequal*, in Mr De Morgan's language, which relation may be denoted by the ordinary sign of equivalence.

Thus we shall have special cases when

$$q_1\equiv 0, \text{ or } q_2\equiv q_1, \text{ or } 1\equiv q_2, \text{ or } q_1+q_2\equiv 1,$$

and double-special cases when

$$q_2\equiv q_1\equiv 0, \quad 1\equiv q_2\equiv q_1, \quad q_1+q_2\equiv 1\equiv q_2.$$

The last of these is of course tantamount to $1\equiv q_2$ with $q_1\equiv 0$. But even this table does not exhaust all the specially notable cases; for in the first of the double-special cases which corresponds to that of a body differing little from a sphere, we may again mark off as extra-double special the case where $\frac{q_1}{q_2}\equiv 0$, and also that where $\frac{q_1}{q_2}\equiv 1$.

It does not fall in with the plan of this paper to investigate these several cases, but they are probably all of them deserving of particular examination.

* We may express the motion in terms of the parameters q_1, q_2 as follows, writing x, y, z for $\cos\alpha$, $\cos\beta$, $\cos\gamma$:

$$\left.\begin{aligned}x^2+y^2+z^2=1,\\ \frac{x^2}{q_1}+\frac{y^2}{q_2}+\frac{z^2}{q_1+q_2}=1,\end{aligned}\right\} \tag{1}$$

$$\frac{dz}{dt}=L\left(\frac{1}{A}-\frac{1}{B}\right)xy=\frac{M}{L}\left(\frac{1}{q_1}-\frac{1}{q_2}\right)xy, \tag{2}$$

$$d\zeta=L\frac{\frac{x^2}{A}+\frac{y^2}{B}}{1-z^2}dt=\frac{M}{L}\left(\frac{1-\frac{z^2}{q_1+q_2}}{1-z^2}\right)dt. \tag{3}$$

In using such tables of treble entry, we may suppose the initial angular velocities about the principal axes to be given, from which and the known moments of inertia the quantities L and M may be calculated, and then by the direct or supplemental method of reduction the value of λ and of the two parameters q_1, q_2 in the equivalent disk, each less than unity, found. 1st. If the reduction is direct—from the given inclination of the axis of the disk to the invariable line—the time t_0 from the epoch can be found by inspection, and then the entries corresponding to $t+t_0$ will give the inclinations at the end of the time t of the principal axes to the invariable line, and the position of the node defined as the intersection of the invariable plane with the plane through the invariable line and the axis of the disk (which axis coincides with a known one of the two extreme axes of the given body), and also the total angular velocity; the corresponding position of the node and value of the total angular velocity of the original body are then known by simple arithmetical computations from the theorems above given, involving λ only for the first, and λ, L, M for the second. 2nd. If the reduction is contrary or supplemental, we have only to substitute the supplemental angles of inclination to the invariable line in determining t_0, and proceed in all other respects as before, taking the supplements of the angles given in the tables in lieu of the angles themselves. In the special case of a body with two equal moments of inertia, were not the simplicity of the motion such as to render tabulation unnecessary, a distinct set of tables of double entry would of course be employed. It is, I think, conceivable that the supposed tables of treble entry might be of some practical value in

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Hence
$$\frac{M}{L}dt=\frac{dz}{\sqrt{(Z_1Z_2)}},$$
where
$$Z_1=\left(1-\frac{1}{q_2}\right)+z^2\left(\frac{1}{q_2}-\frac{1}{q_1+q_2}\right),$$
$$Z_2=\left(1-\frac{1}{q_1}\right)+z^2\left(\frac{1}{q_1}-\frac{1}{q_1+q_2}\right),$$
and
$$d\zeta=\frac{1}{q_1+q_2}\frac{dz}{\sqrt{(Z_1Z_2)}}+\left(1-\frac{1}{q_1+q_2}\right)\frac{dz}{(1-z^2)\sqrt{(Z_1Z_2)}}.$$
The limiting values of z correspond to $Z_1=0$, $Z_2=0$, or, which is the same thing, to the values of z when y and x are successively made zero in the equation (1).

It may be useful to the reader to be enabled to compare the above values of t and ζ in terms of z with the equivalent determination of Legendre, *Exerc. du Cal. Intég.* tome II. p. 334, namely
$$idt=\frac{d\psi}{\sqrt{\{1-c^2(\sin\psi)^2\}}},$$
$$d\phi=\frac{2\tan\mu}{1-m\sin\beta}\cdot\left(\frac{n+1}{2}\frac{d\psi}{\sqrt{\{1-c^2(\sin\psi)^2\}}}-\frac{d\psi}{\{1+(\tan\mu)^2(\sin\psi)^2\}\sqrt{\{1-c^2(\sin\psi)^2\}}}\right);$$
for this purpose it will be necessary to bear in mind that Legendre's A, B, C are not the moments of inertia themselves, but the elements out of whose binary combinations they are formed, and that his middle magnitude is not B but A; the reader will then find it necessary to trace the values of Legendre's i, W, ϵ, ψ, θ, β, m, n, μ, c by the formulæ and definitions given at pages 334, 319, 328, 315, 321, 322, 325, 319 *bis*, 333, and possibly some other which has disappeared from my notes of the Exercises, tome II.

studying by arithmetical or graphical methods the geological phenomenon of evagation of the pole of the earth regarded as a body of irregular form, and in other dynamical problems of a gyroscopical character where an exact determination of the effect of a given disturbing cause might be difficult or unattainable.

The fact that there are no essential differences in the motion of a rigid body of any form and started under any initial circumstances whatever, but such as depend upon the particular values of the two positive proper fractions q_1, q_2, enables us at once to see what are the special cases which alone can arise, and whether or no there is any real distinction to be made between the general cases of the theory. At first sight it would seem that four essential parameters enter into the question, the ratios of the initial values of the partial velocities ω_1, ω_2, ω_3, and the ratios of the constants $A : B : C$, the principal moments of inertia; but one parameter is saved by the substitution of an indefinitely flattened disk for a solid, and another by the introduction of an intrinsic epoch from which the time is reckoned, and thus a table of treble instead of quintuple entry is competent to represent every possible variety of conditions.

The problem that has been treated of in the foregoing pages is one (and possibly the simplest) instance of a well-defined class of dynamical questions subject to a peculiar method of treatment, which consists in the postponement of the determination of the absolute displacement of the moving system until after its displacement relative to a fixed line has been previously determined. The three problems which may be said to form a natural (not merely a historically connected) group, and which offer the most important illustrations of the class in question, are those of the rotation of a free body, of the motion of a particle attracted to two fixed centres of force, and the problem of three bodies. In the first and third of these, the invariable line is a line perpendicular to the invariable plane, determinable by composition of the momenta of the several elements of the system at any instant of time. In the second the invariable line is the line joining the fixed centres; and the distances of the moving point from the two fixed centres or the angles which they make with the line of centres may be expressed by equations complete within themselves, and into which the position of the plane containing the moveable point and the fixed line does not enter. So again in the problem of three bodies, without having recourse to the methods of deformation employed by Jacobi, and those who have followed in his track in treating the question, it is obvious, *à priori*, that one integral may be gained, in the sense of one less being required, by forming a system of equations from which the position of the intersection of the plane of the three bodies with the invariable plane is excluded, equivalent in effect to the so-called "elimination of the node" on Jacobi's method; in which, however, the node so called is not

to be confounded with the intersection above named, but is the mutual intersection of two ideal instantaneous orbits with each other and the invariable plane.

In every ordinary dynamical problem, by a well-known simple contrivance, the *time* element may be preliminarily thrown out of the differential equations of the motion; in the class of which the three noble and celebrated questions here referred to are the conspicuous types, a certain space element is capable of being similarly left out to the end; thus the number of linear differential equations required for the determination of the remaining elements is reduced by two, and if *all* the integrals of this reduced system are capable of being found, then we know, *à priori*, by the theory of the last multiplier, how to reduce to quadratures the values of the two outstanding elements. The process whereby the space coordinate referring to absolute position is, so to say, avoided in this class of dynamical questions, is not, or at least need not be considered as, one of elimination properly so called; elimination is the act of extruding a variable from a system of equations in which it has appeared; the process to be applied in the case before us is one not of extrusion, but of exclusion *ab initio*, or as it may be rendered in a single word, of ab-limination.

I propose at an early moment to return to a consideration of the particular method of ab-limination above indicated as applicable to the problem of three bodies, in the study of which this memoir took its rise.

94.

ON THE MOTION OF A RIGID BODY MOVING FREELY ABOUT A FIXED POINT.

[*Philosophical Proceedings of the Royal Society of London*, XV. (1866—1867), pp. 139—144.]

[Cf. p. 577.]

(*Abstract.*)

THE nature of the present brief memoir will be best conveyed by my giving a succinct account of the principal results which it embodies, in the order in which they occur. The direct solution, in its present form, of the important problem of the motion of a rigid body acted on by no external forces, originating in the admirable labours of Euler, has received the last degree of finish and completeness of which it is susceptible from the powerful analysis of Jacobi; in one sense, therefore, it may be said that the discussion is closed and the question at an end. Notwithstanding this, in the mode of conceiving and representing the general character of the motion, there are certain circumstances which merit attention, and which may be expressed without reference to the formulæ in which the analytical solution is contained.

Poinsot's method of representing the motion by means of his so-called "central ellipsoid" has passed into the every-day language of geometers, and may be assumed to be familiar to all. The centre of this ellipsoid is supposed to be stationary at the point round which any given solid body is turning; its form is determined when the principal moments of inertia of that body are given, and it is supposed accurately to roll without sliding on a fixed plane whose position depends on the initial circumstances of the motion. The associated free body is conceived as being carried along by the ellipsoid, so that its path in space, its continuous succession of changes of position, is thereby completely represented; but no image is thus presented to the mind of the time in which the change of position is effected. I show how this defect in the representation may be remedied, and the time, like the law of displacement, reduced to observation by a slight modification of the apparatus of the central ellipsoid or representative nucleus, as it will

for the moment be more convenient to call it. To steady the ideas, imagine the fixed invariable plane of contact with the nucleus to be horizontal and situated under it; now conceive a portion of its upper surface, say the upper half, to be pared away until it assumes the form of a semi-ellipsoid confocal to the original surface, and that an indefinitely rough plate always remaining horizontal, but capable of turning in its own plane round a vertical axis, which, if produced, would pass through the centre of the ellipsoid, is placed in contact with this upper portion; as the nucleus is made to roll with the under part of its surface upon the fixed plane below, the friction between the upper surface and the plate will cause the latter to rotate round its axis, for the nucleus will not only roll upon the plate above, but at the same time have a swinging motion round the vertical, which will be communicated to the plate. I prove, by an easy application of the theory of confocal ellipsoids, that the time of the free body passing from one position to another will be in a constant ratio to this motion of rotation, which may be measured off upon an absolutely fixed dial face immediately over the rotating-plate; and furthermore I show that the relation between the angular divisions of this dial and the time depends only upon the spinning force which may be supposed to set the free body originally in motion, so that it will hold the same, at whatever distance, by a preliminary adjustment, the rotating-plate may be supposed to be set from the fixed horizontal plane.

Thus, then, we may realize a complete *kinematical* image of all the circumstances of the motion of a free rotating body, and reduce to a purely mechanical measurement the determination of an element hitherto unrepresented, but in reality the most important of all, namely, the *time*.

I then proceed to point out a very singular and hitherto unnoticed *dynamical* relation between the free rotating body and the ellipsoidal top, as I shall now prefer to call Poinsot's central ellipsoid, because I imagine it set spinning like a top upon the invariable plane in contact with it and left to roll of its own accord, the friction between it and the plane being supposed adequate to prevent all sliding. I start with supposing that the density of the top follows any law whatever, and call its principal moments of inertia A, B, C, its semi-axes a, b, c, the relations between these six quantities being left arbitrary.

It is easy to establish that, if a rotating body be acted on by any forces which always meet the axis about which it is at any instant turning, the *vis viva* will remain unaffected by their action; this will be the case in the present instance with the pressure and friction of the invariable plane, the only forces concerned, as we may either leave gravity out of account altogether or suppose the centre of gravity of the top to be at the centre of the ellipsoid, which will come to the same thing. By aid of this principle, conjoined with the two conditions to which the angular velocities of the

associated free body are known to be subject, it is easy to infer that the velocities of this body and its representative top will, throughout the motion, remain in a constant ratio, or, if we please, equal to one another, provided that

$$A : B : C :: \frac{1}{a^2} + \frac{\lambda}{a^4} : \frac{1}{b^2} + \frac{\lambda}{b^4} : \frac{1}{c^2} + \frac{\lambda}{c^4},$$

where λ is an arbitrary constant. If, now, we revert to the natural supposition of the top being of uniform density, it is well known that

$$A : B : C :: b^2 + c^2 : c^2 + a^2 : a^2 + b^2,$$

and these ratios may be identified with those above (although this would not at the first blush be supposed to be the case) by giving a suitable value to the arbitrary constant λ.

Thus, then, Poinsot's central ellipsoid supposed of uniform density and set spinning upon a roughened invariable plane will represent the motion of a free rotating solid, not in space only but also in time; the body and the top may be conceived as continually moving round the same axis and at the same rate at each moment of time*.

The problem of the top is completed in the memoir by applying the general Eulerian equations to determine the friction and pressure, a process which involves some rather onerous but successfully executed algebraical calculations.

I next proceed to account analytically for the kinematical theory established at the outset of the memoir, and in doing so am necessarily led to give greater completeness to it, and at the same time an extension to the existing theory of confocal surfaces of the second order, by introducing the complementary notion of surfaces that I call contrafocal to one another: confocal ellipsoids are those the differences between the squares of whose corresponding principal axes are all three the same; contrafocal ellipsoids I define to be those the sums of the squares of whose corresponding axes are the same. Any two bodies whose central ellipsoids are either confocal or contrafocal I term related—correlated in the one case, contrarelated in the other, and I show that the kinematical construction in question is only another rendering of the first of the propositions herein subjoined concerning bodies so related.

1st. If two correlated bodies be placed with their principal axes respectively parallel and be set spinning by the same impulsive couple, they

* Accordingly, if we conceive any body as lying wholly in the interior of the ellipsoidal top, which is its kinematical exponent, such body will move precisely as if it were free, and consequently its density may be uniformly increased or diminished in any ratio, or it may be entirely removed without affecting the law of the motion of the surrounding crust in relation to space or time.

will move so that the corresponding axes of the one and the other body will continue always equally inclined to the axis of the couple, and their original parallelism at any instant may be restored by turning one of the bodies about this last-named axis through an angle proportional to the time elapsed since the commencement of the motion. Virtually, this amounts to saying that the difference between the displacements of two correlated bodies subject to the same initial impulse is equivalent to a simple uniform motion about the invariable line.

2nd. So, in like manner, if the bodies be contrarelated, the *sum* of their displacements is equivalent to a simple uniform motion about such line.

3rd. In either case alike, the difference between the squared angular velocities of the related bodies is constant throughout the motion.

From these propositions it follows that for all practical intents and purposes the motion of any body is sufficiently represented by the motion of any other one correlated or contrarelated to it. To a spectator on the invariable plane the apparent motion of one rotating body may be made identical with that of any other *related* one by merely making the plane on which he stands turn uniformly round a perpendicular axis. It becomes natural, then, to ascertain whether there is not always some one or more simplest form or forms which may be selected out of the whole couple of infinite series of related bodies, which may conveniently be adopted as the *exemplar* or type of all the rest. Obviously, the best suited for such purpose will be a body reduced to only two dimensions, in other words an indefinitely flattened disk, provided that it be possible in all cases to find a disk correlated or contrarelated to any given solid*.

The algebraical investigation for ascertaining the existence of such disk is the same whichever species of relation is made the subject of inquiry, and leads to the construction of three quadratic equations corresponding to the respective suppositions of the original body becoming indefinitely flattened in the direction of each of its three principal axes in turn; so that for a moment it might be supposed that the number of disks fulfilling the required condition could, according to circumstances, be zero, two, four, or six. But on closer examination, and bearing in mind that negative equally with imaginary moments of inertia are inadmissible, it turns out that there are always two such disks, and no more (except in the case of two of the moments of inertia being equal when the solution becomes unique). Of these two disks, one will be correlated and the other contrarelated to the given

* The peculiar feature in the absolute motion of a disk is, that whilst it is turning in its own plane with a variable velocity, the rate at which it turns about itself is constant, as will at once become evident from eliminating r between the two equations

$$Ap^2+Bq^2+(A+B)\,r^2=M,$$
$$A^2p^2+B^2q^2+(A+B)^2\,r^2=L^2.$$

body, and they will be respectively perpendicular to the axes of greatest and least moments of inertia. We have thus the choice between two methods of reduction to the type form, and this choice is not a matter of unimportance (in nature nothing exists in vain); for by means thereof the motion of any given body subject to any initial conditions can be made to depend upon either at will of the two comprehensive cases (Legendre's 1st and 3rd) to which the motion of a free rotating body is usually referred, so that the distinction between these two cases (corresponding to the two species of Polhodes on either side of the "Dividing Polhode," according to Poinsot's method of exposition) is virtually abrogated.

From the preceding theory, it follows (as also may be made to appear alike from an attentive synoptic view of the commonly received analytical formulæ as from Poinsot's theory of the associated "sliding and rolling cone") that in the problem of the motion of a free body, of whatever form and subject to whatever initial conditions one pleases, there enter but two arbitrary parameters. Calling A, B, $A+B$ the three moments of inertia of one or the other equivalent disks, L the magnitude of the impulsive couple, M the *vis viva*, these two parameters (say p, q) will be $\frac{AM}{L^2}$, $\frac{BM}{L^2}$; if to them we add a quantity $\frac{Mt}{L}$, say τ (where t is the time reckoned, as it may be, from an *intrinsic* epoch as explained in the memoir), all other elements of the motion, as the total or partial velocities and the angles, whatever they may be, selected to determine the position of the rotating body become known functions of these three quantities, p, q, τ, and may be reduced to tables of triple entry, or be graphically represented by a few charts of curves; and it should be noticed that p, the smaller of the two parameters p, q, will be always necessarily included between 0 and 1, and that the other parameter q may, by a due choice of the species of reduction adopted, be *forcibly retained* within the same limits. The five quantities 0, p, q, 1, $p+q$ will then form an ascending series of magnitudes subject only to the liability of the middle term q to become equal to 1 on the one hand, or to p on the other: q becoming unity corresponds to the case of the so-called "Dividing Polhode," Legendre's 2nd case ("*cas très-remarquable*"); and q becoming equal to p is of course the case of the body itself, or its "central ellipsoid," becoming a figure of revolution, in which case the motion is practically the same as that of a uniform circular plate.

Besides these two exceptional cases, the only singular cases properly so called, the quinary scale of magnitudes just exhibited serves to indicate all the more remarkable cases (requiring or inviting particular methods of treatment) which can present themselves in the theory. These may be distinguished into special cases, which arise from any two consecutive terms

becoming (to use Prof. De Morgan's expressive term) subequal, that is, differing from one another by a quantity whose square may be neglected, and double special cases, which arise when any three consecutive terms become subequal; all of which, together with peculiar subcases appertaining to the double special class, perhaps deserve more thorough examination than may have been hitherto accorded to them. I conclude the memoir with pointing out the place which this problem of Rotation appears to me to occupy in dynamical theory, as belonging to a natural and perfectly well defined group of questions, of which the motion of a body attracted to two fixed centres and the renowned problem of three bodies acted on by their mutual attractions are conspicuous instances. This group is characterised by the feature that, as regards them, equations of motion admit of being constructed, from which not only the element of time, as in ordinary mechanical problems, but also an element of absolute space is shut out; supposing the equations thus reduced by two in the number of the variables to have been integrated, Jacobi's theory of the last multiplier serves to reduce both the excluded elements to quadratures and thus to complete the solution. I notice that whilst the time may fairly be said to be eliminated, the space element may be more properly said to undergo the negative process, if it may be so called, of ab-limination; it is not introduced into and then expelled from, but prevented from ever making its appearance at all in the resolving system of differential equations. It is from the study of one of these allied but more difficult questions that the present memoir has taken its rise as a collateral inquiry and elucidatory digression.

95.

ON THE MULTIPLICATION OF PARTIAL DIFFERENTIAL OPERATORS.

[*Philosophical Magazine*, XXXIII. (1867), pp. 48—55.]

In the last Number of the *Magazine** I explained the sense in which I employ the term operator as distinguished from an operant, the distinction being somewhat analogous to the grammatical one between a verb and a noun; for as a combination of the predicate and copula gives rise to a verb which has independent laws of inflexion and regimen, so an operator is a new species of quantity which, springing from the union of an operant and the symbol of operation, becomes amenable to its own proper laws of functional action and subjection†. I found it convenient also to refer to an operator as an *energized operant*‡. At the outset of the paper a proposition was stated inadvertently, regarding any energized function of a set of variables and their corresponding elementary operators, in too general terms. Such function remaining unrestricted in regard to the principal letters $x, y, z, \ldots$ should have been limited to be a *linear* quantic in regard to the elementary operators $\delta_x, \delta_y, \delta_z, \ldots$. If ϕ be any such function, the proposition in question, thanks to the happy introduction of the star symbol, may without any auxiliary definition of the derivatives $\phi_2, \phi_3, \ldots$ employed in the preceding paper, be stated as follows, with perfect freedom from any shade of ambiguity,

$$e^{t\phi *} = \left[e^{(e^{t\phi *}-1)\phi}\right]*,$$

[* p. 567, above.]

† Thus an operator forms a new *part of speech* in algebra. It may be well to notice in this place, in order to prevent error arising hereafter, that the process of energization must in general be indicated, not by the mere apposition of an asterisk, but of brackets and asterisk. Thus, although P turned into an operator may be correctly designated by $P*$, $P*P$ similarly energized will be represented by $[P*P]*$, and not by $P*P*$.

Conversely, denergization will consist in the abstraction of an asterisk and brackets, and not of the former merely. Thus $P*P*$ denergized is not $P*P$ but P^2+P*P, because $P*P*$ is $[P^2+P*P]*$; whereas $P*P*$ *divided by* $*$, a term employed in the sequel in a footnote, is simply $P*P$, so that star division, or destellation as it may be termed, is not to be confounded with denergization.

‡ Or I might have used the word *vitalized* to convey the same idea,—the operator being the operant endued with power of action, but none the less for that capable of being acted upon, calling to mind the relation between dead and living matter. So denergization might be termed *amortization*, a word which exists in the language.

which theorem (t being an arbitrary parameter) contains the general rule for expanding $(\phi *)^n$ in terms of the quantities

$$[\phi * \phi] * ; \ [\phi * \phi * \phi] * ; \ [\phi * \phi * \phi * \phi] *, \ \&c.†$$

† Thus, for example, when

$$\phi = a\delta_b + 2b\delta_c + 3c\delta_d + \ldots,$$

the theorem in the text easily enables us to see that

$$e^{(e^{x\phi *} - 1)\phi} * F(a, b, c, d, \ldots) = F(a, b + ax, c + 2bx + ax^2, d + 3cx + 3bx^2 + ax^3, \ldots),$$

which, as remarked by Mr G. De Morgan and others at the Mathematical Society, may be regarded as a transformation and generalization of the fundamental law of development in Arbogast's theory, sometimes called by the name of Arbogast's first or unreduced method. The identification with the method in question merely requires the supposition that $F(a, b, c, d, \ldots)$ should become a function exclusively of a single one of the letters within the parentheses; but of course we must write the left-hand side of the equation under the unreduced form

$$e^{x\phi *} F(a, b, c, d \ldots).$$

The proof, as noticed by my distinguished mathematical friend Mr Samuel Roberts, of the generalized theorem is virtually implied in the method by which I established long ago the partial differential equations of the invariants to any system of forms; that is, it follows from the observation that the effect upon F of altering x into $x + \delta x$ leaving $a, b, c, \ldots$ unaltered is the same as the effect of leaving x unaltered and altering $b, c, d, \ldots$ into

$$b + a\delta x, \ c + 2b\delta x, \ d + 3c\delta x, \ \ldots.$$

Consequently
$$\frac{\delta F}{\delta x} = \phi * F, \ \frac{\delta^2 F}{\delta x^2} = (\phi *)^2 F, \ \ldots,$$

and therefore, by Maclaurin's theorem,

$$F(a, b + ax, c + 2bx + ax^2, \ldots) = e^{x\phi *} F(a, b, c, \ldots).$$

In memory of the author who appears to have been the first to employ the form which I have called a Protractant, it may hereafter with propriety be termed also an Arbogastiant.

The equivalence of $e^{x\phi *}$ with $[e^{(e^{x\phi *} - 1)\phi}] *$, when ϕ represents an Arbogastiant, or rather a form slightly more general, had been previously stated, but in a much less commodious manner, by Professor Cayley in a memoir contained in *Crelle's Journal*, Vol. XLVII. p. 110. An inspection of this memoir will satisfy the reader how inarticulate was the language of algebra at the not remote epoch when Mr Cayley's paper was written, and how, for want of a distinctive abstract symbol of operativeness, she strove like one lame of speech and tongue-tied, to give intelligible expression to her ideas.

With the *star* sign the restraining ligament has been cut, and henceforth algebra, as far as yet developed, may revel in unbounded freedom of utterance. The rise of this star above the mathematical horizon marks one of the epochs of algebra. It is worth remarking how already it is beginning in its turn to assume the attributes of quantity (*vide* the concluding footnote of this paper, where it is used as a divisor); so that apparently it is destined to run the same course as Newton's fluxional symbol, which is, and of fatal necessity must have been, superseded by the lettered symbols of Leibnitz, which have now long ago, to all intents and purposes, become converted into a new species of algebraical quantity. As soon as it becomes necessary (as will probably before long be the case) to express the specific relation of the star to something which limits and discriminates its mode of application, it must in its turn develope into a third species of symbolical quantity; and so there may be in store for the future of algebra an endless procession of more and more abstract symbols of operation, each successively developing into a more and more subtle species of quantity, suggesting the analogy of successive stages of so-called imponderability in the material world.

A propos of Arbogastiants, it is worthy of a passing notice that if I be any invariant to the form $(a, b, c, \ldots h, k)$, and we write A for the Arbogastiant $(l\delta_k + 2k\delta_h + \ldots)$, then $\frac{(A *)^n}{\Pi n}$ expresses

In like manner, the statement concerning the commutable operators $\phi *$ and $\psi *$, made in a footnote, should have been limited to the case where those two operants, ϕ, ψ, are each of them linear quantics in regard to $\delta_x, \delta_y, \delta_z, \ldots$. The proposition advanced guardedly in the Postscript concerning any lineo-linear functions of $x, y, z, \ldots \delta_x, \delta_y, \delta_z, \ldots$ ("there can be little or no doubt, &c.") I now also wish to be understood as affirming absolutely. I proceed to give a universal theorem for the multiplication of any number of operators, energized functions of $x, y, z, \ldots; \delta_x, \delta_y, \delta_z, \ldots$, freed from all restriction as to linearity of form in respect to the latter set.

The method by which I arrived at this very general theorem was in substance identical with that embodied in the demonstration spontaneously furnished me by my ever ready correspondent Professor Cayley; and as I cannot improve upon his statement, it would be a waste of time to substitute my own words for his. Accordingly, after enunciating the theorem, I shall give the proof of it in the very words of our unrivalled Cambridge Professor, from which it will be seen that in essence this theorem consists in applying the symbolical form of Taylor's theorem to the expansion which, in itself symbolical, contains the generalization of Leibnitz's theorem, thus giving rise to a *symbolism of the second order*, a phenomenon which, it is believed, here for the first time makes its appearance in analysis.

Let $\phi_1, \phi_2, \phi_3, \ldots \phi_r$ be any functions of $x, y, z, \ldots; \delta_x, \delta_y, \delta_z \ldots$, capable of being developed in a series of integer powers of the latter set of variables, where it is of course understood that

$$\delta_x = \frac{d}{dx}, \quad \delta_y = \frac{d}{dy}, \quad \delta_z = \frac{d}{dz}, \ldots;$$

in like manner let

$$\delta'_x = \frac{d}{d\delta_x}, \quad \delta'_y = \frac{d}{d\delta_y}, \quad \delta'_z = \frac{d}{d\delta_z}, \ldots,$$

---

the effect of the substitution $\begin{bmatrix} b, & c, & \ldots & k, & l \\ a, & b, & \ldots & h, & k \end{bmatrix}$ performed upon I. This theorem is an easy consequence of the conjunction of the three circumstances, (1) that if $I_{x,y}$ is what I becomes when for $a, b, c, \ldots k$ we substitute respectively $ax+by$, $bx+cy$, $\ldots kx+ly$, $I_{x,y}$ will be a covariant to the form $(a, b, c, \ldots h, k, l)$, and that consequently the last coefficient in $I_{x,y}$ will be $\frac{(A*)^n}{\Pi n} I$; (2) that this coefficient must bear the same relation to $l, k, \ldots c, b$ as the first does to $a, b, c, \ldots k$; and (3) that an invariant to the form $(l, k, \ldots c, b)$ is identical with the same invariant to the form $(b, c, \ldots k, l)$.

I think I have been informed that Leibnitz was the first to employ the method of the so-called separation of symbols: in his tract on the 'Calculus of Differences,' the poet sage of Collingwood contributed powerfully to its further development; if he should chance to cast his eyes over these pages he will, I fear, stand aghast at the Frankenstein he has thus (it may be unwittingly) played no unimportant part in bringing into existence; or, rather, I should fear, did not all the world know his perfect candour and unstinted sympathy with every form of manifestation of human intelligence.

so that in fact $\delta'_x, \delta'_y, \delta'_z, \ldots$ are abbreviated expressions for $\delta_{\delta_x}, \delta_{\delta_y}, \delta_{\delta_z}, \ldots$ or, if we please so to say, for

$$\frac{d}{d\frac{d}{dx}}, \quad \frac{d}{d\frac{d}{dy}}, \quad \frac{d}{d\frac{d}{dz}}.$$

Let $\delta_{x,i}$, $\delta'_{x,i}$ signify the operants δ_x, δ'_x *restricted* to operate exclusively on ϕ_i; finally, let

$$\Delta_{i,j} = \delta'_{x,i} \,.\, \delta_{x,j} + \delta'_{y,i} \,.\, \delta_{y,j} + \delta'_{z,i} \,.\, \delta_{z,j} + \ldots;$$

then giving to i, j all possible values subject to the inequalities $i < j, j < n+1$, the following equation is true,

$$\phi_1 * \phi_2 * \phi_3 * \ldots \phi_n * = [e^{\Sigma \Delta_{i\,j}} \phi_1 \phi_2 \phi_3 \ldots \phi_n] *.$$

What follows within inverted commas is from Mr Cayley's pen.

"Write $$\xi = \delta_x, \quad \eta = \delta_y,$$
$$A = (x, y)^a (\xi, \eta)^\alpha,$$

namely, A, any function of degrees a, α; and so

$$B = (x, y)^b (\xi, \eta)^\beta, \text{ \&c.},$$

and $$A_{12} = (x_1, y_1)^a (\xi_2, \eta_2)^\alpha, \text{ \&c.},$$

but all suffixes are to be ultimately rejected. Then

$$\begin{aligned} B*A* &= (x_2, y_2)^b (\xi + \xi_1, \eta + \eta_1)^\beta (x_1, y_1)^a (\xi, \eta)^\alpha * \\ &= e^{\xi \delta \xi_1 + \eta \delta \eta_1} (x_2, y_2)^b (\xi_1, \eta_1)^\beta (x_1, y_1)^a (\xi, \eta)^\alpha * \\ &= e^{\Delta_{01}} B_{21} A_{10} * \text{ if } \Delta_{01} = \xi \delta \xi_1 + \eta \delta \eta_1. \end{aligned}$$

Similarly,

$$\begin{aligned} C*B*A* = (x_3, y_3)^c (\xi + \xi_1 + \xi_2, \eta + \eta_1 + \eta_2)^\gamma (x_2, y_2)^b (\xi + \xi_1, \eta + \eta_1)^\beta \\ (x_1, y_1)^a (\xi, \eta)^\alpha * \\ = e^{(\xi_1 + \xi) \delta \xi_2 + (\eta + \eta_1) \delta \eta_2} (x_3, y_3)^c (\xi_2, \eta_2)^\gamma \ldots\ldots\ldots \\ = e^{\Delta_{12} + \Delta_{02}} C_{32} e^{\Delta_{01}} B_{21} A_{10} * \\ = e^{\Delta_{12} + \Delta_{02} + \Delta_{01}} C_{32} B_{21} A_{10}, \text{ and so on.} \end{aligned}$$

This seems the easiest proof of your general theorem."

The reader of sufficient intelligence to understand the theorem itself will have no difficulty in supplying the few missing links between my statement and the above demonstration of it. I will content myself with appending a single example to illustrate its meaning and mode of application.

Let P be any *lineo-linear* function of $x, y, z, \ldots$; $\delta_x, \delta_y, \delta_z, \ldots$; and in general let

$$(P*)^{n-1} P = P_n.$$

Let it be proposed to expand

$$P^i * P^j *.$$

Here, calling
$$P^i = \phi_1, \quad P^j = \phi_2,$$
$$\Delta_{1,2} = \delta'_{x,1}\delta_{x,2} + \delta'_{y,1}\delta_{y,2} + \ldots,$$
it is easily seen† that
$$\begin{aligned}\Delta_{1,2}(\phi_1\phi_2) &= i\,.\,j P^{i-1}\,.\,P^{j-1}\,.\,P{*}P \\ &= ij P^{i+j-2}\,.\,P_2, \\ (\Delta_{1,2})^2(\phi_1\phi_2) &= i(i-1)P^{i-2}\,.\,j(j-1)P^{j-2}\,.\,P_2^2,\end{aligned}$$
....

Hence
$$P^i{*}P^j{*} = P^{i+j}{*} + ijP^{i+j-2}\,.\,P_2{*} + \frac{ij\,(i-1)\,(j-1)}{1\,.\,2}P^{i+j-4}\,.\,P_2^2{*} + \&c.,$$
agreeably to the theorem given for protractors, and stated subsequently to hold good for pertractors in the previous paper, P_2 here denoting what was called $2P_2$ in the passages referred to.

This theorem, it should be observed, remains true when P, remaining a linear quantic in δ_x, δ_y, δ_z, ... is any function whatever of x, y, z.

Let us agree to employ (i), (j) as *umbræ*, such that $(i)^n$, $(j)^n$ shall denote the factorial *quantities*
$$i\,(i-1)\ldots(i-n+1)\,; \quad j\,(j-1)\ldots(j-n+1)$$
for all values of n; then we may express the above theorem under the subjoined condensed form, which will be useful for the better understanding of the sequel,
$$P^i{*}P^j{*} = [e^{\frac{(i)(j)\,P*P}{P^2}}\,.\,P^{i+j}]{*}.$$

Suppose now that we wish to obtain the product of three factors,
$$(P{*})^i, \quad (P{*})^j, \quad (P{*})^k.$$
Call $e^{\Delta_{1,2}+\Delta_{1,3}+\Delta_{2,3}}$, for the sake of brevity, E. The first term in the expansion of E is unity. The second is $\Delta_{1,2} + \Delta_{1,3} + \Delta_{2,3}$, which, applied to $P^i\,.\,P^j\,.\,P^k$, gives
$$\{ij + ik + jk\}\,P^{i+j+k-2}\,.\,(P{*}P).$$
The third term is
$$\tfrac{1}{2}(\Delta^2_{1,2} + \Delta^2_{1,3} + \Delta^2_{2,3} + 2\Delta_{1,2}\,.\,\Delta_{1,3} + 2\Delta_{1,3}\,.\,\Delta_{2,3} + 2\Delta_{1,2}\,.\,\Delta_{2,3})\,;$$
the effect of the application of the first three quantities within the above parentheses is to introduce terms whose sum is
$$\tfrac{1}{2}\{(i^2-i)(j^2-j) + (i^2-i)(k^2-k) + (j^2-j)(k^2-k)\}\,P^{i+j+k-4}(P{*}P)^2\,;$$

† Thus, for example, to fix the ideas, observe that
$$\Sigma\,(\delta'_{x,1}\,.\,\delta_{x,2})\,(ax\delta_y + by\delta_z)\,(cy\delta_z + dz\delta_t) = ax\,.\,c\delta_z + by\,.\,d\delta_t = (ax\delta_y + by\delta_z) * (cy\delta_z + dz\delta_t).$$
So again,
$$\begin{aligned}(\delta'_{y,1}\,.\,\delta_{y,2})^2\,(x\delta_y)^i\,(y\delta_z)^j &= i\,(i-1)\,.\,j\,(j-1)\,.\,(x\delta_y)^{i-2}\,(y\delta_z)^{j-2}\,.\,x^2\,(\delta_z)^2 \\ &= i\,(i-1)\,.\,j\,(j-1)\,(x\delta_y)^{i-2}\,(y\delta_z)^{j-2}\,(x\delta_y * y\delta_z)^2.\end{aligned}$$

the effect of the fourth and fifth quantities is to introduce terms whose sum is

$$\{(i^2 - i)jk + ij(k^2 - k)\} P^{i+j+k-4} \cdot (P{*}P)^2;$$

and the effect of the sixth term is to introduce the terms

$$ik(j^2 - j) P^{i+j+k-4} (P{*})^2 + ijk \cdot P^{i+j+k-3} \cdot P{*}P{*}P;$$

giving altogether for the complete sum

$$[\tfrac{1}{2}\{(i)(j) + (i)(k) + (j)(k)\}^2 P^{i+j+k-4} P_2^{\,2} + ijk P^{i+j+k-3} \cdot P_3 + \ldots]{*}.$$

And in general the effect of the term $(\Delta_{1,2} + \Delta_{2,3} + \Delta_{1,3})^r$ in the expansion of E will be to introduce terms containing all the quantities of the form

$$\frac{(P{*}P)^\beta}{P^{2\beta}} \cdot \frac{(P{*}P{*}P)^\gamma}{P^{3\gamma}} \cdot P^{i+j+k}$$

that can be got consistent with the satisfaction of the equation in integers $2\beta + 3\gamma = r$. The upshot of the calculation is that

$$P^i{*}P^j{*}P^k{*} = \{e^{\Sigma\left\{\frac{(i)(j)P*P}{P^2} + \frac{(i)(j)(k)P*P*P}{P^3}\right\}} P^{i+j+k}\}{*};$$

where it is of course to be understood that (i), (j), (k) are mere *umbræ*, subject to the law above stated for conversion of their powers into factorials of actual quantities.

The law for any number of factors is now obvious, and may be extended to the case where the factors are powers, not of one single operant P, but of different operants P, Q, R, S, *subject to the sole condition of their being linear quantics in regard of* δ_x, δ_y, δ_z, ...; and it will be found that†

$$P^i{*}Q^j{*}R^k{*}S^l{*} \ldots$$
$$= \{e^{\Sigma\left(\frac{(i)P*(j)Q}{PQ} + \frac{(i)P*(j)Q*(k)R}{PQR} + \frac{(i)P*(j)Q*(k)R*(l)S}{PQRS} \ldots\right)} \cdot P^i Q^j R^k S^l \ldots\}{*},$$

† Thus, for the particular case when $P = Q = R\ldots$, we have

$$P^{i_1}{*}P^{i_2}{*}P^{i_3} \ldots P^{i_\omega}{*} = \{(e^{\frac{\Omega}{P*}P})\, P^{\Sigma i}\}{*},$$

where $$\Omega = \{P + (i_1)P{*}\}\{P + (i_2)P{*}\} \ldots \{P + (i_\omega)P{*}\} - P^\omega - \Sigma i P^{\omega-1} \cdot P{*}.$$

Observe that if we convene to understand by

$$\frac{A*}{L} \cdot \frac{B*}{M} \cdots \frac{C}{N} \div {*},$$

the expression $$\frac{A{*}B{*} \ldots {*}C}{LM \ldots N},$$

then if we write $$\frac{(i)P*}{P} = p; \quad \frac{(j)Q*}{Q} = q; \ldots$$

the above theorem takes the form

$$P^i{*}Q^j{*}R^k{*} \ldots = [e^{\Sigma\frac{pq+pqr+\ldots}{*}} P^i \cdot Q^j \cdot R^k \ldots]{*}.$$

Now suppose
$$\phi = P_\alpha^{\,i} \cdot P_\beta^{\,i'} \cdot P_\gamma^{\,i''} \ldots,$$
$$\psi = Q^j \cdot Q^{j'} \ldots,$$
$$\vartheta = R_\alpha^{\,k} \ldots$$

....

....

where in the summation which gives the exponent of e, it is to be understood that the natural order of P, Q, R, S in the numerators is to be maintained.

The formula from which this result has been thus simply derived is of that fundamental character which entitles it to be regarded as a master theorem, that is, rather as a method than an ordinary formula. As already observed, it essentially consists in the union of two known theorems; but these combined and, as it were, duly adjusted and focussed, constitute together an instrument of research as unlike either of its separate elements as a telescope differs in its powers and functions from the pair of lenses out of which it has been formed. And truly the formula in question has a telescopic power in the sense of bringing the remote results of calculation close up to the mental vision.

The very first application made of this instrument, directed to the algebraical firmament, has been rewarded by the discovery of the beautifully simple and genera' expansion given in the text above—a result in beauty and the feeling of wonder it awakens fairly to be paralleled with the spectacle which gladdened the eyes of Galileo when for the first time he pointed *his* telescope to the skies.

---

and write

$$s_1 = \frac{(i)\,P_{\alpha*}}{P_\alpha} + \frac{(i')\,P_{\beta*}}{P_\beta} + \dots$$
$$s_2 = \frac{j)\,Q_{\alpha*}}{Q_\alpha} + \frac{)\,Q_{\beta*}}{Q_\beta} + \dots$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$

then I think there can be little doubt, or, at all events, there is a strong presumption that the following ultra-general theorem holds good :—

$$\phi * \psi * \vartheta * = [e^{\Sigma \frac{s_1 s_2 + s_1 s_2 s_3 + \dots}{*}} \,.\, \phi\psi\vartheta \dots]*.$$

If we suppose all the P's *inter se*, all the Q's *inter se*, &c. to coincide, the above expansion is certainly true, as may be inferred from the expansion proved in the text, conjoined with the known theorem in factorials, that $\{(i)+(j)+\dots\}^n$ is identical with what $(i+j+\dots)^n$ becomes when, in the development of the latter expression for any power of any element, we substitute the corresponding factorial product, that is, when in it for i^q, j^q, ... we substitute $(i)^q$, $(j)^q$....

Even if on examination the above equation should turn out not to be exact, the mere statement of it will be useful in indicating the kind of expression that is applicable. According to the conservative maxim that my universally lamented friend the late Mr Buckle used to be fond of citing, in science even a wrong rule is preferable to anarchy and confusion.

96.

THOUGHTS ON INVERSE ORTHOGONAL MATRICES, SIMULTANEOUS SIGN-SUCCESSIONS, AND TESSELLATED PAVEMENTS IN TWO OR MORE COLOURS, WITH APPLICATIONS TO NEWTON'S RULE, ORNAMENTAL TILE-WORK, AND THE THEORY OF NUMBERS.

[*Philosophical Magazine,* XXXIV. (1867), pp. 461—475.]

PART I.—*Matrices and Sign-successions.*

1. A SELF-RECIPROCAL matrix may be defined as a square array of elements of which each is proportional to its first minor. When the condition is superadded that the sum of the squares of the terms in each row or in each column, or (which comes to the same) that the complete determinant shall be equal to unity, it becomes *strictly* orthogonal; but, by an allowable extension of language, any self-reciprocal matrix may be termed orthogonal when the epithet of *strictness* is withdrawn. The general notion is that of homographic relation between each element and its first minor, that is, the relation $a + bx + c\xi + dx\xi = 0$ between the corresponding terms x and ξ of the matrix and its reciprocal. When $a = 0$ and $d = 0$, we have the case of orthogonalism as above defined*. When $b = 0$ and $c = 0$, so that each term in either matrix is in the inverse ratio of its first minor, we fall upon what I call the case of inverse orthogonalism.

This conception will be found to present itself naturally in the course of certain investigations connected with the calculus of sign-progressions suggested by the form of Newton's rule; and that calculus in its turn leads to a theory of tessellation highly curious in itself, and fruitful of consequences to the calculus of operations and the theory of numbers, furnishing interesting food for thought, or a substitute for the want of it, alike to the analyst at his desk and the fine lady in her boudoir.

* For a matrix of the order 2 the ratio of each element to its reciprocal in an orthogonal matrix is necessarily ± 1. This is a case of exception, and may be disregarded. In all other cases the ratio can be varied *ad libitum.*

2. In a strictly orthogonal matrix the n^2-1 equations resulting from the equal ratios above referred to, on account of the implications existing between them, really amount to no more than $\frac{n^2+n}{2}$ independent conditions, leaving $\frac{n^2-n}{2}$ of the n^2 terms arbitrary. This law, which it would perhaps not be easy to obtain from a direct inspection of the equations, is an instantaneous consequence of the fact that a sum of the squares of n variables may be transformed into a sum of squares of n linear functions of the same by means of an orthogonal substitution,—and that, *vice versâ*, such faculty of transformation is sufficient to establish the character of orthogonalism in the matrix of substitution employed. Consequently the number of conditions to be satisfied is the number of terms in a homogeneous quadratic function of n variables, which is $\frac{n.(n+1)}{2}$. In an *orthogonal* matrix (not *strictly* so) the number of implications is consequently $\frac{(n+2)(n-1)}{2}$.

3. The problem of constructing an inverse orthogonal matrix of any order admits of a general and complete solution. It is to be understood in what follows, that the constant product of any term by its first minor is not to be zero; or, in other terms, the complete determinant of the matrix which is a sum of such products is not to vanish.

First, let us investigate the number of arbitrary elements which enter into any such matrix.

To fix the ideas, consider one of the third order, say

$$\begin{vmatrix} a, & b, & c \\ \alpha, & \beta, & \gamma \\ A, & B, & C \end{vmatrix}$$

and call the reciprocal matrix formed by its first minors

$$\begin{vmatrix} a_1, & b_1, & c_1 \\ \alpha_1, & \beta_1, & \gamma_1 \\ A_1, & B_1, & C_1 \end{vmatrix}.$$

Then
$$\begin{aligned} &aa_1 = bb_1 = cc_1 \\ &= \alpha\alpha_1 = \beta\beta_1 = \gamma\gamma_1 \\ &= AA_1 = BB_1 = CC_1. \end{aligned}$$

These 8 equations are not independent; for we have

$$\begin{aligned} &aa_1 + bb_1 + cc_1 = aa_1 + \alpha\alpha_1 + AA_1 \\ &= \alpha\alpha_1 + \beta\beta_1 + \gamma\gamma_1 = bb_1 + \beta\beta_1 + BB_1 \\ &= AA_1 + BB_1 + CC_1 = cc_1 + \gamma\gamma_1 + CC_1; \end{aligned}$$

which 5 equations in their turn again are not independent, because the sum of the three groups written under one another on the left is equal to the corresponding sum on the right.

Hence we have implication upon implication, so that the number of independent equations is

$$(3^2-1)-(2.3-1)+1=(3-1)^2;$$

and so in general for a matrix of the order n, the number of independent equations is $(n-1)^2$, leaving $2n-1$ of the elements arbitrary.

4. This result is easily verified. For, reverting to the example of the third order, if any inverse orthogonal matrix of that order is multiplied, term to term, by the following one,

$$\begin{vmatrix} l\lambda, & l\mu, & l\nu \\ m\lambda, & m\mu, & m\nu \\ n\lambda, & n\mu, & n\nu \end{vmatrix},$$

the product so formed will evidently retain its character unaltered, since each of the equal products will receive a constant multiplier, $lmn \,.\, \lambda\mu\nu$.

The number of independent quantities thus introduced is 5, namely,

$$l\lambda;\ \frac{m}{l},\ \frac{n}{l};\ \frac{\mu}{\lambda},\ \frac{\nu}{\lambda};$$

and so in the general case we can introduce $(2n-1)$ arbitrary elements. Thus, then, we may without any loss of generality regard only those matrices of the kind in question which are bordered horizontally and vertically by a line of positive units. From these *reduced* forms it is easy to pass to the general forms by term-to-term multiplication with a matrix of the kind above denoted. The question now becomes narrowed to that of determining the number and form of the reduced inverse orthogonal matrices of any given order n,—a problem (if attacked by a direct method) involving the solution of $(n-1)^2$ equations between $(n-1)^2$ unknown quantities.

5. (1) Let n be a prime number. Write down the line of terms

$$1,\quad a,\quad a^2,\ \dots\ a^{n-1},$$

and make a equal in succession to each of the $(n-1)$ roots of $\dfrac{x^n-1}{x-1}=0$. The matrix so formed will be a reduced inverse orthogonal matrix of the nth order.

In the case of $n=3$, it is easy and will be instructive to verify this statement. Calling the required matrix

$$\begin{vmatrix} 1, & 1, & 1 \\ 1, & a, & b \\ 1, & c, & d \end{vmatrix},$$

we obtain the four equations

$$ad - bc = d(a-1) = c(1-b) = a(d-1) = b(1-c),$$

which are equivalent to the following,

$$ad = c = b, \quad bc = d = a.$$

Hence $$a^2d^2 = bc, \text{ or } a^4 = a.$$

Hence rejecting the values $a = 0$ and $a = 1$, either of which would cause the constant product to become zero, we have the two solutions,

$$\begin{aligned}&(1) \quad a = \rho, \quad d = \rho, \quad b = \rho^2, \quad c = \rho^2,\\ &(2) \quad a = \rho^2, \quad d = \rho^2, \quad b = \rho, \quad c = \rho.\end{aligned}$$

There is thus but *one* single type of matrix of this order, namely

$$\begin{vmatrix} 1, & 1, & 1 \\ 1, & \rho, & \rho^2 \\ 1, & \rho^2, & \rho \end{vmatrix}.$$

(2) In like manner, for any prime number n there will be but a single type of matrix, the interior *nucleus* of which is a square matrix of the order $(n-1)$ made up of lines or columns of terms in which each line or column contains the $(n-1)$ powers taken in definite order of the $(n-1)$ prime roots of unity. That such a matrix is inversely orthogonal is not difficult of proof; but it is less easy to establish, what I have scarcely a doubt is true (but which I have not yet attempted to demonstrate), that such matrix, when its lines and columns are permuted in every possible manner, contains the complete solution of the corresponding system of $(n-1)^2$ equations. The number of distinct systems or roots satisfying these equations will be the number of distinct forms which can be obtained by permuting the lines and columns—in a word, the number of distinct *derivatives* (a word it will be found hereafter useful to employ) of any given phase of the nucleus. This number will be easily seen to be

$$(n-1).(n-2)^2.(n-3)^2 \ldots 1^2;$$

for each derivative, when all the permutations are taken of the lines and of the columns, will appear n times repeated. For instance, if ρ be a prime fifth root of unity so that

$$\begin{vmatrix} \rho, & \rho^2, & \rho^3, & \rho^4 \\ \rho^2, & \rho^4, & \rho, & \rho^3 \\ \rho^3, & \rho, & \rho^4, & \rho^2 \\ \rho^4, & \rho^3, & \rho^2, & \rho \end{vmatrix}$$

is the nucleus, if we take

the columns in the order			1, 2, 3, 4,	rows in the order		1, 2, 3, 4,
or	„	„	3, 1, 4, 2	„	„	2, 4, 1, 3,
or	„	„	2, 1, 4, 3	„	„	3, 1, 4, 2,
or	„	„	4, 3, 2, 1	„	„	4, 3, 2, 1,

the resulting derivative is in each case the same. Thus, then, when n is a prime number, the system of $(n-1)^2$ equations which give the terms of the nucleus admits of $\Pi(n-1).\Pi(n-2)$ systems of roots.

It will be seen that this law does not hold when n is a composite number, the rule for which I now proceed to state.

6. (1) I observe that there *will be as many distinct types of solutions as there are distinct modes of breaking up n into factors**.

(2) Let $n = p.q.r\ldots$ be one of the decompositions in question. Write down the disjunctive product

$$(1, a, a^2, \ldots a^{p-1} \between 1, b, b^2, \ldots b^{q-1} \between 1, c, c^2, \ldots c^{r-1} \between \ldots$$

in which the terms are to follow any fixed law of succession. This will produce a line containing $p.q.r\ldots$, that is, n terms.

Let $a, b, c, \ldots$ respectively represent the pth, qth, rth, ... roots of unity; by giving to each of these quantities successively its $p, q, r, \ldots$ values we shall obtain $p.q.r\ldots$, that is, n lines, constituting a matrix of the nth order; the totality of the matrices so formed contain between them the complete solution of the $(n-1)^2$ system of equations.

As an example let $n = 4$.

Here there are two modes of decomposition, namely,

$$n = 4, \quad n = 2.2.$$

Let i, i' denote the two primitive fourth roots of unity, and denote negative unity by $\bar{1}$. The two types will be

$$\begin{vmatrix} 1, & 1, & 1, & 1 \\ 1, & i, & \bar{1}, & i' \\ 1, & \bar{1}, & 1, & \bar{1} \\ 1, & i', & \bar{1}, & i \end{vmatrix} \text{ and } \begin{vmatrix} 1, & 1, & 1, & 1 \\ 1, & \bar{1}, & 1, & \bar{1} \\ 1, & 1, & \bar{1}, & \bar{1} \\ 1, & \bar{1}, & \bar{1}, & 1 \end{vmatrix}.$$

The number of distinct derivatives of the nucleus of the first of these types is $\frac{(1.2.3)^2}{2}$, that is, 18, the divisor 2 originating in the symmetry of the square in respect to its diagonals.

The number of distinct derivatives of the second type, which contains a higher capacity of symmetry than the former (that is, a symmetry persistent under certain permutations of its constituent lines or columns), is 6.

The following Table, in which $+ -$ are substituted for $1, \bar{1}$, will make this evident.

* When n is the νth power of a prime, the number of decompositions becomes the number of *indefinite* partitions of ν.

Phases of nucleus to type 2 . 2 :—

$$
\begin{array}{|ccc|ccc|ccc|}
\hline
- & + & - & + & - & - & - & - & + \\
+ & - & - & - & - & + & - & + & - \\
- & - & + & - & + & - & + & - & - \\
\hline
+ & - & - & - & + & - & - & - & + \\
- & + & - & - & - & + & + & - & - \\
- & - & + & + & - & - & - & + & - \\
\hline
\end{array}
$$

Phases of nucleus to type 4 :—

$$
\begin{array}{|ccc|ccc|ccc|}
\hline
i & - & i' & - & i' & i & i' & i & - \\
- & + & - & + & - & - & - & - & + \\
i' & - & i & - & i & i' & i & i' & - \\
\hline
- & i & i' & i' & - & i & i & i' & - \\
+ & - & - & - & + & i & - & - & + \\
- & i' & i & i & - & i' & i' & i & - \\
\hline
\end{array}
\qquad
\begin{array}{|ccc|ccc|ccc|}
\hline
- & + & - & + & - & - & - & - & + \\
i & - & i' & - & i' & i & i' & i & - \\
i' & - & i & - & i & i' & i & i' & - \\
\hline
+ & - & - & - & + & i & - & - & + \\
- & i & i' & i' & - & i & i & i' & - \\
- & i' & i & i & - & i' & i' & i & - \\
\hline
\end{array}
$$

$$
\begin{array}{|ccc|ccc|ccc|}
\hline
i & - & i' & - & i' & i & i' & i & - \\
i' & - & i & - & i & i' & i & i' & - \\
- & + & - & + & - & - & - & - & + \\
\hline
- & i & i' & i' & - & i & i & i' & - \\
- & i' & i & i & - & i' & i' & i & - \\
+ & - & - & - & + & - & - & - & + \\
\hline
\end{array}
$$

Thus, then, the total number of distinct solutions of our $(4-1)^2$, that is, 9, algebraical equations applicable to this case is $18+6$, or 24. The formula $\Pi(n-1).\Pi(n-2)$ would give only 12. How it should happen that the order of the system of equations for different values of n is not an algebraical, but a transcendental function of n depending on the factors of which n is made up, will become less surprising when it is considered that the quantities equated to zero in any such system, although algebraical in themselves, are not analytical but *tactical* functions of n their degree.

7. It remains to assign the value of the constant product in the reduced form of matrix of the order n, or, which comes to the same thing, the value of the complete determinant of such matrix, which is obviously n times the former quantity.

(1) When n is undecomposed, the value of this determinant, by virtue of a well-known theorem given years ago by Professor Cayley, for expressing the discriminant of an algebraical function as a determinant composed of powers of its roots, is easily recognized to be $i^{(n-1)(n-2)} n^{\frac{n}{2}}$, which we may call Δ_n.

(2) When n is decomposed under the form $pqr\ldots$, the corresponding determinant may easily be proved equal to

$$\Delta_p{}^{qr\ldots} . \Delta_q{}^{pr\ldots} . \Delta_r{}^{pq\ldots} \ldots.$$

Hence the determinant in the general case is

$$(-)^\phi p^{\frac{pqr\ldots}{2}} . q^{\frac{qpr\ldots}{2}} . r^{\frac{rpq\ldots}{2}} \ldots = (-)^\phi n^{\frac{n}{2}},$$

where
$$\phi = n\Sigma \frac{(p-1)(p-2)}{p}.$$

Thus, if each term in any reduced inverse orthogonal matrix of the order n be divided by the square root of n, the fourth power of the resulting determinant is unity for all the types without distinction. If n is decomposed into μ equal factors p, $\phi = \mu(p-1)(p-2)p^{\mu-1}$; so that when $\mu > 1$, the determinant is $\pm i$ if $\mu \equiv 1$ [mod 2], and $p \equiv -1$ [mod 4], and is ± 1 in all other cases. When $\mu = 1$, its value is $(\pm i)$ if $p \equiv -1$ or 0 [mod 4], and ± 1 in the other two cases. When n is undecomposed, the value of the constant product, which is $\frac{1}{n}$ of the determinant, takes the simple form $(i^{n-1} n)^{\frac{n-2}{2}}$.

8. When n is a power of 2, the type corresponding to its decomposition into the equal factors 2 deserves especial consideration. In this type the only roots of unity which appear are 1 and $\bar{1}$; and as each of those numbers is its own arithmetical inverse, the matrix may be said with equal propriety to be inversely orthogonal or directly orthogonal, that is, orthogonal in the sense conveyed in Art. 1. Moreover, on dividing each term by $\sqrt{n}$, it becomes strictly orthogonal, since the sum of the squares of the terms in each row or column then becomes unity.

A very little reflection will make it clear, *à priori*, that using simply + and − in place of +1 and −1, the known theorems relating to the form of the products of two sums of 2, or of 4, or of 8 squares must exhibit instances of orthogonal matrices of this nature. Thus, to begin with the case of the equation

$$(\alpha^2 + \beta^2)(a^2 + b^2) = A^2 + B^2,$$

we may represent the values of A and B by writing the three matrices

$$\begin{vmatrix} \alpha, & \beta \\ \alpha, & \beta \end{vmatrix} \quad \begin{vmatrix} a, & b \\ b, & a \end{vmatrix} \quad \begin{vmatrix} +, & + \\ +, & - \end{vmatrix};$$

on multiplying these three together, term by term, we obtain

$$\begin{vmatrix} +\alpha a + \beta b \\ +\alpha b - \beta a \end{vmatrix},$$

where

$$+\alpha a + \beta b = A,$$
$$+\alpha b - \beta a = B.$$

Moreover the term-to-term product of the second and third matrices, namely, $\begin{vmatrix} a, & b \\ b, & -a \end{vmatrix}$, is an orthogonal matrix.

So again in the equation

$$(\alpha^2 + \beta^2 + \gamma^2 + \delta^2)(a^2 + b^2 + c^2 + d^2) = A^2 + B^2 + C^2 + D^2$$

the three matrices become

$$\begin{vmatrix} \alpha & \beta & \gamma & \delta \\ \alpha & \beta & \gamma & \delta \\ \alpha & \beta & \gamma & \delta \\ \alpha & \beta & \gamma & \delta \end{vmatrix} \quad \begin{vmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{vmatrix} \quad \begin{vmatrix} + & + & + & + \\ + & - & + & - \\ + & - & - & + \\ + & + & - & - \end{vmatrix}.$$

The resulting product,

$$\begin{array}{l} \alpha a + \beta b + \gamma c + \delta d \\ \alpha b - \beta a + \gamma d - \delta c \\ \alpha c - \beta d - \gamma a + \delta b \\ \alpha d + \beta c - \gamma b - \delta a, \end{array}$$

represents in its four lines the respective values of A, B, C, D. Moreover the matrix produced by the product of the second and third, that is,

$$\begin{vmatrix} a, & b, & c, & d \\ b, & -a, & d, & -c \\ c, & -d, & -a, & b \\ d, & c, & -b, & -a \end{vmatrix},$$

is an orthogonal matrix. The same remarks apply to the representation of the product of two sums of eight squares under the form of a sum of eight. Omitting the first matrix, consisting of repetitions of one given set of eight letters, α, β, γ, δ, λ, μ, ν, π, the remaining two matrices may be written as below:

$$\begin{vmatrix} a & b & c & d & l & m & n & p \\ b & a & d & c & m & l & p & n \\ c & d & a & b & n & p & l & m \\ d & c & b & a & p & n & m & l \\ l & m & n & p & a & b & c & d \\ m & l & p & n & b & a & d & c \\ n & p & l & m & c & d & a & b \\ p & n & m & l & d & c & b & a \end{vmatrix} \quad \begin{vmatrix} + & + & + & + & + & + & + & + \\ + & - & - & + & + & - & + & - \\ + & + & - & - & + & - & - & + \\ + & - & + & - & + & + & - & - \\ + & - & - & - & - & + & + & + \\ + & + & + & - & - & - & + & + \\ + & - & + & + & - & - & - & + \\ + & + & - & + & - & + & - & - \end{vmatrix}.$$

The lettered matrix forms (as in the preceding cases) a "*conjugate* system [in Cauchy's sense] of regular substitutions." The right-hand matrix, interpreting + and − to mean plus and minus units, is a direct and inverse orthogonal matrix corresponding to 8 represented as 2.2.2; the lines produced by the term-to-term multiplication of the three matrices give the quantities A, B, C, D, L, M, N, P, which satisfy the equation

$$\Sigma A^2 = (\Sigma \alpha^2)\, \Sigma\, (a^2),$$

and the term-to-term product of the two matrices actually above written is an orthogonal matrix of the 8th order.

9. I now pass to another and more important illustration of such matrices, which presents itself in the application of Newton's rule (or my extension of it) for finding a superior limit to the number of real roots in an algebraical equation. That rule deals with permanences and variations of sign in two series of quantities. It will be more simple to consider the two simultaneous successions of signs obtained by multiplying together the signs of the consecutive terms in the series

$$\begin{array}{l} f, \quad f_1, \quad f_2, \ldots f_n, \\ G, \quad G_1, \quad G_2, \ldots G_n. \end{array}$$

We obtain in this way two series of n signs each, written respectively over one another; and the quantities with which the theory is concerned are the numbers, say π and ϕ, of compound signs $\begin{smallmatrix}+\\+\end{smallmatrix}$ and $\begin{smallmatrix}-\\+\end{smallmatrix}$ which occur in these simultaneous progressions: the f series and G series both consist of functions of x; the increase of π and the decrease of ϕ, when x ascends from one given value a to another b, each of them gives a superior limit to the number of real roots in fx contained between a and b.

It is of course obvious that π corresponds to the number of double permanences, and ϕ to that of variation permanences in the original series of f's and G's. Now it appeared to me desirable, in the same way as double and higher orders of denumerants have been shown in my lectures on Partitions of Numbers to be expressible as linear functions of simple denumerants, so in like manner to get rid of compound variations and permanences, and to express them, or at least their number, by means of simple variations or permanences. This comes to the same thing as finding a means of making the enumeration of the four species of compound signs $\begin{array}{c|c|c|c|} + & - & + & - \\ + & + & - & - \end{array}$, in two simultaneous series of signs, depend on the enumeration of the simple signs + or − in those series themselves, or in series derived from them, or in the two sorts combined.

10. As a first step in the generalization of this question, let us suppose i series of simultaneous progressions of + and − signs, giving rise to 2^i varieties of vertical combinations of sign. Now let the i given series be combined, r and r together, in every possible manner, where r takes all values from 0 to i, both inclusive.

When $r=1$, it is of course understood that the so-called combinations are the original i series themselves.

When $r=0$, it is to be understood that a series exclusively of the signs + is intended.

When r is not 0, nor 1, let the r series corresponding to any r-ary combination be multiplied term-to-term together.

When $r=0$, the + succession, and when $r=1$ the given n series are to be reckoned as the corresponding products. The number of series of signs so obtained will of course be

$$1+i+\frac{i(i-1)}{2}+\ldots=2^i.$$

By the sum of any series let us understand the number of signs +, less the number of signs −. When the i given series are written over one another, each of the 2^i varieties of columns that can be formed of the signs + and − will occur a certain number of times. I shall show that these 2^i numbers are linear functions of the 2^i sums last mentioned. Of this theorem, on account of its importance, I shall give a rigorous proof.

As a matter of typographical convenience, I write the columnar combinations of sign in horizontal in lieu of their proper vertical order, as, for example, $+ + -$ in lieu of $\begin{matrix}+\\+\\-\end{matrix}$, and, moreover, use such horizontal line enclosed within brackets to signify the number of the recurrences of the corresponding combination; thus $(+ - - +)$ means the number of times the combination $\begin{pmatrix}+\\-\\-\\+\end{pmatrix}$ occurs in four given simultaneous progressions. Again, as regards the sums, s will denote the sum of the line of *plus* signs, which is of course the same as the number of terms in each progression, i the number of columns, and $s_{p,q,r}\ldots$ will denote the sum of the line formed by the multiplication together of the pth, qth, rth, ... lines of the given i set of lines. This being premised, and using each of the symbols $\lambda_1, \lambda_2, \lambda_3, \ldots$ to denote + or −, as the case may be, the number of recurrences of each species of combination in terms of the *sums* is expressed by the following formula,

$$[\lambda_1, \lambda_2, \ldots \lambda_i]=\frac{1}{2^i}(s+\Sigma\lambda_p s_p+\Sigma\lambda_p\lambda_q s_{p,q}+\Sigma\lambda_p\lambda_q\lambda_r s_{p,q,r}+\&c.),$$

as I shall proceed to prove. But first, to make the meaning of this formula more clear, let us suppose $i=2$, the formula then gives the following equations:—

$(+\,+)$, that is the number of combinations $\begin{bmatrix}+\\+\end{bmatrix}$, $=\frac{1}{4}\{s+s_1+s_2+s_{1,2}\}$,

$(+\,-)$, " " " " $\begin{bmatrix}+\\-\end{bmatrix}$, $=\frac{1}{4}\{s+s_1-s_2-s_{1,2}\}$,

$(-\,+)$, " " " " $\begin{bmatrix}-\\+\end{bmatrix}$, $=\frac{1}{4}\{s-s_1+s_2-s_{1,2}\}$,

$(-\,-)$, " " " " $\begin{bmatrix}-\\-\end{bmatrix}$, $=\frac{1}{4}\{s-s_1-s_2+s_{1,2}\}$.

11. Now for the proof of the general formula.

For shortness call the quantity $s+\Sigma\lambda_p s_p+\Sigma\lambda_p\lambda_q s_{p,q}\ldots$ (where the signs $\lambda_1, \lambda_2, \ldots \lambda_i$ are all supposed to be given) E.

Let us consider the effect of the existence of any single column of signs $\mu_1, \mu_2, \ldots \mu_i$ in the given i progressions upon the value of E; besides contributing the signs $\mu_1, \mu_2, \ldots \mu_i$ respectively to the series $s_1, s_2, \ldots s_i$ this column will contribute to the series

$$s_{\theta_1, \theta_2, \ldots \theta_j} \text{ the sign } \mu_{\theta_1}\mu_{\theta_2}\ldots\mu_{\theta_j}.$$

Hence altogether it will contribute to E

$$(1+\lambda_1\mu_1)(1+\lambda_2\mu_2)\ldots(1+\lambda_i\mu_i) \text{ units};$$

and thus the total value of E, depending on the entire number of columns of all kinds, will be

$$\Sigma\,[\{(1+\lambda_1\mu_1)(1+\lambda_2\mu_2)\ldots(1+\lambda_i\mu_i)\}\,.\,(\mu_1\mu_2\ldots\mu_i)],$$

where the λ system is given, but the μ system is variable.

But any factor $(1+\lambda_q\mu_q)$ is zero unless $\lambda_q=\mu_q$. Hence for any system of values of μ not coincident with the λ system the corresponding multiplier of $(\mu_1\mu_2\ldots\mu_i)$ vanishes, and for that system it becomes 2^i. Hence

$$E=2^i(\lambda_1, \lambda_2, \ldots \lambda_i),$$

as was to be proved.

12. These formulæ admit of a useful application to Newton's rule.

The two superior limits to the number of roots included between (a) and (b) which it (or my extension of it) furnished are $\Delta\,(+\,+)$ and $-\Delta\,(+\,-)$, where Δ refers to the ascent from a to b, and the series are those mentioned in Art. 9. Hence, calling the two limits λ, λ', remembering that s is constant,

$$\lambda=\frac{\Delta s_1+\Delta s_2+\Delta s_{1,2}}{4},$$

$$\lambda'=\frac{\Delta s_1-\Delta s_2+\Delta s_{1,2}}{4};$$

so that the limits are

$$\tfrac{1}{4}\{\Delta s_1 + \Delta s_{1,2}\} \pm \frac{\Delta s_2}{4}.$$

The mean of these is $\frac{1}{4}(\Delta s_1 + \Delta s_{1,2})$, which *à fortiori* is also a superior limit. Here s_1 refers to the series

$$f, f_1, f_2, \ldots f_n,$$

and $s_{1,2}$ refers to the series

$$fG, f_1G_1, f_2G_2, \ldots f_nG_n,$$

which I have called, in an article in this *Magazine**, the H series. If p is the number of permanences in the f, and ϕ in the H series, it is readily seen that

$$\frac{\Delta s_1 + \Delta s_{1,2}}{4} = \frac{\Delta p + \Delta\phi}{2}.$$

Hence, since λ and λ' are each of them superior limits, it follows as an immediate consequence that $\frac{\Delta p + \Delta\phi}{2}$ is so likewise; but this assertion conveys no new information, and ought not to be treated as a new theorem, as I inadvertently stated it to be; the fact, however, of its being implied in what was previously known is so far from being immediately evident, that M. Angelo Genocchi has followed me in regarding the theorem as an independent one, and devoted an article to the demonstration of it as such in the *Nouvelles Annales* for January of this year †.

13. The complete system of relations between the two sets of 2^i quantities given by the theorem in Art. 10 may it is evident be expressed by means of the inverse orthogonal matrix (also orthogonal) whose type corresponds to $2 . 2 . 2 \ldots$ (i terms). Thus, for example, for the case of $i = 3$, we may write—

[* p. 542 above.]

† If we call ν the number of real roots in f comprised between a and b, we know from Fourier's theorem that $\nu = \Delta p - 2\theta$, where θ is the number of times that an *even* change occurs in the value of p as we pass from a to b, this change being always in the positive direction. And, again, as I have shown in the article in the *Philosophical Magazine* above referred to,

$$\nu = \frac{\Delta p + \Delta\phi}{2} - \vartheta,$$

where ϑ is the total number of *times* that ϕ undergoes a change within the same interval,—such change being always even, on account of the two terminals of the G series being both positive—the one extremity being a positive constant, and the other the square of f. This change, however, is sometimes additive and sometimes ablative, ϕ not necessarily *increasing* always (as p does) on ascending from a to b: thus the two unknown transcendants θ and ϑ are connected by the simple relation

$$2\theta - \vartheta = \frac{\Delta p - \Delta\phi}{2}.$$

Of course each evanescence of a term in the f or G series between two terms of like sign is to be reckoned as a distinct *time* of change. I also make abstraction of the singular cases where several consecutive terms vanish together in either series.

	s	s_1	s_2	s_3	$s_{1,2}$	$s_{1,3}$	$s_{2,3}$	$s_{1,2,3}$
$2^3 . (+ + +)$	+	+	+	+	+	+	+	+
$2^3 . (+ + -)$	+	+	+	−	+	−	−	−
$2^3 . (+ - +)$	+	+	−	+	−	+	−	−
$2^3 . (+ - -)$	+	+	−	−	−	−	+	+
$2^3 . (- + +)$	+	−	+	+	−	−	+	−
$2^3 . (- + -)$	+	−	+	−	−	+	−	+
$2^3 . (- - +)$	+	−	−	+	+	−	−	+
$2^3 . (- - -)$	+	−	−	−	+	+	+	−

The meaning of this Table is self-apparent. Thus, for example, if we wish to find the value of $(- + +)$, that is, the number of recurrences of $\begin{vmatrix} \mp \\ + \\ + \end{vmatrix}$ in the three given series, we read it off from the 5th line above and find it equal to

$$\frac{s - s_1 + s_2 + s_3 - s_{1,2} - s_{1,3} + s_{2,3} - s_{1,2,3}}{8}.$$

The Table of signs itself is obviously the matrix corresponding to the product 2 . 2 . 2.

From the fact of this Table being orthogonal, we infer that the two sets of quantities are (to a numerical multiplier *près*) the same linear functions, the first set of the second, and the second of the first.

14. The theorem of Art. 10 may be extended to simultaneous progressions of signs denoting any root of +, as for example, $+\rho$, ρ^2, where ρ is a cube root of + instead of + and −. Let each series be supposed to consist of qth roots of +, and let $(\lambda_1, \lambda_2, \ldots \lambda_i)$ denote the number of recurrences of the column $\begin{vmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_i \end{vmatrix}$ in which each λ is some qth root of +; then there will be q^i quantities of the form $(\lambda_1, \lambda_2, \ldots \lambda_i)$. Again, we may form series by combining together not merely the given i series themselves, but their squares, cubes, &c. up to the $(q-1)$th powers, and form the term-to-term products of all the series entering into any such combination; in this way, including s (the series constituted exclusively of + signs), we shall obtain q^i series, the

general symbol for the sum of the terms in any one of which, when we substitute the roots of 1 for the corresponding roots of +, may be written $[s_1^{q_1}, s_2^{q_2}, s_3^{q_3}, \dots s_i^{q_i}]$, where each s is a qth root of +, and each q with a subscript is some one of the numbers in the series 0, 1, 2, ... $(q-1)$. If now we understand by the above bracket, when $q_1, q_2, \dots q_i$ are all zero, the value corresponding to s in the particular case previously considered, that is, the number of terms in each series, the relation between the two sets of q^i numbers is given by the equation

$$(\lambda_1, \lambda_2, \dots \lambda_i) = \frac{1}{q^i} \Sigma \frac{[s_1^{q_1}, s_2^{q_2}, s_3^{q_3}, \dots s_i^{q_i}]}{(\lambda_1^{q_1}, \lambda_2^{q_2}, \lambda_3^{q_3}, \dots \lambda_i^{q_i})} * .$$

If we write out a Table expressing these relations in a manner similar to that employed for the particular case of $q=2$ in a preceding article, we shall obtain a square array of signs (q^i to a side) which will form an inverse orthogonal matrix corresponding to the type $q \cdot q \cdot q \dots$ (to i terms).

* The reader will please to observe that the terms included under the sign of summation are in general not real but complex numbers formed with the qth roots of unity. Their *sum*, however, is necessarily a real number, being the number of recurrences of the column of signs $\lambda_1, \lambda_2, \dots \lambda_i$ in the given system of sign-progressions. The proof of the theorem is precisely the same as for the case previously considered, where $q=2$; namely, it may be shown that the sum above denoted by Σ is equal to

$$\Sigma \left[\frac{\left\{1-\left(\frac{\lambda_1}{\mu_1}\right)^q\right\} \left\{1-\left(\frac{\lambda_2}{\mu_2}\right)^q\right\} \dots \left\{1-\left(\frac{\lambda_i}{\mu_i}\right)^q\right\}}{\left(1-\frac{\lambda_1}{\mu_1}\right)\left(1-\frac{\lambda_2}{\mu_2}\right)\dots\left(1-\frac{\lambda_i}{\mu_i}\right)} \right] [\mu_1 \, \mu_2 \dots \mu_i],$$

each of the q^i terms of which new sum vanishes except that one in which the variable μ system is identical with the given λ system of the qth roots of unity, for which term the fraction becomes equal to q^i.

97.

ON THE SUCCESSIVE INVOLUTES TO A CIRCLE.

[*Norwich British Association Report* (1868), pp. 10, 11.]

FROM the first involute of a circle we may derive a family of parallel curves forming the second involutes of the circle; from each of these again families, the totality of which will form the third involutes, and so on continually.

The author had been led by circumstances to study the arco-radial or semi-intrinsic equation of these curves, and had arrived at certain conclusions concerning its form which subsequent investigations have verified: it turns out that the general equation between the arc s and radius vector r of the general involute of the nth degree will be found by taking F, any rational integer function of x of the nth degree, and eliminating x between the equations

$$r^2 = F^2 + \left(\frac{dF}{dx}\right)^2, \quad s = \int dxF + \frac{dF}{dx}.$$

It follows, as the author had surmised, that the general arco-radial equation for the involute of the nth order when n exceeds unity, is of the degree $(n+1)$ in r^2 and $2n$ in s. Of course, in the case of n equal to unity, the degrees sink to 1 in r^2 and 1 in s. The second involute formed by unwrapping from the cusp of the first may be termed the natural second involute, but is not the most simple of the family; this, which is at the normal distance of half the radius externally from the one last named, is of the third degree in r and the second in s. It may be derived from the curve which a fixed point in a wall at half the length of the radius of a wheel from the ground marks in the wheel as it rolls along the face of the wall by doubling the vectorial angles and taking the squares of the radii vectores. From the arco-radial it is easy to pass to the general polar equation to the n-ary involute; the equation between p, the perpendicular from the centre and q the polar subtangent, is also very easily obtained, being, in fact, no other than the result of eliminating between $Fx = p$, $F'x = q$, Fx being any quantic in x of the nth degree, so that this equation will be of the $(n-1)$th order in q, and the nth order in p.

In the *Philosophical Magazine* for October and December, and in the *Proceedings of the Mathematical Society of London* [below], will be found further developments of the theory of these circular involutes, which it is proposed to term Cyclodes.

98.

NOTE ON SUCCESSIVE INVOLUTES TO A CIRCLE*.

[*Philosophical Magazine*, XXXVI. (1868), pp. 295—306.]

IT is surprising that the families and groups of families of forms capable of being educed by successive involution from a circle should not have attracted the attention of geometers. I find, if any, not more than a passing allusion to their existence in Dr Salmon's *Higher Plane Curves,* in the *Integral Calculus* of Mr Todhunter, or in the memoirs of the late Dr Whewell in the *Cambridge Philosophical Transactions* (vols. VIII. and IX.), although these latter are exclusively devoted to the study of various curves of mechanical and kinematical origin by aid of the so-called intrinsic equation, which is in fact the natural expression of, and key to, the properties of such like curves. And yet this form of equation almost instantaneously furnishes the general polar equation to the entire system of circular involutes, and exhibits at once their leading properties†.

Foremost amongst these stands the algebraical form of equation (and that a quantic), which connects not only the arc, but also the squared radius vector with the angle of contingence, and consequently the two former with one another. In a marvellous and, so to say, transcendental fashion, these curves participate in the nature of algebraical curves—their apses, cusps, and points of retrocession being counted by the order of the involution, and becoming imaginary in pairs.

I need hardly say that by a second involute I mean an involute of the first, by a third an involute of a second, and so on in continual progression. To any given curve all its first involutes form a system of parallel curves, so

* The germ of this Note was communicated to the Mathematical Section of the British Association at the Norwich Meeting.

† Professor Rankine and Mr Merrifield have made a useful application of the *second* involute of the circle to the calculation of the stability of the finite rotation of vessels. In the Turkey carpet under my eyes whilst this is being written, I perceive graceful and complicated figures of winding and intersecting scrolls and convolutions, which render it, I think, not at all improbable that the successive involutes of the circle would furnish or suggest many patterns available for decorative purposes: the enormous variety of each kind of involute, which of course increases with the order of derivation, adds to the probability of this conjecture.

that in general the number of *form*-parameters to a curve will be augmented by i when we pass to its general involute of the ith order. In the case of the circle, however, owing to its homogeneity, the first involute, like the curve itself, contains only one *form*-parameter (it being, in other words, a property of the first involute, that when rotated round a certain point, the curves so generated continue always parallel to each other); and so the number of *form*-parameters in the general ith involute will contain i, and not $i+1$ parameters, as the general formula would require.

I shall use ϕ, s, r, θ to denote the angle of contingence, arc, radius vector, and vectorial angle of the curves under consideration.

Starting from the circle $s = a\phi$, a set of corresponding successive involutes will, as is well known, be represented by

$$s_1 = \frac{a\phi^2}{2} + b\phi; \qquad s_2 = \frac{a\phi^3}{6} + \frac{b\phi^2}{2} + c\phi,$$

and so on, according to the obvious law

$$s_i = \int d\phi \,.\, s_{i-1}.$$

Now in general for any *curve* whatever, if we call p the perpendicular on the tangent from an arbitrary pole, q the projection of the radius on the tangent, we have

$$q = -\frac{dp}{d\phi}; \tag{1}$$

also

$$p + \frac{d^2p}{d\phi^2} = \frac{ds}{d\phi}, \tag{2}$$

$$p^2 + \left(\frac{dp}{d\phi}\right)^2 = r^2. \tag{3}$$

These equations are of course not new*; they are given by Mr Todhunter

* We have only to take P, P', two consecutive points, and on PP', $P'T$, the tangents at P, P', draw perpendiculars from an arbitrary point O, and we obtain at once, by inspection,

$$\delta p = -q\,\delta\phi, \quad \delta q + \delta s = p\,\delta\phi,$$

whence

$$q = -\frac{dp}{d\phi}, \quad p + \frac{d^2p}{d\phi^2} = \frac{ds}{d\phi}.$$

Or, again, proceeding analytically, we have

$$x - A = \int ds \cos\phi, \quad y - B = \int ds \sin\phi;$$

whence, integrating by parts,

$$x - A = G\cos\phi - G'\sin\phi,$$
$$y - B = G\sin\phi + G\ \cos\phi,$$

where

$$G = s - s'' + \ldots; \quad G' = s' - s''' + \ldots;$$

whence

$$r = G^2 + G'^2 \text{ and } G + G'' = \frac{ds}{d\phi}.$$

in the later editions of his *Integral Calculus*, accompanied with a reference to another English treatise, from which he has taken them; but in themselves easily as they can be obtained, they contain the whole theory of the remarkable curves to which this note refers. In the case before, $\frac{ds}{d\phi}=F\phi$, where F is (a quantic in, that is) a rational integral function of ϕ. Hence we have for the solution of (1), (2),

$$p=F-F''+F''''+\ldots+A\cos\phi+B\sin\phi,$$

$$-q=F'-F'''+\ldots-A\sin\phi+B\cos\phi;$$

wherefore r^2 is known in terms of F and the arbitrary constants A and B, whose values depend on the position of the origin from which r is reckoned, by a due choice of which they may be made to vanish. One will readily suppose that this eligible position of the pole must be the centre of the generating circle; and the proof is as follows:

If $r_1, r_2, \ldots r_i$ be any radii vectores corresponding to $s_1, s_2, \ldots s_i$, it is well known, and follows from the definition of the involute, that

$$r^2_{i+1}=r_i^2-2r_is_i\frac{dr_i}{ds_i}+s_i^2. \tag{4}$$

From which also we may deduce

$$q=r\frac{dr}{ds}=G', \quad p^2=r^2-q^2=G^2.$$

This last demonstration would at first sight seem to be only valid for the case of the G series coming to an end, that is, of $\frac{ds}{d\phi}$ being a rational integral function in ϕ; but it would be quite legitimate to infer at once from it the *universality* of the equations above written connecting r^2 with s and ϕ; for we may write down the general differential equation of the second order

$$d\cos^{-1}\frac{dr}{ds}-d\theta=d\phi,$$

that is,

$$\frac{d\cdot\frac{dr}{ds}}{\sqrt{\{(ds)^2-(dr)^2\}}}-\frac{\sqrt{(ds^2-dr^2)}}{r}=d\phi,$$

in which $\frac{ds}{d\phi}$ may be considered as given, and r or r^2 to be determined. The equations in question, having been shown to be true for a form $\frac{ds}{d\phi}$ containing an indefinite number of arbitrary constants, evidently can only amount to a transformation of, and may be used in supersession of the equation last written. It may be worth while to set out this latter under a more familiar form of notation. If, then, we use y for r and x for ϕ, and call $\frac{ds}{d\phi}=X$ (any function of x), it becomes

$$\frac{-Xy''+X'y'}{X\sqrt{(X^2-y'^2)}}+\frac{\sqrt{(X^2-y'^2)}}{y}=1,$$

an apparently very complicated form of equation, but admitting of the simple solution $y^2=u^2+u'^2$, where u satisfies the linear differential equation $u+u''=X$.

Now, suppose that for any number i the origin has been so chosen that

$$r_i^2 = (s_{i-1} - s_{i-3} + \dots)^2 + (s_{i-2} - s_{i-4} + \dots)^2,$$

then
$$r_i \frac{dr_i}{ds_i} = \frac{1}{s_{i-1}} \left\{ \begin{array}{l} (s_{i-1} - s_{i-3} \dots)(s_{i-2} - s_{i-4} \dots) \\ + (s_{i-2} - s_{i-4} \dots)(s_{i-3} - s_{i-5} \dots) \end{array} \right\}$$
$$= s_{i-2} - s_{i-4} + \dots .$$

Hence
$$r^2_{i+1} = (s_i - s_{i-2} + s_{i-4} \dots)^2 + (s_{i-1} - s_{i-3} + \dots)^2;$$

and the supposed relation, if true for any value i, is true for all superior values; but when the origin is at the centre, $s_1 = a\phi$, $r_1^2 = a^2$; and consequently the equation (4) is true universally*.

Thus, then, when the origin is at the centre, whilst for the ith involute to a circle the arc is any quantic $\int d\phi F$ of degree $(i+1)$ in ϕ, r^2 is a quantic in ϕ of degree $2i$ of the particular form $(G\phi)^2 + (G'\phi)^2$, where $G\phi$ may be supposed, if we please, to be any quantic of the order i in ϕ†; and then $\dfrac{ds}{d\phi}$ the radius of curvature at s, ϕ, is expressed by $G\phi + G'\phi$‡.

* The above result might have been deduced more directly from the equation $\dfrac{dp_i}{d\phi} = p_{i-1}$, which is true for any curve and its evolute. In fact this paper need never have been written (for all that it contains is a straightforward inference from four equations which may be found scattered up and down in elementary treatises), had it been the custom to regard those equations as forming collectively a connected apparatus. I mean the four following, where the unaccented and accented s and p refer to any curve and its evolute respectively, ϕ being the angle of contingence common in magnitude to the two:

$$(1)\quad \frac{ds}{d\phi} = p + \frac{d^2p}{d\phi^2}, \qquad (2)\quad r^2 = p^2 + \left(\frac{dp}{d\phi}\right)^2,$$

$$(3)\quad s' = \frac{ds}{d\phi}, \qquad (4)\quad p' = \frac{dp}{d\phi},$$

the last of them more familiarly known under the form

$$p'^2 = r^2 - p^2.$$

The third and fourth equations show respectively that s and p are each quantics in ϕ; the first gives the connexion between the constants which enter into these quantics, and the second, combined with the first, the relation between s and r (in other words, the rectification of the curve), that between r and θ (where θ is the vectorial angle) being contained in a fifth equation,

$$\theta = \phi + \sin^{-1}\frac{p}{r}.$$

† Accordingly we see that the spiral of Archimedes, as is well known, is the locus of the feet of the perpendiculars upon the tangents to the first involute from the centre of the circle; and, much more generally, if we substitute for each radius vector in this spiral any given quantic thereof, we obtain the corresponding first pedal to an involute whose order of derivation is the degree of the quantic. For example, by squaring or cubing the radius vector of the spiral of Archimedes (of course leaving the vectorial angle unchanged), we may form the pedal to particular species of the second and third involutes respectively.

‡ Since the radius of curvature, radius vector, and perpendicular on the tangent arc are all known rational integral functions of the same quantity, it becomes a simple problem of elimination to determine the central force competent to make a body describe an involute of any order

It may be here noticed that substituting for ϕ, $\phi+\lambda$, where λ is arbitrary, amounts only to a rotation of the curve through the angle λ, so that, as regards the intrinsic form of the curve, no generality is sacrificed by imposing one condition upon the coefficients in $G\phi$, or, if we please, in making any of the coefficients in it except the first to vanish.

From what precedes, and from the general theory of elimination, it follows that in general the relation between i^2 and s is expressed by a rational integral equation of the degree $(i+1)$ in the former and $2i$ in the latter. But this is subject to an obvious exception in the case of $i=1$; for then, calling $G\phi = a\phi$, we have

$$\frac{ds}{d\phi} = a\phi, \quad s = \frac{a\phi^2}{2} + b, \text{ and } r^2 = a^2\phi^2 + a^2 = 2as + (a^2 - 2ab);$$

so that the degrees in r^2 and s are here 1 and 1 in lieu of 2 and 2, as given by the general rule.

As regards the polar equation to the general involute,

$$r^2 = (G\phi)^2 + (G'\phi)^2,$$

it is obvious that, agreeably to the well-known case for the first involute,

$$\theta = \sin^{-1}\frac{dr}{ds} + \phi = \sin^{-1}\frac{G'}{r} + \phi^*,$$

where G' and ϕ are given by the solution of an algebraical equation of the $2i$th degree, and which will therefore *usually* be incapable of expression in

to a circle. Thus it will be found that the half-pitch second involute may be described under the action of a central force varying as the inverse cube of the shortest distance from the generating circle. So, again, the first involute may be described under the action of a central force, the component of which in the direction of the tangent to the generating circle (or say the centrifugal force) varies as the inverse cube of this tangent, the centre of force in each case being of course situated at the centre of the circle.

* ϕ and G' will form $2i$ systems of values. Will they be all applicable to the true involute, and how about the sign to be given to r? It must, I think, be a matter of some delicacy and difficulty to answer these questions. For take even the first involute, where

$$\theta = \sin^{-1}\frac{a}{r} + \sqrt{\left(\frac{r^2-a^2}{a^2}\right)},$$

we know, as a matter of fact, that if the first term is made to decrease as r increases, the positive sign of the square roots only must be employed, and of course the negative sign if the first term increases with r. Were we to reverse this rule, instead of the involute we should obtain what may be termed the counter-involute; that is, a figure formed by points, each the *opposite* of every point in the involute in respect to its centre of curvature. Or, again, if a pair of parallel rulers were made always to touch a circle at opposite points, and the under parallel to *roll* round the circle, whilst each point in this line describes an involute, each point in the one above would describe a counter-involute. Or, again, if a string, by aid of a pin, were unwrapped *back* upon itself from a circle, the extremity would describe the extraneous curve. From this last observation it would seem as if the forced intrusion of a foreign curve into the polar equation of the involute resulted from the impossibility of affixing an absolute sign to the length of an arc—the condition of drawing a tangent always equal in length to the varying arc of a curve admitting of satisfaction without breach of continuity in two distinct modes.

finite terms beyond the second involute. In the case of this involute the reducing equation is not a general biquadratic, but a form involving only square and no cube roots—being in fact reducible to a quadratic in ϕ^2, as will at once be seen from the fact that we may write

$$(\alpha\phi^2+\gamma)^2+4\alpha^2\phi^2=r^2.$$

Since $\dfrac{ds}{d\phi}$, that is $G+G''$, is a quantic of the degree i or ϕ, we learn that there may be i cusps to the ith involute, or any less number differing from i by an even integer. Also, since $\dfrac{dr^2}{ds}=G'$, the number of apses (in regard to the centre) may be any number inferior to and differing from i by an odd integer*. Also, since G represents the perpendicular on the tangent, the number of points where the tangent passes through the centre will follow the same law (although, of course, the two numbers need not be equal) as the number of the cusps. The cusps, of course, can only exist at points where the involute meets the parent curve.

Between any two cusps of an involute evidently must be comprised an odd number of the cusps of its parent curve; but, of course, not *vice versâ*; thus, for example, in the second involute, if there are no cusps, it will easily be seen that the curve possesses a simple loop enclosing the cusp of the first involute (its evolute), and consequently cutting the two branches of the latter, and so in general the disappearance of consecutive cusps in any involute will give rise to loops enclosing those cusps of the parent curve on the branches adjoining to which (on each side), cusps of the derived curve are wanting; (by a branch, I mean, of course, the portion of curve included between any two cusps, or between either of the two terminal cusps and infinity;) whether the absence of cusps of the involute on $2i$ consecutive branches of the parent curve implies the necessary existence of i distinct loops, one round every alternate one of the $2i-1$ cusps in which those branches meet, requires further consideration. It is clear that in an analytical sense the length of the arc of the parent curve included between any two cusps of the second curve must be taken as zero; the correct view (at least, for the purposes of this theory) being that the angle of convergence *continually* increases or decreases up to positive or negative infinity as we

* Thus we see that the apsidal distances from the centre are the arithmetical magnitudes of the roots of the equation formed by equating to zero the *discriminant* of $G+r=0$, which is of course of the degree $(i-1)$ in r.

If we consider the apses and cusps of any involute to form a combined group, an odd number of the points of this group will always be included between any two intersections of the curve with a circle concentric with the parent circle; for the limiting equation to $G^2+G'^2-r^2=0$ is $G'(G+G'')=0$. Every point in this combined group is a point of maximum or minimum elongation from the centre.

pass in one direction from point to point in a curve. Accordingly we ought not to say, as is usually done, that at a cusp the tangent is suddenly reversed in direction, but, rather, that the increment of the arc on passing through a cusp changes sign, as it ought to do according to first principles; for the *flow* of the incremental arc, from being concurrent with, becomes opposite to that of the rotating tangent line which carries it, or *vice versâ*. Thus in the common cycloid (a curve of infinite length to the eye and with an infinite number of cusps) we have $s = c \cos \phi$, which, subject to this interpretation, is perfectly true and self-consistent for the whole extent of the curve from infinity to infinity. In that case we have a visible representation of *quantity* undergoing an infinite number of periodic changes, although the *subject matter* of the quantity is continually changing and never recurs. In the case of the involutes of the circle, the number of those periodic changes is of course finite and equal to the number of the cusps. If $A, B, C, D, \ldots L$ be the cusps in natural order on the curve whose involute is to be found, and if we call x the radius of curvature of the point of the involute corresponding to A (x being taken positive when this radius is in the position into which it would be brought by unwinding a string from the infinite branch adjacent to A), and if we form the series $x - AB + BC - CD \ldots \pm KL$, where AB, $BC, \ldots$ are the arithmetical lengths of the branches, it is clear that at each term of this series in which a change of sign in the sum takes place the involute will have a cusp; if the number of branches is odd and x is negative, the sum may remain negative at whatever term we stop, and then there will be no cusp in the involute so engendered; but when the number of points $A, B, C, \ldots L$ is even, then it is easy to see that one of the infinite branches must contain a cusp of the involute and the other be vacant. The second involute, whether cusped or not, manifestly consists of two parts symmetrically arranged about its apse. If we form a third involute by unwinding from this apse as origin, the figures so formed will again be symmetrical, and the cusps will lie at the vertices of an isosceles triangle; and now *every* involute of this symmetrical third involute will again be symmetrical, and so on continually, the number of conditions imposed on the parameters in order to ensure symmetry in the involute of the ith order being thus the integer part of $\frac{i-1}{2}$. When this symmetry obtains, the algebraical equation requisite for determining the polar equation depends on the solution of an equation of only the ith instead of the $2i$th degree; for it is clear that in this case the functions G^2 and G'^2 may be made to contain only powers of ϕ^2. Thus we may very easily write down the general polar equation to the absolutely general second involute, and might, if it were worth while, do as much for the symmetrical class of third and fourth involutes, of which the former will contain two and the latter three arbitrary parameters, by solving a cubic and biquadratic equation in these two cases respectively.

It is easy to see also that the *arco-radial* equation to the symmetrical involute of an odd order is of only half the degrees in s and r^2 that it is of in the general case, and for the symmetrical involute of an even order, although of the same degrees in r^2 and s as in the general case, involves only the even powers of s.

A few words upon the second involute, and I have done; for it is difficult to deal with theory in any detail so as to be intelligible, or even safe, without the suggestive and regulative aid of drawn figures, which I have not yet been able to obtain in a form fit for use.

The two principal classes to distinguish in the second involute are the cusped and uncusped species. The cusped second involute winds round the parent curve upon which the extremities of its finite branch rest. The uncusped species crosses itself, and intersects each branch of the first involute of which it encloses the cusp, its node being on one side of it and its apse on the other. The transition case is when the unwinding begins from the cusp of the first involute; the second involute so obtained has a very singular point at that cusp, which may be regarded as a coincident pair of cusps*.

The general connecting equations for this involute may be put under the form

$$r^2 = \left(\frac{a}{2}\phi^2 + b\right)^2 + a^2\phi^2,$$

$$\frac{ds}{d\phi} = \frac{a}{2}\phi^2 + (a + b),$$

$$\theta = \sin^{-1}\frac{a\phi}{r} + \phi,$$

where a is the radius of the circle; and there will be a loop or cusps according as $a + b$ is positive or negative; when $b = -a$, we have the transition case, for which

$$r^2 = \frac{a^2}{4}\phi^4 + a^2, \quad s = \frac{a}{6}\phi^3,$$

and the arco-radial equation becomes

$$(r^2 - a^2)^3 = \frac{81}{4}a^2s^4.$$

* Mr Crofton has noticed, in an ingenious paper published in the *Mathematical Messenger*, that this involute is the locus of the centres of all the circles cutting orthogonally the originating circle and the parent first involute. This is seen very easily as follows:

$$p = a\left(\frac{\phi^2}{2} - 1\right), \quad s' = p + p'' = a\frac{\phi^2}{2},$$

$$r^2 = p^2 + p'^2 = a^2\frac{\phi^2}{4} + a, \text{ or } r^2 - a^2 = s'^2,$$

showing that the tangents to the circle and first involute from any point in the second are equal to one another.

Another and more remarkable case occurs when $G^2 + G'^2$ becomes a perfect square, for then the degrees in s and r will sink to half their usual values: this occurs when $b = -\frac{a}{2}$, which is the case of a looped curve bisecting at its apse the radius drawn from the centre of the circle to the generating first involute.

We have in that case

$$G = \frac{a}{2}(\phi^2 - 1),$$

$$r = \frac{a}{2}(\phi^2 + 1) \quad \text{or} \quad \phi = \sqrt{\left(\frac{2r-a}{a}\right)},$$

$$s = \frac{a}{2}\left(\frac{\phi^3 + 3\phi}{3}\right),$$

whence $$9as^2 = (2r-a)(r+a)^2,$$

and $$\theta = \sin^{-1}\sqrt{\left(\frac{2ar-a^2}{r^2}\right)} + \sqrt{\left(\frac{2r-a}{a}\right)} = \text{vers}^{-1}\frac{a}{r} + \sqrt{\left(\frac{2r-a}{a}\right)},$$

a form even simpler than that of the first involute.

We may write this equation under the form

$$\theta = -2\cos^{-1}\sqrt{\left(\frac{\frac{1}{2}a}{r}\right)} + \sqrt{\left(\frac{r-\frac{1}{2}a}{\frac{1}{2}a}\right)};$$

or, turning round the line from which θ is reckoned through a quarter of a revolution,

$$\theta = 2\sin^{-1}\sqrt{\left(\frac{\frac{1}{2}a}{r}\right)} + \sqrt{\left(\frac{r-\frac{1}{2}a}{\frac{1}{2}a}\right)}.$$

Let now $$\theta = 2\vartheta, \quad \frac{a}{2}\cdot r = \rho^2$$

(which is the same thing as if for x and y we substituted $x^2 - y^2$ and $2xy$), then

$$\vartheta = \sin^{-1}\frac{\frac{1}{2}a}{\rho} + \frac{\sqrt{\left\{\rho^2 - \left(\frac{a}{2}\right)^2\right\}}}{a}.$$

This is the polar equation to a known curve (of the kind used by Captain Moncrieff in his barbette gun-carriage). It is of the class of curves generated by a fixed point on a wheel rolling on a plane. Such a curve may be termed the *convolute* of a circle of a *pitch* denoted by the ratio of the distance of the fixed point *below* the centre to the radius of the revolving circle; thus a convolute of zero-pitch is the spiral of Archimedes, a convolute of unit pitch the first involute to the circle: the general equation to a convolute, when the distance below the centre is d and the radius a, is given by the

Rev. James White in the last September Number of the *Educational Times* and is easily shown to be

$$\vartheta = \sin^{-1}\frac{d}{\rho} + \frac{\sqrt{(\rho^2 - d^2)}}{a}.$$

Similarly, we may define the pitch of the second involute to be the ratio of the distance of its apse from the centre to the radius; and then we are conducted to the observation that whilst the convolute of full pitch is the first involute, the convolute of half pitch, on applying to it one of the simplest forms of M. Chasles's or Mr Roberts's method of transformation (given in Dr Salmon's *Higher Plane Curves*, p. 236), namely, doubling the vectorial angle and squaring the radius vector, becomes converted into the second involute of half pitch. Since for this curve

$$r = \frac{a}{2}(\phi^2 + 1) = \frac{ds}{d\phi},$$

we see that it may be completely defined, without reference to any theory of involutes, as the curve whose radius of curvature at any point is equal to its radius vector reckoned from a given origin. It is the curve which completely satisfies the equation $rd\cos^{-1}\dfrac{dr}{ds} = s$, the two arbitrary parameters which the complete integral of this equation should contain being furnished by the linear magnitude and angle of swing of the curve round the given origin*.

I conclude with the remark that if we regard the s and r^2 of the successive involutes as *rectilinear* coordinates to a variable point, the arco-radial

* This evolute possesses the property, which serves to characterize it completely, of cutting the originating *circle* (its second evolute) orthogonally. For when $r^2 = a^2$, $G^2 = 0$, that is, the tangent to the curve passes through the centre. Moreover, since $G = 0$ gives $\phi = 1$, it follows that the curve cuts out of the circle an arc equal in length to the diameter. Summarizing such of its principal properties as have fallen in our way, we see that it bisects the line joining the centre of the originating circle and the cusp of the first involute; that it cuts the said circle orthogonally; that its radius of curvature is everywhere equal to its elongation from the centre; that it is a trajectory to a central force varying as the inverse cube of the shortest distance from the periphery of the originating circle; that its arco-radial equation is of only half the number of dimensions of the general involute of the same order; and that by the simplest form of quadratic transformation (namely, that which leaves unaltered the inclination of the tangent to the radius vector) it changes into the half-pitch circular convolute; not to add that its polar equation is even simpler than that of the first involute. Certainly, then, as it seems to me, it ought to take permanent rank among the spirals which have a specific name on the geometrical register; and for want of a better, with reference to the place where its properties first came into relief, it might be termed the *Norwich spiral*. Where it meets the first involute we have

$$\left(\frac{\phi^2+1}{2}\right)^2 = \frac{r^2}{a^2} = \phi^2 + 1,$$

or

$$(\phi^2+1)(\phi^2-3) = 0;$$

so that at the real intersections the radius vector is $2a$, and the perpendicular on the tangent, namely, $\left(\dfrac{\phi^2-1}{a}a\right)$, is a, showing that the tangent and radius vector at those points are inclined to each other at an angle of 30°.

equation will represent a peculiar class of unicursal algebraical curves. Thus the first involute will represent a pair (one for each branch of the curve) of coincident right lines, and the general second involute (taking r^2 and s^2 as the coordinates) a pair of coincident semicubical parabolas.

In making s vary continuously on passing a cusp, the corresponding abscissa from increasing must begin to decrease, or *vice versâ*, according to the principles previously noticed.

Thinking of the recovery of the cusps and apses from the arco-radial equation, I have been led to consider a morphological property of a more general class of unicursal equations, which I think is likely to bear valuable fruit, and may possibly form the subject of another communication.

99.

ON SUCCESSIVE INVOLUTES TO CIRCLES.—SECOND NOTE*.

[*Philosophical Magazine*, XXXVI. (1868), pp. 459—466.]

SINCE the appearance of the former Note on this subject, I have enjoyed the inestimable advantage of securing the cooperation of my all-accomplished and omni-capable friend Mr Spottiswoode, to whose kindness and skill my readers are indebted for the beautiful figures given in the following pages, which I shall proceed briefly to describe, and which, as far as I can learn, offer the first examples of the actual visible representation of any derived involutes of the circle beyond those of the first order. I propose, for want of a better word, provisionally to give the name of Cyclodes (suggested by Professor Cayley) to these spirals. They may be considered a genus of a more general class of spirals which I propose to name algebraical spirals, defined by the condition that the perpendicular on the tangent from a certain fixed point (which may be termed its pole) is a rational algebraical function of the angle of contingence; so that a cyclode may be said to be an *integral* algebraical spiral, that is one in which the perpendicular on the tangent becomes a rational integral function of the angle of contingence.

I find in a certain question, presently to be alluded to, the theory of the class so indisputably bound up with that of the genus, as to persuade me of the importance of the theory of the former being gone into by someone who has leisure for the investigation, and of the desirableness of an organic description being discovered or devised for the rational fractional case. The peculiar feature of the cyclode class is the absence of points of inflection, real or imaginary. The cusps of cyclodes are strictly analogous to the asymptotes in algebraical curves, like them entering and disappearing in pairs, creating

* The thought foreshadowed in the concluding paragraph of the former note leads to the following theorem.

Let f, ϕ, ψ be quantics in α, β; F the unicursal function obtained by elimination of α, β between

$$x=f, \quad y=\phi, \quad z=\psi;$$

$\Delta_x . F$ the discriminant of F regarded as a quantic in x and 1; $J(\phi, \psi)$ the Jacobian of ϕ, ψ; R the result of eliminating ϕ, ψ between

$$y=\phi, \quad z=\psi, \quad J(\phi, \psi)=0;$$

Q the product of all the homogeneous linear functions of y, z which vanish at the double points of F; then I say (and the proof is all but self-evident) $\Delta_x . F=R . Q^2$.

partial interruptions of continuity, and thus separating the curve into distinct branches*. In the same way as the order of an algebraical curve is deter-

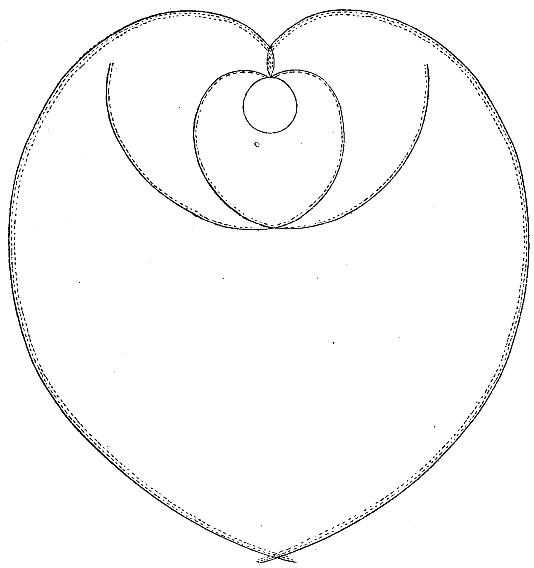

Fig. 1.

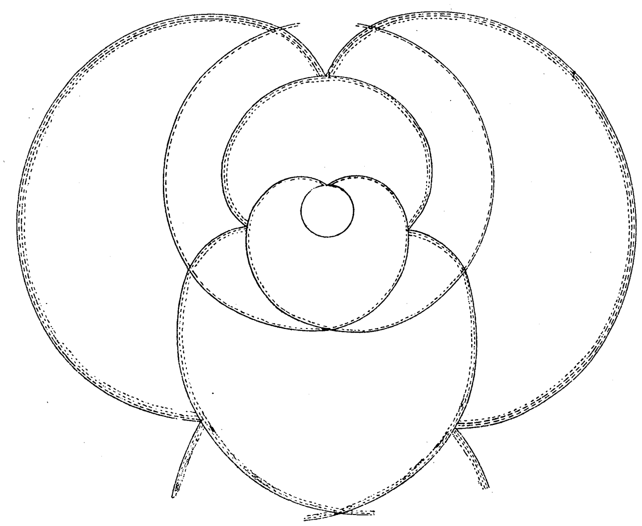

Fig. 2.

mined by the number of its intersections with any right line, so that of any such spiral may be characterized by half the number of its intersections with

* Parallelism for cyclodes bears some analogy to projection for algebraic curves, and operates in the way of addition or diminution upon the cusps as the latter process does upon the asymptotes.

any circle having its centre at the pole. When the rational fraction which expresses the value of the perpendicular is of the degree m in the numerator and n in the denominator, the order will thus become the *dominant* of the two quantities $m+n$, $2n$.

Figs. 1, 2, 4, 5, exhibit examples of cyclodes of the first, second, and third orders, distinguished respectively, where required, by the number

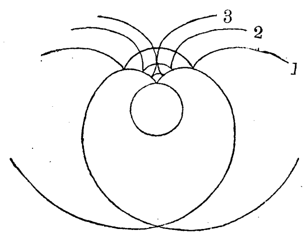

Fig. 3.

1. Triple tangent.
2. Double tangent passes through Cusp.
3. Double tangents coincide.

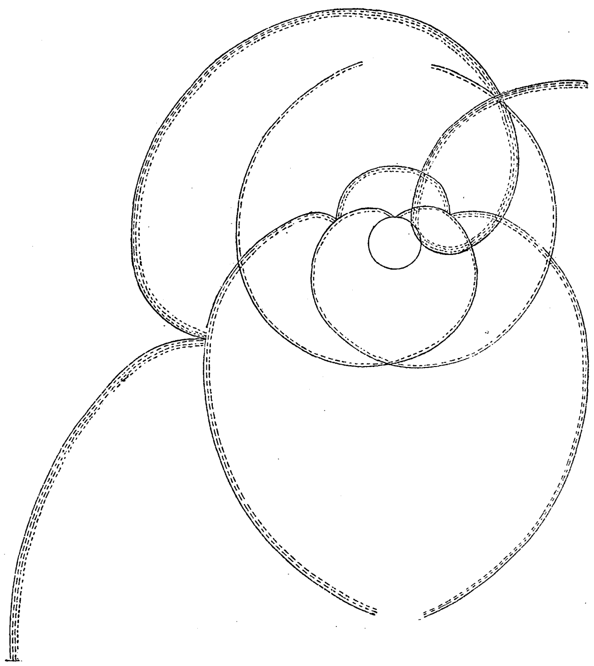

Fig. 4.

of accompanying dotted lines*. Let us consider more closely those of the second order, which separate themselves into two classes, the cusped and

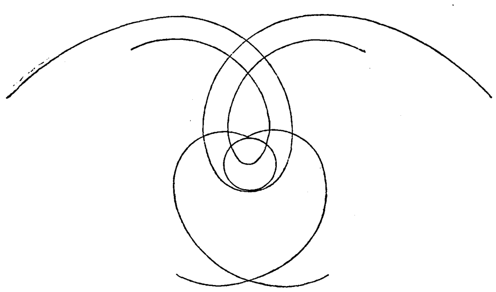

Fig. 5.

* In figs. 5 and 3, which refer to cyclodes of the second order exclusively, it has not been thought necessary to adjoin the dotted lines.

uncusped. The cusped class are the analogues of the hyperbola, the uncusped class of the ellipse, and the very remarkable secondary cyclode whose tail (to use the late Dr Whewell's expression) is zero, and which may be termed the natural one of the order, is the analogue of the parabola. In the former Note I spoke of "points of retrocession"; instead of points of retrocession, I propose to call these "points of radiation" or "radiant-points"; the intervention of the cusps prevents the happening of the supposed "retrocession" at such points. This error illustrates the danger of, so to say, fighting in the dark, that is, reasoning from general mental impressions in the absence of all suggestive visible representation of geometrical forms*.

Let us compare the cusped cyclodes of the second order in figs. 2, 4, where the *tail* † is negative, with the *natural* one in fig. 1, for which this tail is zero. It is interesting and instructive to trace the passage from the one to the other by following the fortunes of the double tangents, which in the figs. 2 and 4 will be seen to connect the sort of Moorish arch that constitutes the middle finite branch with each of the adjoining infinite branches.

How is either of such double tangents to be determined? At the two points where it meets the curve the angle of contingence is not the same, but has increased by 180° in passing from the one to the other. Accordingly p, the perpendicular, if represented by $F\phi$ when the double tangent is regarded as a tangent at one point, will be represented by $-F(\phi+\pi)$ when that line is regarded as a tangent at the other point of contact, and the equation for finding ϕ becomes $F\phi+F(\phi+\pi)=0$. Thus we see incidentally that p must become zero, that is that a point of a radiation must necessarily exist somewhere between the two points of contact. And here I may remark incidentally that this throws light on the notable equation, applicable to any curves whatever,

$$\frac{ds}{d\phi}=p+\frac{d^2p}{d\phi^2};$$

for $\dfrac{d^2p}{d\phi^2}$, by virtue of the remark made in a footnote to the former paper on this subject, is the perpendicular to a tangent to the second evolute at a point corresponding to that, in the curve itself, for which p is the perpendicular to the tangent; but at corresponding points in a curve and its second evolute, the tangents, although parallel in *direction*, are opposite

* I believe I am correct in saying that in like manner a mistake made by Steiner in his description of a surface viewed only by himself "in the depths of his inner consciousness," was first discovered by Professor Kummer after the construction of an actual model. So impossible is it to *prove demonstration*, and to make oneself absolutely safe against the fallacy of ignoring entities on the one hand, or unduly assuming their existence on the other.

† In general the *tail* is the distance of the cusp of the first involute from the corresponding points of the involutes successively engendered therefrom.

in *flow*. Hence $p+\frac{d^2p}{d\phi^2}$ $\left(\text{and not } p-\frac{d^2p}{d\phi^2}\right)$ is the distance between these two tangents; and it is obvious that such distance is identical with the radius of curvature corresponding to the perpendicular p; so that, viewed in this light, the differential equation above written is reduced to a *truism*. Returning to our cyclode of the second order, we may write its equation under the form

$$p=\frac{a}{1.2}(\phi^2-\gamma),$$

where a is the radius of the base-circle, ϕ is zero at the apse, that is the point which divides the curve into two equal and symmetrical branches. Hence to find the nearest double tangent* we have

$$\phi^2-\gamma+(\phi+\pi)^2-\gamma=0.$$

Putting $\phi+\frac{\pi}{2}=\psi$, this equation becomes

$$\psi^2=\gamma-\frac{\pi^2}{4}.$$

The *tail* of the secondary cyclode (say τ) is obviously the distance of the apse (in respect of the centre) from the node of the parent first cyclode; and its length is $\frac{a}{2}(\gamma-2)$, the distance of the apse from the centre being $\frac{a}{2}\gamma$. As γ increases towards 2, ψ decreases; that is, either double tangent tends more and more towards the horizontal; the Moorish arch therefore sinks, and the adjoining haunches rise until when $\phi=0$, that is $\gamma=\frac{\pi^2}{2}$, the two double tangents are in direct opposition and merge into a triple tangent touching the arch at its centre.

As γ continues to decrease by ϕ becoming negative, the central arc sinks below the level of the adjoining branches, and the double tangents slope more and more towards its extremities, until at length they pass through the cusps; when this takes place the tangent to the second cyclode becomes perpendicular to the parent cyclode, and consequently touches the originating circle so that $p=-a$, that is

$$\frac{a}{2}(\phi^2-\gamma)=-a,$$

that is
$$\left(\psi-\frac{\pi}{2}\right)^2-\psi^2-\frac{\pi^2}{4}=-2,$$

that is
$$\psi=\frac{2}{\pi},\quad \gamma=\frac{\pi^2}{4}+\frac{4}{\pi^2},\quad \tau=\frac{a}{2}\left(\frac{\pi}{2}-\frac{2}{\pi}\right)^2.$$

* For of course we may write in general

$$(\phi^2-\gamma)+[\{\phi+(2i+1)\pi\}^2-\gamma]=0,$$

i being any integer, and ϕ will give the direction of a double tangent.

As γ goes on decreasing, the double tangents quit the Moorish arch altogether and connect the two infinite branches, which turn their protuberances towards each other more and more, until finally they touch and the double tangents coincide. This happens when $\phi = -\frac{\pi}{2}$, that is $\gamma = \frac{\pi^2}{4}$.

As the tail goes on still to decrease, the double tangents become imaginary, the infinite branches intersect and cut out a lune, one extremity of which, the two cusps of the cyclode under consideration, and the cusp of the parent cyclode, together form a quadrangle, which continually contracts its dimension until finally it vanishes with the tail and the central arc, and the four points merge into the remarkable *round* point indicated in fig. 1, corresponding to the parabolic or transition case between the cusped and uncusped species. This paradoxical point is a mere creature of the reason, and can by no effort be made sensible to the understanding. Observe that, in this point, the curve dips its beak, so to say, into the cusp of the parent first involute, and yet touches the original circle. Professor Cayley informs me he has met with the same kind of point in an investigation into the form of the parallels to an ellipse, and proposes to call it a triangular point, as consisting of the union of a node and two cusps. At this point, in the case before us, we have

$$p = \frac{a\phi^2}{2} - a, \quad \frac{ds}{d\phi} = p + p'' = \frac{a\phi^2}{2};$$

so that, it will be observed, $\frac{d^2s}{d\phi^2}$, as well as $\frac{ds}{d\phi}$, vanishes when ϕ is made zero.

This gives me occasion to make a remark which I do not remember having seen in the text-books, namely, that for any curve, while *in general* $\frac{ds}{d\phi} = 0$ indicates the existence of a cusp, this law is subject to the exception that if a succession of such derivatives

$$\frac{ds}{d\phi}, \quad \frac{d^2s}{d\phi^2}, \quad \frac{d^3s}{d\phi^3}, \ldots$$

all vanish simultaneously, there will not be a cusp in fact unless the last of the series is of an *odd* order.

Fig. 5 exhibits the critical cases (1) of the double tangents in opposition, (2) on the point of quitting the central branch, (3) in coincidence. Mr Spottiswoode informs me that this figure has not been drawn with the same attention to mechanical exactitude as the other figures.

In fig. 5 are seen examples of the uncusped species. The Norwich spiral (of which a word or two more presently) belongs to this species, but is not drawn; its apse lies midway between the centre of the circle and the cusp of

the first cyclode. In fig. 2 is seen an example of a symmetrical tricuspidal cyclode of the third order; in fig. 4, of a unicuspidal cyclode of the same order, where a loop replaces the missing cusps.

To return to the Norwich spiral; its radius of curvature ρ has been shown in the preceding rule to be always equal to its radius vector r, reckoned from the centre of the circle. Now it is easy to see that whilst $\int d\phi\rho$ represents the arc of any curve, $\int d\phi r$ will represent the corresponding arc of its first pedal; so that the spiral in question possesses the remarkable property (capable, one would think, of some practical kinematic application) that these two arcs always remain equal to each other. More generally, if $p^2 + p'^2$*, where p is a rational integral function of ϕ, and p' its first derivative in respect to ϕ, is a perfect square, the arc of the curve and of its pedal will always remain algebraically related. Here, then, we are led to consider the possibility of satisfying this diophantine condition for cyclodes beyond the second order. At a first glance the problem might seem to be impossible. For if the condition is satisfied by $p = F\phi$, a rational integral quantic in ϕ of the order n, it obviously will be satisfied also by $F(\phi + \lambda)$, λ being an arbitrary constant; and consequently we have only $(n-1)$ and not n disposable constants (or ratios) wherewith to satisfy the n conditions involved in a function of ϕ of order $2n$ being a perfect square.

This objection, however, is only apparent, and may at once be seen so to be, at all events as regards cyclodes of an even order—say, of order $2m$. For we may suppose

$$p = F(\phi + \lambda)\{f(\phi + \lambda)\}^2,$$

a quantic of the order m in $(\phi + \lambda)^2$, then

$$F^2 + F'^2 = f^2 + 4(\phi + \lambda)^2 f'^2$$

is a quantic of the order $2m$ in $(\phi + \lambda)^2$, and the m disposable constants in f are sufficient to make this a perfect square. Thus, then, the n conditions are not absolutely incompatible. Still the disproof of the incompatibility might *seem* to involve the necessity of F being a function of $(\phi + \lambda)^2$, that is of the cyclode being of the symmetrical kind. Moreover, if the problem be attacked by a direct exoscopic method for cyclodes of the second, fourth, and sixth orders, it will be found that the only cyclodes which possess the required property

* It will be remembered that $r^2 = p^2 + p'^2$. I may remark incidentally that this equation enables us to extend the well-known one, $p^2 = r^2 - a^2$, applicable to the first cyclode: the general theorem which includes this as a particular case is obviously

$$p^2 = r^2 - r'^2 + r''^2 + \ldots \pm a^2,$$

p being the perpendicular on the tangent of a cyclode of any order, and r, r', r'', ... the distances of the corresponding points in the cyclode and its successive evolutes from the centre of the originating circle.

are of the symmetrical kind, namely, for the second order, $p=\frac{a}{2}(\phi^2-1)$, for the fourth, $p=\frac{a}{2}(\phi^2-4)^2$, and for the sixth,

$$p=\frac{a}{2}(\phi^2-9)^3\text{*}, \quad \text{or} \quad p=\frac{a}{2}(\phi^2-9)(\phi^2-36)^2.$$

The inference, then, might appear to be almost irresistible as to the necessity of the symmetrical form holding good. But it is *not* so; *it is* true that only cyclodes of *even* orders are reducible, that is capable of giving r as a *rational* integral function of ϕ; but after the sixth order, that is beginning with the eighth, non-symmetrical reducible cyclodes come into existence, and, as the order rises, become infinitely more numerous than those of the symmetrical kind.

Calling $2m$ the order, every distinct mode of making the partitions of numbers expressed by the two simultaneous equations

$$\begin{Bmatrix} x_1+x_2+\ldots+x_i=m \\ y_1+y_2+\ldots+y_i=m \end{Bmatrix},$$

where i takes all possible values, gives rise to a system of equations yielding in general many solutions; and it is only when $x_1=y_1$, $x_2=y_2, \ldots, x_i=y_i$ that the solutions are of the symmetrical kind. Moreover, even in that case, in *general*, and subject only to rare cases of exception, the reducing system of equations gives two distinct groups of solutions, one corresponding to

* It is very easy to see that there is always one *reducible* symmetrical cyclode of the order $2m$ defined by the equation

$$p=\frac{a}{(2m)\,!}(\phi^2-m^2)^m,$$

corresponding to which

$$r=\frac{a}{(2m)\,!}(\phi^2-m^2)^{m-1}(\phi^2+m^2).$$

Thus, when $m=2$,

$$p=\frac{a}{24}(\phi^2-4)^2,$$

$$r=\frac{a}{24}(\phi^4-16);$$

when $\phi=0$, we have

$$p=\frac{2a}{3}, \quad p'=0, \quad p''=-\frac{2a}{3};$$

whence we may derive the following construction:—Draw an *uncusped* secondary cyclode with a tail equal to one-third of the radius; unwind from this a ternary cyclode beginning from the apse, which will become a cusp in the cyclode so engendered; and from this last cyclode, beginning at its cusp, again unwind a new cyclode, which will possess a *triangular* point at the apse of its atavian secondary cyclode. This will be a quartic reducible cyclode, and, as regards *form* (irrespective of position and magnitude), the only one that exists. By the way, it may be noticed that a system of coordinates consisting of the vectorial angle and angle of contingence furnishes what may be termed a *form* equation, that is, one in which actual magnitude is ignored. Thus, for example, $\tan\theta=k\tan\phi$ is the *form* equation to a conic.

symmetrical and the other to non-symmetrical cyclodes*. This wonderful theory, this outlying and unexplored region of geometry, in which the two great continents of algebra and arithmetic trend towards and come into contact at more than one point with one another, forms the subject of a communication to be brought by the author of this Note before the Mathematical Society of London, simultaneously and under the same roof with Mr Norman Lockyer's announcement to the Royal Society of his equally, but not more surprising and certain to be prolific discovery of the sun's unsuspected chromosphere, the analogue of the ocean of forms of which the isolated power-forms $[(\phi^2 - n^2)^n]$ correspond to the piled-up rose-coloured prominences.

* It is to be understood that every x and y must be an *actual* integer, *zero* being for this purpose to be regarded, not as a number, but as a negation of number. Furthermore, if the x and y numbers are not only respectively equal each to each, but have all the same value (as for example, *unity*), the corresponding system of equations become *incompatible*; or, to speak more philosophically, the order of the system becomes *zero*, which here *per contra* ought to be regarded as a number rather than as a negation of number; for the order of the system of equations is always lowered, not only by every x becoming equal to every y, but also by any number of x's or of y's becoming equal to each other; so that the order of the system sinking to zero, in consequence of all the x's and all the y's becoming equal, is only an extreme instance of this general law. If we go to the wider case of algebraical spirals, where $p = \frac{f(\phi)}{F(\phi)}$, the difference between the degrees of f and F being still an even integer $2m$, where m is positive or negative, and require $p^2 + \left(\frac{dp}{d\phi}\right)^2$ to be made a perfect square, precisely the same method of solution is applicable as when F is of the degree zero. If we call the degrees of f and ϕ κ and k respectively, so that $\kappa - q = 2m$, we have to make

$$x_1 + x_2 + \ldots + x_\epsilon - \xi_1 - \xi_2 - \ldots - \xi_\eta = m,$$
$$y_1 + y_2 + \ldots + y_\lambda - \eta_1 - \eta_2 - \ldots - \eta_\mu = m,$$

$\epsilon + \eta = \lambda + \mu = i$, where i takes all possible values,

$$x_1 + x_2 + \ldots + x_\epsilon + y_1 + y_2 + \ldots + y_\lambda = \kappa,$$
$$\xi_1 + \xi_2 + \ldots + \xi_\eta + \eta_1 + \eta_2 + \ldots + \eta_\mu = q.$$

Every such system of partitions gives rise to a system of equations containing solutions of the diophantine problem in question, that is, the problem of making r a rational function of ϕ. When the degree of p in ϕ, that is, $\kappa - q$ (and consequently m) is zero, the order of all the equation-systems undergoes a marked depression.

100.

PRESIDENTIAL ADDRESS TO SECTION 'A' OF THE BRITISH ASSOCIATION.

[*Exeter British Association Report* (1869), pp. 1—9.]*

LADIES AND GENTLEMEN,—

A few days ago I noticed in a shop window the photograph of a Royal mother and child, which seemed to me a very beautiful group; on scanning it more closely, I discovered that the faces were ordinary, or, at all events, not much above the average, and that the charm arose entirely from the natural action and expression of the mother stooping over and kissing her child which she held in her lap; and I remarked to myself that the homeliest features would become beautiful when lit up by the rays of the soul—like the sun "gilding pale streams with heavenly alchemy." By analogy, the thought struck me that if a man would speak naturally and as he felt on any subject of his predilection, he might hope to awaken a sympathetic interest in the minds of his hearers; and, in illustration of this, I remembered witnessing how the writer of a well-known article in the *Quarterly Review* so magnetized his audience at the Royal Institution by his evident enthusiasm that, when the lecture was over and the applause had subsided, some ladies came up to me and implored me to tell them what they should do to get up the Talmud; for that was what the lecture had been about.

Now, as I believe that even Mathematics are not much more repugnant than the Talmud to the common apprehension of mankind, and I really love my subject, I shall not quite despair of rousing and retaining your attention for a short time if I proceed to read (as, for greater assurance against breaking down, I shall beg your permission to do) from the pages I hold in my hand.

It is not without a feeling of surprise and trepidation at my own temerity that I find myself in the position of one about to address this numerous and distinguished assembly. When informed that the Council of the British Association had it in contemplation to recommend me to the General Committee to fill the office of President of the Mathematical and Physical

[* The Address was also reprinted by the Author in a volume issued by Longmans, Green and Co., London, 1870, of which the earlier portion deals with the Laws of Verse. Some additional notes to the Address there given are reproduced at the end of this volume.]

Section, the intimation was accompanied with the tranquilizing assurance that it would rest with myself to deliver or withhold an address as I might think fit, and that I should be only following in the footsteps of many of the most distinguished of my predecessors were I to resolve on the latter course.

Until the last few days I had made up my mind to avail myself of this option, by proceeding at once to the business before us without troubling you to listen to any address, swayed thereto partly by a consciousness of the very limited extent of my oratorical powers, partly by a disinclination, in the midst of various pressing private and official occupations, to undertake a kind of work new to one more used to thinking than to speaking (to making mathematics than to talking about them), and partly and more especially by a feeling of my inadequacy to satisfy the expectations that would be raised in the minds of those who had enjoyed the privilege of hearing or reading the allocution (which fills me with admiration and dismay) of my gifted predecessor, Dr Tyndall, a man in whom eloquence and philosophy seem to be inborn, whom Science and Poetry woo with an equal spell*, and whose ideas have a faculty of arranging themselves in forms of order and beauty as spontaneously and unfailingly as those crystalline solutions from which, in a striking passage of his address, he drew so vivid and instructive an illustration.

From this lotus-eater's dream of fancied security and repose I was rudely awakened by receiving from the Editor of an old-established journal in this city a note containing a polite but peremptory request that I should, at my earliest convenience, favour him with a "copy of the address I proposed to deliver at the forthcoming Meeting." To this invitation, my first impulse was to respond very much in the same way as did the "Needy knife-grinder" of the *Antijacobin*, when summoned to recount the story of his wrongs to his republican sympathizer, "Story, God bless you, I have none to tell, Sir!" "Address, Mr Editor, I have none to deliver."

I have found, however, that increase of appetite still grows with what it feeds on, that those who were present at the opening of the Section last year,

* So it is said of Jacobi, that he attracted the particular attention and friendship of Böckh, the director of the philological seminary at Berlin, by the zeal and talent he displayed for philology, and only at the end of two years' study at the University, and after a severe mental struggle, was able to make his final choice in favour of mathematics. The relation between these two sciences is not perhaps so remote as may at first sight appear, and indeed it has often struck me that metamorphosis runs like a golden thread through the most diverse branches of modern intellectual culture, and forms a natural link of connexion between subjects in their aims so unlike as Grammar, Ethnology, Rational Mythology, Chemistry, Botany, Comparative Anatomy, Physiology, Physics, Algebra, Music, all of which, under the modern point of view, may be regarded as having morphology for their common centre. Even singing, I have been told, the advanced German theorists regard as being strictly a development of recitative, and infer therefrom that no essentially new melodic themes can be invented until a social cataclysm, or the civilization of some at present barbaric races, shall have created new necessities of expression and called into activity new forms of impassioned declamation.

and enjoyed my friend Dr Tyndall's melodious utterances, would consider themselves somewhat ill-treated if they were sent away quite empty on the present occasion, and that, failing an address, the Members would feel very much like the guests at a wedding-breakfast where no one was willing or able to propose the health of the bride and bridegroom.

Yielding, therefore, to these considerations and to the advice of some officially connected with the Association, to whose opinions I feel bound to defer, and unwilling also to countenance by my example the too prevailing opinion that mathematical pursuits unfit a person for the discharge of the common duties of life and cut him off from the exercise of Man's highest prerogative, "discourse of reason and faculty of speech divine,"—rather, I say, than favour the notion that we Algebraists (who regard each other as the flower and salt of the earth) are a set of mere calculating-machines endowed with organs of locomotion, or, at best, a sort of poor visionary dumb creatures only capable of communicating by signs and symbols with the outer world, I have resolved to take heart of grace and to say a few words, which I hope to render, if not interesting, at least intelligible, on a subject to which the larger part of my life has been devoted.

The President of the Association, Prof. Stokes, is so eminent alike as a mathematician and physicist, and so distinguished for accuracy and extent of erudition and research, that I felt assured I might safely assume he would, in his Address to the Association at large, take an exhaustive survey, and render a complete account of the recent progress and present condition and prospects of Mathematical and Physical Science. This consideration narrowed very much and brought almost to a point the ground available for me to occupy in this Section; and as I cannot but be aware that it is as a cultivator of pure mathematics (the subject in which my own researches have chiefly, though by no means exclusively lain*) that I have been placed in this Chair, I hope the Section will patiently bear with me in the observations I shall venture to make on the nature of that province of the human reason and its title to the esteem and veneration with which, through countless ages it has

* My first printed paper was on Fresnel's Optical Theory, published in the *Philosophical Magazine*; my latest contribution to the *Philosophical Transactions* is a memoir on the "Rotation of a Free Rigid Body." There is an old adage, "purus mathematicus, purus asinus." On the other hand, I once heard the great Richard Owen say, when we were opposite neighbours in Lincoln's-Inn Fields (doves nestling among hawks), that he would like to see *Homo Mathematicus* constituted into a distinct subclass, thereby suggesting to my mind sensation, perception, reflection, abstraction, as the successive stages or phases of protoplasm on its way to being made perfect in Mathematicised Man. Would it sound too presumptuous to speak of perception as a quintessence of sensation, language (that is, communicable thought) of perception, mathematic of language? We should then have four terms differentiating from inorganic matter and from each other the Vegetable, Animal, Rational, and Supersensual modes of existence.

been and, so long as Man respects the intellectual part of his nature, must ever continue to be regarded*.

It is said of a great party leader and orator in the House of Lords that, when lately requested to make a speech at some religious or charitable (at all events a non-political) meeting, he declined to do so on the ground that he could not speak unless he saw an adversary before him—somebody to attack or reply to. In obedience to a somewhat similar combative instinct, I set to myself the task of considering certain recent utterances of a most distinguished member of this Association, one whom I no less respect for his honesty and public spirit than I admire for his genius and eloquence†, but from whose opinions on a subject which he has not studied I feel constrained to differ. Goethe has said—

> "Verständige Leute kannst du irren sehn
> In Sachen, nämlich, die sie nicht verstehn."

Understanding people you may see erring—in those things, to wit, which they do not understand.

I have no doubt that had my distinguished friend, the probable President-elect of the next Meeting of the Association, applied his uncommon powers of reasoning, induction, comparison, observation, and invention to the study of mathematical science, he would have become as great a mathematician as he is now a biologist; indeed he has given public evidence of his ability to grapple with the practical side of certain mathematical questions; but he has not made a study of mathematical science as such, and the eminence of his position and the weight justly attaching to his name render it only the more imperative that any assertions proceeding from such a quarter, which may appear to me erroneous, or so expressed as to be conducive to error, should not remain unchallenged or be passed over in silence‡.

He says "mathematical training is almost purely deductive. The mathematician starts with a few simple propositions, the proof of which is so obvious that they are called self-evident, and the rest of his work consists of

* Mr Spottiswoode favoured the Section, in his opening address, with a combined history of the progress of Mathematics and Physics; Dr Tyndall's address was virtually on the limits of Physical Philosophy; the one here in print is an attempted faint adumbration of the nature of Mathematical Science in the abstract. What is wanting (like a fourth sphere resting on three others in contact) to build up the Ideal Pyramid is a discourse on the Relation of the two branches (Mathematics and Physics) to, their action and reaction upon, one another, a magnificent theme with which it is to be hoped some future President of Section A will crown the edifice and make the Tetralogy (symbolizable by $A+A'$, A, A', $A \,.\, A'$) complete.

† Although no great lecture-goer, I have heard three lectures in my life which have left a lasting impression as masterpieces on my memory—Clifford on Mind, Huxley on Chalk, Dumas on Faraday.

‡ In his *éloge* of Daubenton, Cuvier remarks, "Les savants jugent toujours comme le vulgaire les ouvrages qui ne sont pas de leur genre."

subtle deductions from them. The teaching of languages, at any rate as ordinarily practised, is of the same general nature—authority and tradition furnish the data, and the mental operations are deductive." It would seem from the above somewhat singularly juxtaposed paragraphs that, according to Prof. Huxley, the business of the mathematical student is from a limited number of propositions (bottled up and labelled ready for future use) to deduce any required result by a process of the same general nature as a student of language employs in declining and conjugating his nouns and verbs—that to make out a mathematical proposition and to construe or parse a sentence are equivalent or identical mental operations. Such an opinion scarcely seems to need serious refutation. The passage is taken from an article in *Macmillan's Magazine* for June last, entitled "Scientific Education—Notes of an After-dinner Speech," and I cannot but think would have been couched in more guarded terms by my distinguished friend had his speech been made *before* dinner instead of *after*.

The notion that mathematical truth rests on the narrow basis of a limited number of elementary propositions from which all others are to be derived by a process of logical inference and verbal deduction, has been stated still more strongly and explicitly by the same eminent writer in an article of even date with the preceding in the *Fortnightly Review*, where we are told that "Mathematics is that study which knows nothing of observation, nothing of experiment, nothing of induction, nothing of causation." I think no statement could have been made more opposite to the undoubted facts of the case, that mathematical analysis is constantly invoking the aid of new principles, new ideas, and new methods, not capable of being defined by any form of words, but springing direct from the inherent powers and activity of the human mind, and from continually renewed introspection of that inner world of thought of which the phenomena are as varied and require as close attention to discern as those of the outer physical world (to which the inner one in each individual man may, I think, be conceived to stand in somewhat the same general relation of correspondence as a shadow to the object from which it is projected, or as the hollow palm of one hand to the closed fist which it grasps of the other), that it is unceasingly calling forth the faculties of observation and comparison, that one of its principal weapons is induction, that it has frequent recourse to experimental trial and verification, and that it affords a boundless scope for the exercise of the highest efforts of imagination and invention.

Lagrange, than whom no greater authority could be quoted, has expressed emphatically his belief in the importance to the mathematician of the faculty of observation; Gauss has called mathematics a science of the eye, and in conformity with this view always paid the most punctilious attention to preserve his text free from typographical errors; the ever to be lamented

Riemann has written a thesis to show that the basis of our conception of space is purely empirical, and our knowledge of its laws the result of observation, that other kinds of space might be conceived to exist subject to laws different from those which govern the actual space in which we are immersed, and that there is no evidence of these laws extending to the ultimate infinitesimal elements of which space is composed. Like his master Gauss, Riemann refuses to accept Kant's doctrine of space and time being forms of intuition, and regards them as possessed of physical and objective reality. I may mention that Baron Sartorius von Waltershausen (a member of this Association) in his biography of Gauss ("Gauss zu Gedächtniss"), published shortly after his death, relates that this great man used to say that he had laid aside several questions which he had treated analytically, and hoped to apply to them geometrical methods in a future state of existence, when his conceptions of space should have become amplified and extended; for as we can conceive beings (like infinitely attenuated bookworms* in an infinitely thin sheet of paper) which possess only the notion of space of two dimensions, so we may imagine beings capable of realising space of four or a greater number of dimensions†. Our Cayley, the central luminary, the Darwin of the English school of mathematicians, started and elaborated at an early age, and with happy consequences, the same bold hypothesis.

Most, if not all, of the great ideas of modern mathematics have had their origin in observation. Take, for instance, the arithmetical theory of forms, of which the foundation was laid in the diophantine theorems of Fermat, left without proof by their author, which resisted all the efforts of the myriad-minded Euler to reduce to demonstration, and only yielded up their cause of being when turned over in the blowpipe flame of Gauss's transcendent genius; or the doctrine of double periodicity, which resulted from the observation by Jacobi of a purely analytical fact of transformation; or Legendre's law of reciprocity; or Sturm's theorem about the roots of equations, which, as he informed me with his own lips, stared him in the face

* I have read or been told that eye of observer has never lighted on these depredators, living or dead. Nature has gifted me with eyes of exceptional microscopic power, and I can speak with some assurance of having repeatedly seen the creature wriggling on the learned page. On approaching it with breath or finger-nail it stiffens out into the semblance of a streak of dirt, and so eludes detection.

† It is well known to those who have gone into these views that the laws of motion accepted as a fact suffice to prove in a general way that the space we live in is a flat or level space (a "homaloid"), our existence therein being assimilable to the life of the bookworm in an *unrumpled page*: but what if the page should be undergoing a process of gradual bending into a curved form? Mr W. K. Clifford has indulged in some remarkable speculations as to the possibility of our being able to infer, from certain unexplained phenomena of light and magnetism, the fact of our level space of three dimensions being in the act of undergoing in space of four dimensions (space as inconceivable to us as our space to the supposititious bookworm) a distortion analogous to the rumpling of the page to which that creature's powers of direct perception have been postulated to be limited.

in the midst of some mechanical investigations connected with the motion of compound pendulums; or Huyghens' method of continued fractions, characterized by Lagrange as one of the principal discoveries of "that great mathematician, and to which he appears to have been led by the construction of his Planetary Automaton"; or the New Algebra, speaking of which one of my predecessors (Mr Spottiswoode) has said, not without just reason and authority, from this Chair, "that it reaches out and indissolubly connects itself each year with fresh branches of mathematics, that the theory of equations has almost become new through it, algebraic geometry transfigured in its light, that the calculus of variations, molecular physics, and mechanics" (he might, if speaking at the present moment, go on to add the theory of elasticity and the highest developments of the integral calculus) "have all felt its influence."

Now this gigantic outcome of modern analytical thought, itself, too, only the precursor and progenitor of a future still more heaven-reaching theory, which will comprise a complete study of the interoperation, the actions and reactions, of algebraic forms (Analytical Morphology in its absolute sense), how did this originate? In the accidental observation by Eisenstein, some score or more years ago, of a single invariant (the Quadrinvariant of a Binary Quartic) which he met with in the course of certain researches just as accidentally and unexpectedly as M. Du Chaillu might meet a Gorilla in the country of the Fantees, or any one of us in London a White Polar Bear escaped from the Zoological Gardens. Fortunately he pounced down upon his prey and preserved it for the contemplation and study of future mathematicians. It occupies only part of a page in his collected posthumous works. This single result of observation (as well entitled to be so called as the discovery of Globigerinæ in chalk or of the Confoco-ellipsoidal structure of the shells of the Foraminifera), which remained unproductive in the hands of its distinguished author, has served to set in motion a train of thought and to propagate an impulse which have led to a complete revolution in the whole aspect of modern analysis, and whose consequences will continue to be felt until Mathematics are forgotten and British Associations meet no more.

I might go on, were it necessary, piling instance upon instance to prove the paramount importance of the faculty of observation to the process of mathematical discovery*. Were it not unbecoming to dilate on one's personal experience, I could tell a story of almost romantic interest about my

* Newton's Rule was to all appearance, and according to the more received opinion, obtained inductively by its author. My own reduction of Euler's problem of the Virgins (or rather one slightly more general than this) to the form of a question (or, to speak more exactly, a set of questions) in simple partitions was, strange to say, first obtained by myself inductively, the result communicated to Prof. Cayley, and proved subsequently by each of us independently, and by perfectly distinct methods.

own latest researches in a field where Geometry, Algebra, and the Theory of Numbers melt in a surprising manner into one another, like sunset tints or the colours of the dying dolphin, "the last still loveliest" (a sketch of which has just appeared in the *Proceedings of the London Mathematical Society**), which would very strikingly illustrate how much observation, divination, induction, experimental trial, and verification, causation, too (if that means, as I suppose it must, mounting from phenomena to their reasons or causes of being), have to do with the work of the mathematician. In the face of these facts, which every analyst in this room or out of it can vouch for out of its own knowledge and personal experience, how can it be maintained, in the words of Professor Huxley, who, in this instance, is speaking of the sciences as they are in themselves and without any reference to scholastic discipline, that Mathematics "is that study which knows nothing of observation, nothing of induction, nothing of experiment, nothing of causation"?

I, of course, am not so absurd as to maintain that the habit of observation of external nature will be best or in any degree cultivated by the study of mathematics, at all events as that study is at present conducted; and no one can desire more earnestly than myself to see natural and experimental science introduced into our schools as a primary and indispensable branch of education: I think that that study and mathematical culture should go on hand in hand together, and that they would greatly influence each other for their mutual good. I should rejoice to see mathematics taught with that life and animation which the presence and example of her young and buoyant sister could not fail to impart, short roads preferred to long ones, Euclid honourably shelved or buried "deeper than did ever plummet sound" out of the schoolboy's reach, morphology introduced into the elements of Algebra—projection, correlation, and motion accepted as aids to geometry—the mind of the student quickened and elevated and his faith awakened by early initiation into the ruling ideas of polarity, continuity, infinity, and familiarization with the doctrine of the imaginary and inconceivable.

It is this living interest in the subject which is so wanting in our traditional and mediaeval modes of teaching. In France, Germany, and Italy, everywhere where I have been on the Continent, mind acts direct on mind in a manner unknown to the frozen formality of our academic institutions schools of thought and centres of real intellectual cooperation exist; the relation of master and pupil is acknowledged as a spiritual and a lifelong tie, connecting successive generations of great thinkers with each other in an unbroken chain, just in the same way as we read, in the catalogue of our French Exhibition, or of the Salon at Paris, of this man or that being the pupil of one great painter or sculptor and the master of another. When followed out in this spirit, there is no study in the world which brings into

* Under the title of "Outline Trace of the Theory of Reducible Cyclodes." [Below, p. 663.]

more harmonious action all the faculties of the mind than the one of which I stand here as the humble representative, there is none other which prepares so many agreeable surprises for its followers, more wonderful than the changes in the transformation-scene of a pantomime, or, like this, seems to raise them, by successive steps of initiation, to higher and higher states of conscious intellectual being.

This accounts, I believe, for the extraordinary longevity of all the greatest masters of the Analytical art, the Dii Majores of the mathematical Pantheon. Leibnitz lived to the age of 70; Euler to 76; Lagrange to 77; Laplace to 78; Gauss to 78; Plato, the supposed inventor of the conic sections, who made mathematics his study and delight, who called them the handles or aids to philosophy, the medicine of the soul, and is said never to have let a day go by without inventing some new theorems, lived to 82; Newton, the crown and glory of his race, to 85; Archimedes, the nearest akin, probably, to Newton in genius, was 75, and might have lived on to be 100, for aught we can guess to the contrary, when he was slain by the impatient and ill-mannered sergeant, sent to bring him before the Roman general, in the full vigour of his faculties, and in the very act of working out a problem; Pythagoras, in whose school, I believe, the word mathematician (used, however, in a somewhat wider than its present sense) originated, the second founder of geometry, the inventor of the matchless theorem which goes by his name, the precognizer of the undoubtedly mis-called Copernican theory, the discoverer of the regular solids and the musical canon, who stands at the very apex of this pyramid of fame, (if we may credit the tradition) after spending 22 years studying in Egypt, and 12 in Babylon, opened school when 56 or 57 years old in Magna Græcia, married a young wife when past 60, and died, carrying on his work with energy unspent to the last, at the age of 99. The mathematician lives long and lives young; the wings of his soul do not early drop off, nor do its pores become clogged with the earthy particles blown from the dusty highways of vulgar life.

Some people have been found to regard all mathematics, after the 47th proposition of Euclid, as a sort of morbid secretion, to be compared only with the pearl said to be generated in the diseased oyster, or, as I have heard it described, "une excroissance maladive de l'esprit humain." Others find its justification, its "raison d'être," in its being either the torch-bearer leading the way, or the handmaiden holding up the train of Physical Science; and a very clever writer in a recent magazine article, expresses his doubts whether it is, in itself, a more serious pursuit, or more worthy of interesting an intellectual human being, than the study of chess problems or Chinese puzzles. What is it to us, they say, if the three angles of a triangle are equal to two right angles, or if every even number is, or may be, the sum of two primes, or if every equation of an odd degree must have a real root.

How dull, stale, flat, and unprofitable are such and such like announcements! Much more interesting to read an account of a marriage in high life, or the details of an international boat-race. But this is like judging of architecture from being shown some of the brick and mortar, or even a quarried stone of a public building, or of painting from the colours mixed on the palette, or of music by listening to the thin and screechy sounds produced by a bow passed haphazard over the strings of a violin. The world of ideas which it discloses or illuminates, the contemplation of divine beauty and order which it induces, the harmonious connexion of its parts, the infinite hierarchy and absolute evidence of the truths with which it is concerned, these, and such like, are the surest grounds of the title of mathematics to human regard, and would remain unimpeached and unimpaired were the plan of the universe unrolled like a map at our feet, and the mind of man qualified to take in the whole scheme of creation at a glance.

In conformity with general usage, I have used the word mathematics in the plural; but I think it would be desirable that this form of word should be reserved for the applications of the science, and that we should use mathematic in the singular number to denote the science itself, in the same way as we speak of logic, rhetoric, or (own sister to algebra*) music. Time was when all the parts of the subject were dissevered, when algebra, geometry, and arithmetic either lived apart or kept up cold relations of acquaintance confined to occasional calls upon one another; but that is now at an end; they are drawn together and are constantly becoming more and more intimately related and connected by a thousand fresh ties, and we may confidently look forward to a time when they shall form but one body with one soul. Geometry formerly was the chief borrower from arithmetic and algebra, but it has since repaid its obligations with abundant usury; and if I were asked to name, in one word, the pole-star round which the mathematical firmament revolves, the central idea which pervades as a hidden spirit the whole corpus of mathematical doctrine, I should point to Continuity as contained in our notions of space, and say, it is this, it is this! Space is the *Grand Continuum* from which, as from an inexhaustible reservoir, all the fertilizing ideas of modern analysis are derived; and as Brindley, the engineer, once allowed before a parliamentary committee that, in his opinion, rivers were made to feed navigable canals, I feel almost tempted to say that one principal reason for the existence of space, or at least one principal function which it discharges, is that of feeding mathematical invention. Everybody knows what a wonderful influence geometry has exercised in the hands of

* I have elsewhere (in my 'Trilogy' [above, p. 419, *footnote*] published in the *Philosophical Transactions*) referred to the close connexion between these two cultures, not merely as having Arithmetic for their common parent, but as similar in their habits and affections. I have called "Music the Algebra of sense, Algebra the Music of the reason; Music the dream, Algebra the waking life,—the soul of each the same!"

Cauchy, Puiseux, Riemann, and his followers Clebsch, Gordan, and others, over the very form and presentment of the modern calculus, and how it has come to pass that the tracing of curves, which was once to be regarded as a puerile amusement, or at best useful only to the architect or decorator, is now entitled to take rank as a high philosophical exercise, inasmuch as every new curve or surface, or other circumscription of space is capable of being regarded as the embodiment of some specific organized system of continuity*.

The early study of Euclid made me a hater of Geometry, which I hope may plead my excuse if I have shocked the opinions of any in this room (and I know there are some who rank Euclid as second in sacredness to the Bible alone, and as one of the advanced outposts of the British Constitution) by the tone in which I have previously alluded to it as a school-book; and yet, in spite of this repugnance, which had become a second nature in me, whenever I went far enough into any mathematical question, I found I touched, at last, a geometrical bottom: so it was, I may instance, in the purely arithmetical theory of partitions; so, again, in one of my more recent studies, the purely algebraical question of the invariantive criteria of the nature of the roots of an equation of the fifth degree: the first inquiry landed me in a new theory of polyhedra; the latter found its perfect and only possible complete solution in the construction of a surface of the ninth order and the subdivision of its infinite content into three distinct natural regions.

Having thus expressed myself at much greater length than I originally intended on the subject, which, as standing first on the muster-roll of the Association, and as having been so recently and repeatedly arraigned before the bar of public opinion, is entitled to be heard in its defence (if anywhere) in this place,—having endeavoured to show what it is not, what it is, and what it is probably destined to become, I feel that I must enough and more than enough have trespassed on your forbearance, and shall proceed with the regular business of the Meeting.

Before calling upon the authors of the papers contained in the varied bill of intellectual fare which I see before me, I hope to be pardoned if I direct attention to the importance of practising brevity and condensation in the delivery of communications to the Section, not merely as a saving of valuable time, but in order that what is said may be more easily followed and listened to with greater pleasure and advantage. I believe that immense good may be done by the oral interchange and discussion of ideas which takes place in the Sections; but for this to be possible, details and long descriptions should be reserved for printing and reading, and only the general outlines and broad

* M. Camille Jordan's application of Dr Salmon's Eikosi-heptagram to Abelian functions is one of the most recent instances of this reverse action of geometry on analysis. Mr Crofton's admirable apparatus of a reticulation with infinitely fine meshes rotated successively through indefinitely small angles, which he applies to obtaining whole families of definite integrals, is another equally striking example of the same phenomenon.

statements of facts, methods, observations, or inventions brought before us here, such as can be easily followed by persons having a fair average acquaintance with the several subjects treated upon. I understand the rule to be that, with the exception of the author of any paper who may answer questions and reply at the end of the discussion, no member is to address the Section more than once on the same subject, or occupy more than a quarter of an hour in speaking.

In order to get through the business set down in each day's paper, it may sometimes be necessary for me to bring a discussion to an earlier close than might otherwise be desirable, and for that purpose to request the authors of papers, and those who speak upon them, to be brief in their addresses. I have known most able investigators at these Meetings, and especially in this Section, gradually part company with their audience, and at last become so involved in digressions as to lose entirely the thread of their discourse, and seem to forget, like men waking out of sleep, where they were or what they were talking about. In such cases I shall venture to give a gentle pull to the string of the kite before it soars right away out of sight into the region of the clouds. I now call upon Dr Magnus to read his paper and recount to the Section his wondrous story on the Emission, Absorption, and Reflection of Obscure Heat*.

POSTSCRIPT.—The remarks on the use of experimental methods in mathematical investigation led to Dr Jacobi, the eminent physicist of St Petersburg, who was present at the delivery of the address, favouring me with the annexed anecdote relative to his illustrious brother C. G. J. Jacobi.

"En causant un jour avec mon frère défunt sur la necessité de contrôler par des expériences réitérées toute observation, même si elle confirme l'hypothèse, il me raconta avoir découvert un jour une loi très-remarquable de la théorie des nombres, dont il ne douta guère qu'elle fût générale. Cependant par un excès de précaution ou plutôt pour faire le superflu, il voulut substituer un chiffre quelconque réel aux termes généraux, chiffre qu'il choisit au hasard ou, peut-être, par une espèce de divination, car en effet ce chiffre mit sa formule en défaut; tout autre chiffre qu'il essaya en confirma la généralité. Plus tard il réussit à prouver que le chiffre choisi par lui par hasard, appartenait à un système de chiffres qui faisait la seule exception à la règle.

"Ce fait curieux m'est resté dans la mémoire, mais comme il s'est passé il y a plus d'une trentaine d'années, je ne rappelle plus des détails.

"M. H. JACOBI."

"EXETER, 24. *Août*, 1869."

* Curiously enough, and as if symptomatic of the genial warmth of the proceedings in which seven sages from distant lands (Jacobi, Magnus, Newton, Janssen, Morren, Lyman, Neumayer) took frequent part, the opening and concluding papers (each of surpassing interest, and a letting-out of mighty waters) were on Obscure Heat, by Prof. Magnus, and on Stellar Heat, by Mr Huggins.

101.

ON PROFESSOR CHRISTIAN WIENER'S STEREOSCOPIC REPRESENTATION OF THE CUBIC EIKOSI-HEPTAGRAM.

[*Exeter British Association Report* (1869), p. 15.]

THE author produced stereographic drawings sent over to him by Professor Christian Wiener, of Karlsruhe, of the famous complex of 27 right lines lying on a cubic surface, discovered by Salmon and rediscovered by Steiner. Dr Wiener, at the request of Professor Clebsch, of Heidelberg, had actually built up a suitable cubic surface, and marked the lines in colours upon it; from this model the stereograms produced had been photographed.

102.

ON THE SUCCESSIVE INVOLUTES TO A CIRCLE.

[*Exeter British Association Report* (1869), p. 15.]

THE author referred to his communication to the Section at the Meeting held last year at Norwich "On the General Theory of the Successive Involutes to a Circle now called Cyclodes," and went on to give an account of a particular kind of cyclode, the simplest of their respective orders, which, from the lowering of the degree which takes place in their arco-radial equation, are termed reducible cyclodes. He referred to his researches for determining their number and groupings for any order of derivation, and to a new class of theorems in the Partition of Numbers in which these researches have eventuated.

A sketch of his conclusions is contained in a Number recently published of the *Proceedings of the London Mathematical Society,* copies of which were distributed among the Members of the Section present [p. 663, below].

103.

OUTLINE TRACE OF THE THEORY OF REDUCIBLE CYCLODES*, THAT IS A PARTICULAR FAMILY OF SUCCESSIVE INVOLUTES TO A CIRCLE WHOSE DETERMINATION DEPENDS ON THE SOLUTION OF AN ALGEBRAICO-DIOPHANTINE EQUATION, AND OF THE NUMBER AND CLASSIFICATION OF THE FORMS OF SUCH FAMILY FOR ANY GIVEN ORDER OF SUCCESSION†.

[*Proceedings of the London Mathematical Society*, II. (1869), pp. 137—160.]

La conquête de la vérité exige le concours harmonique de toutes les facultés humaines.

ALBERT RÉVILLE.

(1) A CYCLODE is the continued [nth] involute of a circle. The centre of the circle is the pole of the cyclode.

(2) If x, y are the *inclination* and *polar distance* of the tangent to a cyclode, $y = (x, 1)^n$ is its equation, where n is the number of unwindings, or,

* See *Philosophical Magazine* (October and December, 1868) [pp. 630, 641, above] for preliminary notions on involutes in general and on the successive involutes to a circle. Attached to the second of these notes is a valuable plate containing beautifully drawn specimens of cyclodes of the first three orders, executed by Mr Spottiswoode.

† It may be right to mention that the results arrived at in the denumerational portion of this paper have been obtained almost exclusively by processes of observation, comparison, experiment, and induction, satisfying the sense of the perfect and beautiful; and consequently the theory must be regarded as essentially non-mathematical, it having been authoritatively laid down by a great naturalist and eminent public instructor in the *Fortnightly Review* for June 1869, that Mathematics is "that study which knows nothing of observation, nothing of experiment, nothing of induction, nothing" [does any science know much?] "of causation." Lagrange has expressed himself pointedly (in a passage which I cannot at the moment recall) in the very opposite sense to Professor Huxley; so certainly would Fermat, Euler, Gauss, Jacobi, Abel, Cauchy, Eisenstein, Kronecker, Plücker, Riemann, and the other great masters of our art, had their opinion on the subject been appealed to. In a subjective sense this paper owes its origin to my attention having been drawn by the Rev. James White, of Trinity Church, Woolwich, to Capt. Moncrieff's self-reversing gun-carriage, the rack in which for steadying and regulating the motion is the curve which would be traced on the plane of a wheel rolling on a rail by a point fixed on above or below the rail. When the point is level with the centre of the wheel, the curve traced is Archimedes' Spiral, when level with the rail the first Involute of the circle, and when midway between the rail and the centre a *Square root* (in the quaternion sense) of that Second

so to say, the order of the involution. When the radius of the circle is unity and the axis of reference is the line joining the pole to the cusp of the $(n-1)$th evolute of the cyclode (that is the first involute of the generating circle), the coefficients of x^n, x^{n-1} become 1 and 0 respectively. In what follows, it is to be understood that in general this simplified form is the one referred to.

(3) If r is radius, ρ radius of curvature, s length of arc reckoned from any origin on the curve,

$$r^2 = y^2 + \left(\frac{dy}{dx}\right)^2,$$

$$s = \int \rho dx = \int y dx + \frac{dy}{dx}.$$

Hence every cyclode is rectifiable, s and r^2 being unicursally related.

(4) The arco-radial equation is of the form $F(s, r^2) = 0$. When $y^2 + y'^2 = \square$, s and r become unicursally related, so that

$$F(s, r^2) = \Phi(s, r)\,\Phi(s, -r),$$

and the cyclode is said to be *reducible**.

(5) Order is the degree of the parabolic function $(x, 1)^n$. Class is the number of distinct factors in $(x, 1)^n$, that is the number of *distinct* tangents that can be drawn to a cyclode from its pole; for in the case of such tangents $y = 0$.

(6) Reducible cyclodes form a *family*: the *class* and the *order* of every reducible cyclode must both be *even*. They are of two *kinds*, symmetrical and unsymmetrical. The simplified form for the symmetrical kind is $(x^2, 1)^\nu$.

involute to the circle whose apse is midway between the centre of the circle and the cusp of the first involute, to which second involute or cyclode I have given the name of the Norwich Spiral, to mark the fact of its birth at the Meeting of the British Association held last year in that city. The theory has since grown upwards and outwards faster than I have been able to climb after it, like the *bean-stalk* of profound significance in the well-known child's story. A very elegant and simple (because philosophically conceived) solution of the problem of finding the form of the rack (which I call a *Convolute* to the circle, and which the Arsenal people mistakenly but naturally enough supposed to be a Cycloidal curve) was given by Mr White in the *Educational Times* (*vide* Reprint for latter half of 1868). The problem was originally proposed many years ago by Mr Earnshaw, as a Senate-House problem, suggested to him by seeing a boy trundle a wheel against a brick wall in the streets of Cambridge.

* σ (the length of the arc of the pedal in respect of the pole) is $\int dx \, . \, r$; so that s, σ, x, y, r are all unicursally related in the case of *reducible* cyclodes. The equation in the text, $y^2 + \left(\frac{dy}{dx}\right)^2$ = a perfect square, is the algebraico-diophantine equation referred to in the title. A more general equation, towards the solution of which I have made a few steps, is the following—

$$y^2 + y'^2 = \eta^2 + \lambda\eta'^2;$$

but I do not at present perceive its geometrical bearing.

A symmetrical cyclode is divided into two equal and similar parts by the line joining the pole to the cusp of its $(n-1)$th evolute. Non-symmetrical reducible cyclodes are left- and right-handed; they appear in pairs, the equations to the two of a pair being $y=Fx$, $y=F(-x)$ respectively.

(7) Let ν be divided in all possible ways (say ω ways) into a given number μ of parts; let the *partitionments* be $P_1, P_2, \dots P_\omega$. Then $(P_1, P_2, \dots P_\omega)^2$ developed contains terms of the form $P_j . P_j$, and of the form $P_j P_k$, $P_k P_j$. Every such binary combination may be called a *Diptych* of a group of reducible cyclodes of order 2ν and class 2μ; and the total of such groups will be the total of reducible cyclodes of order 2ν and class 2μ. The sum of these totals for all classes from 2 to 2ν will be the *ensemble* of reducible cyclodes of order 2ν.

(8) The *alæ* of a diptych are the two sets of terms which form its two sides. These *alæ* will sometimes be represented as folded together, and sometimes expanded, as convenience may dictate. Under these *alæ* each diptych, as will be seen, shelters a group or brood of cyclodes. The relation between the *alæ* and the parabolic functions $(x, 1)^n$ of the reducible cyclodes proper to the *diptych* is as follows. Suppose that

$$i, j, k, \dots \mid i_1, j_1, k_1, \dots$$

is the diptych. Then by hypothesis

$$i+j+k \dots = i_1+j_1+k_1 \dots = \nu,$$

the semi-order.

$i, j, k \dots$ are the same in number (that number being μ the semi-class) as $i_1, j_1, k_1, \dots$

$y = UU_1$* is the equation to every cyclode proper to the given diptych, where

$$U = (x+a)^i\ (x+b)^j\ (x+c)^k \dots$$
$$U_1 = (x+a_1)^{i_1} (x+b_1)^{j_1} (x+c_1)^{k_1} \dots$$

To find $a, b, c, \dots$; $a_1, b_1, c_1, \dots$ write

$$V = (\lambda+a)(\lambda+b)(\lambda+c)\dots, \qquad V_1 = (\lambda+a_1)(\lambda+b_1)(\lambda+c_1)\dots,$$

$$E = \left(2i\frac{d}{da} + 2j\frac{d}{db} + 2k\frac{d}{dc}\dots\right), \quad E_1 = \left(2i_1\frac{d}{da_1} + 2j_1\frac{d}{db_1} + 2k_1\frac{d}{dc_1}\dots\right),$$

and let $EV = E_1V_1 = V - V_1$ for all values of λ. This implies

$$2\Sigma i = 2\Sigma i_1 = \Sigma a - \Sigma a_1,$$
$$\Sigma 2i\,(b+c\dots) = \Sigma 2i_1\,(b_1+c_1+\dots) = \Sigma ab - \Sigma a_1 b_1,$$
$$\Sigma 2i\,(bc\ \dots\ ) = \Sigma 2i_1\,(b_1 c_1\ \dots\ ) = \Sigma abc - \Sigma a_1 b_1 c_1,$$

....

....

* The U and U_1 may be called the two alar constituents, or more simply the segments of the parabolic function appertaining to any reducible cyclode; and we see that the radial tangents (or tangent radii from the centre) of every such cyclode constitute two distinct groups corresponding to these segments.

The equality $\Sigma i = \Sigma i_1$ has been presupposed between the elements of the diptych. Remain $2\mu - 1$ equations between the 2μ quantities $a, b, c, \ldots a_1, b_1, c_1, \ldots$.

Hence the quantities themselves are, as they should be, indeterminate; but their differences will be determinate (as is easily proved)*. If we employ the simplified form, then in addition to the $2\mu - 1$ equations above given, there will be the equation

$$\Sigma a + \Sigma a_1 = 0,$$

which will make the system perfectly determinate.

(9) If $i, j, k, \ldots i_1, j_1, k_1$ are all unequal, the number of solutions will be

$$1 \times 1 . 2 \times 2 . 3 \ldots \times (\mu - 1) . \mu,$$

that is $\Pi(\mu - 1)\Pi(\mu)$.

(10) The effect of interchanging the *alæ* of the diptych (when unsymmetrical) is to convert every a into $-a$, and every a_1 into $-a_1$; but when the *alæ are alike* this change leaves the solutions unaltered. The solutions proper to a conformable diptych (that is one in which the *alæ* agree term for term) resolve themselves into two groups:—one group in which

$$a = -a_1, \quad b = -b_1, \quad c = -c_1, \ldots,$$

which is the group of symmetrical reducible cyclodes; and a second group, in which the solutions appear in *pairs* of the form

$$\begin{array}{cccccccc} a, & b, & c, & \ldots & a_1, & b_1, & c_1, & \ldots \\ -a_1, & -b_1, & -c_1, & \ldots & -a, & -b, & -c, & \ldots \end{array}$$

Thus non-symmetrical reducible cyclodes are of two sorts:—congeminate, where the left- and right-handed curves belong to the same system of equations; and contrageminate, where the left- and right-handed curves belong to opposite systems. The former are associated with the symmetrical species, inasmuch as they belong to diptychs which engender along with them the symmetrical species; the latter belong to unconformable diptychs, that is with dissimilar *alæ*†. We now see more clearly the exactitude of the representation before stated of the diptychs of a given class by $(P_1, P_2, \ldots P_\omega)^2$, and the meaning of the coefficient 2 which affects the product $P_j . P_k$.

(11) The equation-system for unsymmetrical reducible cyclodes may be

* The property of *reducibility* is of course unaffected by the rotation of a cyclode round the pole; hence, not the roots of the parabolic function, but their differences ought to determine the existence of this property.

† The equation-system belonging to a conformable diptych may be resolved into two distinct systems containing respectively the symmetrical and the non-symmetrical solutions.

said to be *dualistic*. For symmetrical cyclodes it is *monistic* *, and takes the form

$$\begin{aligned} \nu &= \Sigma a, \\ \Sigma i\,(b+c+d\,\ldots\,) &= 0, \\ \Sigma i\,(bc+bd+cd\,\ldots\,) &= \Sigma abc, \\ \Sigma i\,(bcd\,\ldots\,) &= 0, \\ \Sigma i\,(bcde\,\ldots\,) &= \Sigma abcde, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

Hence if $i, j, k, \ldots$ are all unlike, the number of solutions for a monistic system belonging to a diptych of the μth class † is the product of μ terms of the fluctuating progression

$$1\,.\,1\,.\,3\,.\,3\,.\,5\,\ldots$$

(12) The number of solutions of a determinate algebraic system of equations may be called its denumerant. The denumerant may be algebraical or arithmetical.

In estimating the former, all solutions count, whether or not deducible from one another by interchange between the unknowns.

In estimating the latter, solutions which become identical by permuting the unknowns are regarded as one and the same solution. Thus, for example, the algebraical denumerant of $[x^2+y^2=c,\ x^4+y^4=e]$ is 8, its arithmetical denumerant is 4.

In the theory under consideration, it is always the arithmetical ‡ denumerant which is intended, unless the contrary is expressed. Thus we see

* The *dualistic* system for a conformable diptych resolves itself into the *monistic* system above given, which contains the symmetrical solutions, and another system (with which I do not think it desirable to encumber this sketch, but) which I have easily succeeded in isolating, and which contains the congeminate solutions. The congeminate denumerants do not, however, appear to lend themselves to a distinct law of summation apart from the general mass as the symmetrical ones do.

† It will be convenient to refer to the diptych of a cyclode of the order 2ν and class 2μ as being itself of the order ν and class μ.

‡ It is a singular and advantageous fact, quite contrary to what might have been expected from analogy, that it is these denumerants (corresponding to the actual number of distinct geometrical forms) and not the algebraical denumerants (the ordinary measures of the so-called orders of the various systems of equations) which lend themselves to summation. In the common theory of partitions *per contra*, if we wish to express their number algebraically, we must reckon permuted as distinct arrangements; this is as if we should substitute algebraical for arithmetical denumeration in the theory under consideration. This fact is partly to be accounted for by considering that the diophantine equation to be solved, namely, $f^2+f'^2=\phi^2$, may be attacked *exoscopically*, the coefficients in f and ϕ being posited as the unknowns, in which scheme of procedure permutations of equal factors in f would not present themselves at all: my original tentative efforts were made in this the seemingly most natural direction to follow;

that one cause of the *Reduction* of the denumerant of a diptychical system of equations is the *consolidation* of several solutions into a single one. A second cause of reduction, arising from roots passing off to infinity, may be termed *evaporation.* For example, in the equation-system $[x - ky = c, x^3 - y^3 = e]$ the denumerant is 3; but when $k = 1$, one root passes off to infinity and the denumerant becomes 2.

Consolidation and evaporation may go on simultaneously: thus, for example, in the system $[x + ky = c,\ x^3 + y^3 = e]$, when $k = 1$, the denumerant is reduced first by evaporation to 2, and then by consolidation to 1.

(13) The problem of which I shall indicate a complete method of solution is that of obtaining the denumerant to any given diptych whatever.

The original problem proposed for solution was to find the *total* number of reducible cyclodes of any given order; not only this, but the number of the same of any given *class,* as well as *order,* admits of easy *statement.*

If $D(n, m)$ be used to denote the total number of reducible cyclodes of the order $2n$ and class $2m$ (obtainable from a given circle), then

$$D(r+s, r) = D(r+s, s) = \frac{\Pi(r+s-1)\,\Pi(r+s-2)}{\Pi r\,\Pi(r-1)\,\Pi s\,\Pi(s-1)}.$$

Calling $D(r+s, r) = Q_{r,s}$, the annexed Table exhibits the values of $Q_{r,s} = Q_{s,r}$* for different values of r and s.

r \ s	1	2	3	4	5	6
Q						
1	1	1	1	1	1	1
2	1	3	6	10	15	21
3	1	6	20	50	105	196
4	1	10	50	175	498	1176
5	1	15	105	496	1764	5292
6	1	21	196	1176	5292	19404

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but how futile it would have been to have persevered in this course, and how all but impossible to disentangle, by aid of it, the classes and genera and diptychical groups, is apparent from the very nature of the results here brought to light. Of all the analytical questions left unsolved by my predecessors or contemporaries which I have yet grappled with (and they are neither few nor facile), this presents beyond all comparison the hardest knot I have ever succeeded in untying. But what will not yield to patient contemplation! Nature, though coy, is kind, and does not predetermine to mock the pursuit of her worshippers.

* That $Q_{r,s} = Q_{s,r}$ implies $D(n, r) = D(n, n-r)$,—a truly remarkable theorem, which exists also when Δ replaces D.

$D(m, n)$ will consist of a certain number of symmetrical cyclodes, say $\Delta_{m,n}$, and $\dfrac{D(m, n) - \Delta(m, n)}{2}$ pairs of unsymmetrical cyclodes, giving a total of $\dfrac{D(m, n) + \Delta(m, n)}{2}$ absolutely distinct forms when left- and right-handed similar curves are treated as identical.

(14) Let $$\Delta(r+s, r) = q_{r,s},$$
then $q_{r,s}\,(=q_{s,r})$ is the quantity in the rth line and sth column of the Table B subjoined, deduced by zigzag multiplication from the adjacent Table A*.

TABLE A.

1	1	1	1	1	1
1	2	3	4	5	6
1	2	3	4	5	6
1	3	6	10	15	21
1	3	6	10	15	21
1	4	10	20	35	56

TABLE B.

r \ s	q† 1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	1	2	2	3	3
3	1	2	4	6	9	12
4	1	2	6	9	18	24
5	1	3	9	18	36	60
6	1	3	12	24	60	100

(15) $D(n)$, $\Delta(n)$ being used to denote the absolute number of reducible cyclodes and the number of symmetrical reducible cyclodes, respectively, of the order $2n$ of all classes, we have

$$D(n) = \frac{\Pi(2n-2)}{\Pi n \,.\, \Pi(n-1)},$$

$$\Delta(2k+1) = \frac{\Pi(2k)}{(\Pi k)^2}, \quad \Delta(2k) = \tfrac{1}{2}\Delta(2k+1).$$

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* Calling
$$\left.\begin{array}{l} 1,\ 1,\ \ 1,\ \ 1,\ \ 1,\ \ldots = L \\ 1,\ 2,\ \ 3,\ \ 4,\ \ 5,\ \ldots = L_1 \\ 1,\ 3,\ \ 6,\ 10,\ 15,\ \ldots = L_2 \\ 1,\ 4,\ 10,\ 20,\ 35,\ \ldots = L_3 \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\},$$
the Table A itself is the product of zigzag multiplication of
$$\begin{Bmatrix} L & L_1 & L_2 & L_3 & L_4 & \ldots \\ 1 & 1 & 1 & 1 & 1 & \ldots \end{Bmatrix}.$$

† It is easily seen that when

$$r = 2h+1,\ s = 2k+1;\ q_{r,s} = \left(\frac{\Pi(h+k)}{\Pi h \Pi k}\right)^2,$$

$$r = 2h,\qquad s = 2k+1;\ q_{r,s} = \frac{\Pi(h+k)\,\Pi(h+k-1)}{\Pi h \Pi(h-1)\,(\Pi k)^2},$$

$$r = 2h,\qquad s = 2k;\qquad q_{r,s} = \frac{\{\Pi(h+k-1)\}^2}{\Pi(h-1)\,\Pi h \Pi(k-1)\,\Pi(k)}.$$

The total number of reducible cyclodes and of reducible symmetrical cyclodes for all orders from 2 to 20 inclusive is exhibited in the Table below, where n is the semi-order.

n	Δ	D
1	1	1
2	1	1
3	2	2
4	3	5
5	6	14
6	10	42
7	20	132
8	35	429
9	70	1430
10	125	5236

The ratio of the successive values of Δ alternates between 2 and a variable quantity which converges ascendingly towards 2; that of the successive values of D converges (also ascendingly) towards 4. It will be observed that D and Δ agree, until $2n = 8$, showing that the non-symmetrical reducible cyclodes only begin to make their appearance with the eighth order of involution, that is after the generating tight string has been unwound 8 times in succession.

(16) The *number of distinct species of diptychs of any given* class *is limited,* being independent of the order; *and every diptych of such species will have the same denumerant.* The doctrine of *genus and species* is founded on the notion of *parallel equalities*.*

A parallel equality is one existing between the sum of any number of elements in one *ala* of a diptych, and the sum of a like number of elements in

* The foundation of the *conception* of the doctrine of parallel equalities lies in the easily demonstrable fact that the leading coefficient in the final resolving equation of the *monistic* system of equations proper to any conformable diptych $(a, b, c \ldots l)^2$ is made up exclusively of powers of linear factors of the form $a-b$, $a+b-c-d$, $a+b+c-d-e-f$, &c., and that the same for the *dualistic* system proper to a general diptych $[a, b, c \ldots l \mid \alpha, \beta, \gamma \ldots \lambda]$ is made up exclusively of powers of linear factors of the form $a-\alpha$, $a+b-\alpha-\beta$, $a+b+c-\alpha-\beta-\gamma$, &c.: this fact is sufficient for proving that evaporation can only commence to take place by virtue of the existence of one or more parallel equalities, but is not sufficient to demonstrate the truth of the grand law that not only the *existence*, but the *amount* of evaporation is governed solely and exclusively by the nature of the parallel equality or system of parallel equalities (when several coexist either separately or mutually superimposed) in any given diptych. For a proof of what is stated in this note in respect of the leading coefficient of the monistic system, see *Phil. Mag.* for May 1869 [p. 689, below]; I am also in possession of the corresponding proof and theory for the dualistic system.

the opposite *ala*; that is, it is an inter-alar equinomial relation of equality to which it will sometimes be convenient to refer to as a *parallelism.* Two antagonistic parties of creditors to a bankrupt's estate, equal in number and claiming for equal amounts, gives an image of the parallelism in question. By the definition of a diptych there is necessarily one such parallel equality, namely between all the elements in one *ala* and all in the other*.

Any two diptychs in which the entire system of parallel equations is the same for each are said to belong to the same *genus,* and their *algebraical* denumerants will be the same. When in two diptychs belonging to the same genus all the equalities between the elements of each *ala* taken alone in the one are matched by like equalities between the analogous elements of the corresponding *ala* in the other, they are said to belong to the same *species,* and their *arithmetical* denumerants will be identical†.

(17) Now let it be proposed to determine the denumerants, for greater simplicity, say the total denumerants (including non-symmetrical as well as symmetrical cyclodes), of the several species belonging to any given class. There are as many unknowns, say, $d_1, d_2, \dots d_s$, as there are species (s) included in such class: the total number of diptychs of that class belonging to any given order increases with the order, being represented by $(P_1, P_2, \dots P_\omega)^2$, where $P_1, P_2, \dots P_\omega$ denote the partitionments of the number n into m parts. For every assumed value of n we shall be able to obtain the value of $D(P_1, P_2, \dots P_\omega)^2$ as a known linear function of $d_1, d_2, \dots d_s$; but $D(P_1, P_2, \dots P_\omega)^2$ is a known function of n and m‡; hence we may obtain the values of a sufficient number of linear functions of $d_1, d_2, \dots d_s$ to obtain each of them separately; and we know *à priori* that each is integer and positive.

Thus, regarding the denumerants as gas in relation to the equation-systems viewed as the gross matter from which this gas is extracted (a simile suggested by Mr Clifford), we witness in this theory the interesting phenomenon of this gas becoming again condensed, as it were (like solidified carbonic acid gas), into a sort of crystallized matter finer than that from

* Consequently any partial inter-alar parallel equality implies a second complementary one; in other words, all the parallel equalities, excepting the implied total one, are conjugate and make their appearance in pairs.

† Thus we see that the elements of the diptych being all increased or diminished by the same quantity, or, more generally, being replaced respectively by the same linear function of each of them, will not affect the genus or species, as such substitution will leave the identities and parallel equalities unaltered; so that we may say that the species of a Reducible Cyclode is, as it were, a semi-invariant of the indices of the factors of its parabolic function.

‡ See Art. 13, where $r=m$, $s=n-m$, and $D(r+s, r)$ means the same as $D(P_1, P_2, \dots P_\omega)^2$ in the text above.

which it was originally generated, the denumerants of the original equation-system becoming the unknowns of a new (linear) equation-system.

[Chasles's theory (with Zeuthen's, Cayley's, and Salmon's extensions) of conics which satisfy given conditions, the new theory of geometric Unicursal or successive Quadric Transformation, and this theory of Cyclodes, all belong to a peculiar branch of algebra to which might be given the name of pneumatic analysis; and they have many features in common.]

In the same way it may obviously be shown that the denumerants of the *symmetrical* cyclodes proper to the distinct species of conformable diptychs of any given class may be deduced from the solution of a system of linear equations as many in number as such species. In fact, however, only as many linear functions (in either case) need to be formed as there are distinct *genera*; for the denumerants (arithmetical) of the several *species* of the same genus are known, *à priori*, in terms of each other, since their algebraical denumerants are identical.

(18) A most important and wonderful relation exists between $D(r+s, r)$ or $\Delta(r+s, r)$ and the corresponding denumerants of the conformable diptych $[a^r, b^s]^2$, where a^r means a, r times repeated, and b^s means b, s times repeated. [Observe that any such diptych whose class $r+s$ is given belongs to a single genus; for we cannot have

$$\rho a + \sigma b = \rho' a + \sigma' b \quad \text{and} \quad \rho + \sigma = \rho' + \sigma',$$

unless $a = b$, which is of course excluded.] Namely *the total denumerant of this diptych is equal to* $D(r+s, r)$, *and its special denumerant is equal to* $\Delta(r+s, r)$.

(19) A very good example of *evaporation* is afforded by the diptych $[a^r]^2$; for the algebraical denumerant without evaporation would be $\Pi r \,.\, \Pi(r-1)$; but the actual algebraical denumerant must contain $(\Pi r)^2$. Hence the evaporation is *total*, that is there is no reducible cyclode of any order or class whose parabolic function is made up of distinct factors raised all to the same power (and, as a particular case, made up exclusively of distinct simple factors), except when the cyclode is of the second degree, for $\Pi r \,.\, \Pi(r-1)$ contains $(\Pi r)^2$ when $r=1$; and in fact the equation of the Norwich spiral, which gave birth to this theory, is in its simplified form $y=(x+1)(x-1)$. This is analogous to the multiplicity (in Hirst's and Cremona's theory) of the *principal points* of any nucleus (as such system is termed by Professor Cayley) when, and not before, the degree of the transformation transcends the second. In the present theory we see that, except for the Norwich spiral, in all reducible cyclodes a certain number of the points, the tangents to which pass through the pole, must be cusps of a more or less elevated order.

(20) As an example of the method of classification, grounded on parallel equalities, I subjoin the complete system of genera and species of diptychs of the third (corresponding to cyclodes of the 6th) class, with their corresponding denumerants*.

Genus (1)	Denumerant	Genus (2)	Denumerant	Genus (5)	Denumerant
$a\ b\ c$ $a\ \beta\ \gamma$	12	$a\ b\ c$ $a\ \beta\ \gamma$	10	$a\ b\ c$ $a\ a\ \gamma$	4
$a\ b\ c$ $a\ \beta\ \beta$	6	$a\ b\ c$ $a\ \beta\ \beta$	5	$a\ b\ b$ $a\ a\ \gamma$	2
$a\ b\ c$ $a\ a\ a$	2	Genus (3)† $a\ b\ c$ $a\ b\ c$	7	Genus (6) $a\ b\ c$ $a\ a\ a$	
$a\ b\ b$ $a\ a\ \beta$	3				
$a\ b\ b$ $a\ a\ a$	1	Genus (4) $a\ b\ b$ $a\ b\ b$	1	Genus (7) $a\ a\ a$ $a\ a\ a$	0

Of the above 7 genera, the 3rd and 4th alone give rise to symmetrical cyclodes, the Δ (that is the symmetrical) denumerant for the 3rd being 3 and the 4th 1.

(21) All species of conformable diptychs necessarily belong to distinct genera; that is the genera of conformable diptychs each contain only a single species. The conformable species of the fourth‡ class are as follows: [the numbers under Δ signify the number of *symmetrical* cyclodes proper to each genus, and under E the amount of evaporation].

* The *algebraical* denumerant, it will be noticed, is the same for all the species of the same genus. Thus, for example, in genus (1) for the first species this is 12, for the second $\Pi 2 . 6$, for the third $\Pi 3 . 2$, for the fourth $\Pi 2 . \Pi 2 . 3$, and for the last $\Pi 2 . \Pi 3 . 1$. The first genus, it will be seen, contains four, the second and fifth two, the third, fourth, sixth, and seventh one *species* respectively.

† The *special* denumerant means the number of *symmetrical* cyclodes contained by a diptych. The diptych $(a, b, c)^2$ having the special denumerant 3 and the total denumerant 7, we see that the number of congeminate pairs is $\frac{7-3}{2}$, that is, 2. So the number of congeminate *pairs* to $(a, b, c, d)^2$ is 36; and I have almost conclusive grounds for believing that in general the value of $D - \Delta$ for $(a_1, a_2, \ldots a_i)^2$ is $(i-2)\{\Pi(i-1)\}$; so that the number of right- and left-handed (congeminate) pairs contained in such diptych is

$$\frac{(i-2)\{\Pi(i-1)\}^2}{2}.$$

‡ There are upwards of 40 distinct *genera* of the 4th class when unconformable diptychs are taken into account. It is a problem of great interest to determine the number of genera and species contained in any given class.

Species	Δ	E†
$(a\ b\ c\ d)^2$	9	0
$(a\ b\ b\ d)^2$	4	1
$(a\ a\ c\ c)^2$	1	5
$(a\ b\ b\ b)^2$	1	3
$(a\ a\ a\ a)^2$	0	9
$(a\ b\ c\ b+c-a)^2$	7	2
$(a\ b\ b\ 2b-a)^2$	3	3

(22) The conformable species of the fifth class will be most easily represented concretely by means of actual numbers, so as to avoid the necessity of employing equations to denote the parallel equalities‡.

Types of species	Δ	E
$(1\ 2\ 3\ 5\ 8)^2$	45	0
$(1\ 2\ 3\ 5\ 5)^2$	21	3
$(1\ 2\ 3\ 3\ 3)^2$	6	9
$(\dot{1}\ \dot{2}\ \dot{2}\ 2\ \dot{3})^2$	5	15
$(1\ 2\ 2\ 5\ 5)^2$	9	9
$(1\ 1\ 2\ 2\ 2)^2$	2	21
$(1\ 1\ 1\ 1\ 2)^2$	1	21
$(1\ 1\ 1\ 1\ 1)^2$	0	45
$(\dot{1}\ \dot{2}\ \dot{3}\ \dot{4}\ 8)^2$	43	2
$(\dot{1}\ \overset{\cdot\prime}{2}\ \overset{\cdot\prime}{4}\ \overset{\cdot\prime}{5}\ \overset{\prime}{7})^2$	41	4
$(\overset{\cdot *}{1}\ \overset{\cdot\prime *}{2}\ \overset{\cdot\prime}{3}\ \overset{\cdot\prime *}{4}\ \overset{*\prime}{5})^2$	39	6
$(1\ \dot{3}\ \dot{4}\ \dot{4}\ \dot{5})^2$	20	5
$(\dot{1}\ \dot{2}\ \dot{3}\ \dot{4}\ 4)^2$	19	7
$(\overset{\cdot\prime}{1}\ \dot{2}\ \overset{\cdot\prime}{2}\ \overset{\cdot\prime}{3}\ \overset{\prime}{4})^2$	18	9
$(\dot{1}\ \dot{2}\ \dot{2}\ \dot{3}\ 3)^2$	8	13

† It should be stated that the E column is deduced from the Δ column, and not *vice versâ*. It might have been imagined, *ă priori*, that the amount of evaporation could be determined independently, in the first instance, and the reduction of the denumerants inferred therefrom; but it appears as a datum of experience that the formulæ of denumeration are always simpler than those of evaporation, although, of course, theoretically speaking, the evaporating solutions ought to be obtainable by calculating the solutions for which the new homogenizing variable takes the value zero, or, which means the same thing, for which $1=0$. In a word, although in the order of nature the evaporation is the cause of the reduction, in the order of our knowledge the reduced denumerant, coming first, plays the part of a subtractor, and the evaporation that of a remainder.

‡ Since the above was sent to press, I have hit upon a very simple expedient for representing the parallels of any form of diptych by means of marks of connotation. This method is used in

(23) Call the cyclodic order 2Ω, and the class 2χ.

Then for $\Omega = 1, 2, 3, 4, 5, 6, 7$ the diptychs grouped in classes, omitting the *vacuous* ones [that is those for which the evaporation is *total*], will be as follows:—

$$
\begin{array}{lll}
\Omega = 1 & \chi = 1 & (1)^2 \\[4pt]
\Omega = 2 & \chi = 1 & (2)^2 \\[4pt]
\Omega = 3 & \chi = 1 & (3)^2 \\
 & \chi = 2 & (1\ 2)^2 \\[4pt]
\Omega = 4 & \chi = 1 & (4)^2 \\
 & \chi = 2 & (1\ 3,\ 2\ 2)^2 \\
 & \chi = 3 & (1\ 1\ 2)^2 \\[4pt]
\Omega = 5 & \chi = 1 & (5)^2 \\
 & \chi = 2 & (1\ 4,\ 2\ 3)^2 \\
 & \chi = 3 & (1\ 1\ 3,\ 1\ 2\ 2)^2 \\
 & \chi = 4 & (1\ 1\ 1\ 2)^2 \\[4pt]
\Omega = 6 & \chi = 1 & (6)^2 \\
 & \chi = 2 & (1\ 5,\ 2\ 4,\ 3\ 3)^2 \\
 & \chi = 3 & (1\ 1\ 4,\ 1\ 2\ 3,\ 2\ 2\ 2)^2 \\
 & \chi = 4 & (1\ 1\ 1\ 3,\ 1\ 1\ 2\ 2)^2 \\
 & \chi = 5 & (1\ 1\ 1\ 1\ 2)^2 \\[4pt]
\Omega = 7 & \chi = 1 & (7)^2 \\
 & \chi = 2 & (1\ 6,\ 2\ 5,\ 3\ 4)^2 \\
 & \chi = 3 & (1\ 1\ 5,\ 1\ 2\ 4,\ 1\ 3\ 3,\ 2\ 2\ 3)^2 \\
 & \chi = 4 & (1\ 1\ 1\ 4,\ 1\ 1\ 2\ 3,\ 1\ 2\ 2\ 2)^2 \\
 & \chi = 5 & (1\ 1\ 1\ 1\ 3,\ 1\ 1\ 1\ 2\ 2)^2 \\
 & \chi = 6 & (1\ 1\ 1\ 1\ 1\ 2)^2.
\end{array}
$$

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the sequel; but I have not thought it worth while to disturb the text in this place otherwise than by introducing marks of connotation to distinguish the terms of the parallel equalities. It ought to be noticed that when any element of a parallel equality occurs more than once, the mark of connotation is to be placed indifferently over any (but only one) of the group of quantities thus repeated.

I may take advantage of this opportunity to refer, although a little out of order, to Mr Clifford's extremely ingenious idea of making the proof of the existence of a root to an algebraical function fx of degree n depend on the same for the degree $\frac{n(n-1)}{2}$, and consequently in the last resort on the same for a function of an odd degree, for which case it flows direct from the principle of continuity. Equating to zero each coefficient in the linear remainder of fx divided by x^2+px+q, we obtain two simultaneous equations of known degrees (dependent on the value of n) in p and q. Now the pinch of the required proof consists in our being able to show that the p, q system satisfying these equations has exactly $\frac{n(n-1)}{2}$ values, neither more nor less; so that it is requisite to establish, *à priori*, a certain definite amount of evaporation, which I believe no one has yet quite succeeded in doing. This example affords an

Expanding we obtain the following set of diptychs, the numbers opposite to which are their symmetrical and total denumerants respectively. The Table extends from $\Omega=1$ to $\Omega=7$ inclusive.

Diptych	Δ	D
$(1)^2$	1	1
$(2)^2$	1	1
$(3)^2$	1	1
$(1\ 2)^2$	1	1
$(4)^2$	1	1
$(1\ 3)^2$	1	1
1 3 . 2 2	0	1
$(2\ 2)^2$	0	0
$(1\ 1\ 2)^2$	1	1
$(5)^2$	1	1
$(1\ 4)^2$	1	1
(1 4 . 2 3)	0	2
$(2\ 3)^2$	1	1
$(1\ 1\ 3)^2$	1	1
(1 1 3 . 1 2 2)	0	2
$(1\ 2\ 2)^2$	1	1
$(1\ 1\ 1\ 2)^2$	1	1

Diptych	Δ	D
$(6)^2$	1	1
$(1\ 5)^2$	1	1
(1 5 . 2 4)	0	2
$(2\ 4)^2$	1	1
(1 5 . 3 3)	0	1
(2 4 . 3 3)	0	1
$(3\ 3)^2$	0	0
$(1\ 1\ 4)^2$	1	1
(1 1 4 . 1 2 3)	0	4
$(1\ 2\ 3)^2$	3	7
1 1 4 . 2 2 2	0	1
1 2 3 . 2 2 2	0	1
$(2\ 2\ 2)^2$	0	0
$(1\ 1\ 1\ 3)^2$	1	1
(1 1 1 3 . 1 1 2 2)	0	4
$(1\ 1\ 2\ 2)^2$	1	1
$(1\ 1\ 1\ 1\ 2)^2$	1	1*
$(7)^2$	1	1
$(1\ 6)^2$	1	1
(1 6 . 2 5)	0	2
$(2\ 5)^2$	1	1
(1 6) (3 4)	0	2
(2 5 . 3 4)	0	2
$(3\ 4)^2$	1	1

instructive corroboration of the remark in the last preceding foot-note, concerning evaporation coming second in the order of our knowledge, although first in the order of nature. The arrangement of the diptychical denumerants for a given order and class in the text which follows is columnar. The more natural arrangement (and one so likely to be fruitful in consequences, that I much regret not thinking of it in time) is obviously that of a square, of which each side will contain as many places as there are partitionments corresponding to the order and class. In that diagonal which separates the square into two symmetrical sets of figures (and nowhere else) there will be double entries, corresponding to the D and Δ denumerants respectively of the conformable diptychs.

* Thus for the 12th order and 4th class the number of symmetrical reducible cyclodes is $1+1$, that is, 2, and the total number is $1+2.2+1+2.1+2.1+0=10$; so for the 12th order

Diptych	Δ	D
$(1\ 1\ 5)^2$	1	1
$(1\ 1\ 5\ .\ 1\ 2\ 4)$	0	4
$(1\ 2\ 4)^2$	3	7
$(1\ 1\ 5\ .\ 1\ 3\ 3)$	0	2
$(1\ 2\ 4\ .\ 1\ 3\ 3)$	0	5
$(1\ 3\ 3)^2$	1	1
$(1\ 1\ 5\ .\ 2\ 2\ 3)$	0	3
$(1\ 2\ 4\ .\ 2\ 2\ 3)$	0	4
$(1\ 3\ 3\ .\ 2\ 2\ 3)$	0	2
$(2\ 2\ 3)^2$	1	1
$(1\ 1\ 1\ 4)^2$	1	1
$(1\ 1\ 1\ 4\ .\ 1\ 1\ 2\ 3)$	0	6
$(1\ 1\ 2\ 3)^2$	4	16
$(1\ 1\ 1\ 4\ .\ 1\ 2\ 2\ 2)$	0	3
$(1\ 1\ 2\ 3\ .\ 1\ 2\ 2\ 2)$	0	7
$(1\ 2\ 2\ 2)^2$	1	1
$(1\ 1\ 1\ 1\ 3)^2$	1	1
$(1\ 1\ 1\ 1\ 3\ .\ 1\ 1\ 1\ 2\ 2)$	0	4
$(1\ 1\ 1\ 2\ 2)^2$	2	6
$(1\ 1\ 1\ 1\ 1\ 2)^2$	1	1*

The above Table gives a complete conspectus of the groups of reducible cyclodes of all the even (the only admissible) orders up to the 14th inclusive.

(24) The diophantine equation $y^2 + y'^2 = \square$ may be transformed into

$$1 + \left(\frac{y'}{y}\right)^2 = \square.$$

and 8th class these numbers are $1+1$ and $1+2.4+1$, that is, 2 and 10 as before, as should be the case according to article (13). The equations to be resolved are of the degree D for unconformable, and of the degrees Δ and $\frac{D-\Delta}{2}$ for conformable diptychs; hence up to the 12th order of reducible cyclodes, inclusive, we have not to deal with equations of a higher degree than the 4th, and consequently all the reducible cyclodes of order inferior to 14 admit of explicit algebraical representation.

* Looking at the 3rd and 4th classes, the sums of the numbers in the D column will be found to be 30 and 34 respectively; but on doubling the numbers printed opposite to the unconformable diptychs (which represent not single but paired cyclodes), the number is the same (as it ought to be) for each class, namely, 50, that being the number in the 3rd column and 4th line (or *vice versâ*) of the Q Table, Art. 13. Similarly the corresponding number for the 2nd and 5th classes, doubling as before, becomes 15 instead of 9 for the one, and 11 for the other. It is a noticeable fact that the number of pairs is not the same in two conjugate classes.

In this equation we may suppose

$$y = (x+a)^{\alpha}(x+b)^{\beta}(x+c)^{\gamma} \dots,$$

$\alpha, \beta, \gamma \dots$ being now *general* quantities and no longer necessarily integers.

If $$\eta = \log y = \Sigma\alpha \log(x+a),$$

x, η are the coordinates of a compound logarithmic wave.

If the number of terms in Σ is even, and the quantities $a, b, c \dots$ be divided arbitrarily into two sets containing a like number of terms, as, say, $a, b, c \dots, a', b', c' \dots, \alpha, \beta, \gamma, \dots, \alpha', \beta', \gamma'$ being corresponding values of the coefficients,

$$1 + \left(\frac{d\eta}{dx}\right)^2 = \square,$$

provided that

$$2\alpha = \frac{(a-a')(a-b')(a-c')\dots}{(a-b)(a-c)\dots}, \quad -2\alpha' = \frac{(a'-a)(a'-b)(a'-c)\dots}{(a'-b')(a'-c')\dots} *,$$

and similarly for $\beta, \gamma \dots; \beta', \gamma' \dots$.

From these equations it follows that $\Sigma\alpha = \Sigma\alpha'$ as before.

(25) Conversely, provided

$$\alpha + \beta + \gamma \dots = \alpha' + \beta' + \gamma' \dots,$$

$\alpha, \beta, \gamma \dots, \alpha', \beta', \gamma' \dots$ may be treated as a diptych, and $a, b, c \dots, a', b', c' \dots$ be deduced by equations of the same form as before†: this explains a seeming paradox, that when $y = x^n + cx^{n-2} + \dots$, so that it contains only $(n-1)$ disposable constants, we can satisfy the equation

$$y^2 + y'^2 = (x^n + hx^{n-1} + kx^{n-2} + \dots)^2,$$

which involves the satisfaction of $2n$ equations by $(2n-1)$ unknowns.

(26) It is well worthy of notice that if s is the length of the arc of the compound logarithmic wave

$$\eta = \Sigma\alpha \log(x+a) + \Sigma\alpha' \log(x+a'),$$

* Thus by substitution and multiplication we obtain the important equality

$$\frac{\zeta(abc\dots)}{\zeta(a'b'c'\dots)} = \pm\sqrt{\left(\frac{\alpha'\beta'\gamma'\dots}{\alpha\beta\gamma\dots}\right)},$$

from which it follows that, although the *maximum denumerant* of a diptych of the class μ is $\Pi(\mu-1)\,\Pi(\mu)$ (see Art. 9), the degree of the resolving equation need never exceed $\frac{\Pi(\mu-1)\,\Pi(\mu)}{2}$.

[ζ I use here, as of old elsewhere, to denote the product of the differences of the quantities which it precedes taken in some certain prescribed order of succession.]

† I mean that the equation-system above written may be *transformed* into the equation-system of article (8).

where the α and α' quantities form a diptych, if s be the length of the arc reckoned from a certain point in the curve

$$s - x = \Sigma\alpha \log (x + a) - \Sigma\alpha' \log (x + a').$$

Such a curve is called a rectifiable compound logarithmic wave; and if the a and a' series and the α, $-\alpha'$ series agree term for term, it is said to be symmetrical.

(27) From what has been stated in article (18), it follows that the total number of reducible cyclodes of the order $2n$ is the same as the total number of rectifiable compound logarithmic waves that can be formed with $2n$ non-coincident wavelets of any two given species; and this remarkable equality also subsists when only the symmetrical cyclodes and the symmetrical waves form the subject of comparison. If the class $2m$ be given as well as the order $2n$ of the cyclodes, both the above equalities continue to apply when $2m$ of the wavelets are limited to be of one, and $2n - 2m$ of the other species.

(28) The diptychical equations may be applied to the solution of the diophantine equation $y^2 + y'^2 = \square$ when y is any rational function of x of which the numerator and denominator are each of a given order.

The distinction of class, genus, and species will be still applicable as before, but there will be a fall in the number of solutions when the degree of y in x is zero, that is when the given orders of numerator and denominator are the same, so that the rational fraction of the degree is zero: apart from this cause of difference, the number of solutions corresponding to any given diptych will be governed in all cases solely by the principle of parallel equalities.

(29) It is obvious that the theorems giving the value of the D and Δ* denumerants for any given class, combined with the theorems giving their sum for any given class and order, amount to independent theorems in the theory of partitions; and as regards the Δ denumerants these theorems belong to the subject of partitions of a comparatively simple character. For instance, the table of Δ for the fourth class enables us to affirm that understanding x, y, z, t to be all distinct quantities, and meaning by $\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6$ the denumerants (that is the number of positive-integer solutions, *zeros* excluded) of the indeterminate equations or equation pairs below written

$$x + y + z + t = n; \quad x + y + 2z = n; \quad x + y = \frac{n}{2}; \quad x + 3y = n;$$

$$\begin{bmatrix} x + y + z + t = n \\ x + y = z + t \end{bmatrix}; \quad \begin{bmatrix} x + y + 2z = n \\ x + y = 2z \end{bmatrix};$$

* I shall henceforth call D the total and Δ the special denumerant, the *total* referring to solutions of either kind indifferently, the *special* to symmetrical solutions exclusively.

we have

$$9(\delta_1-\delta_5)+4(\delta_2-\delta_6)+\delta_3+\delta_4+7\delta_5+3\delta_6=\frac{(k+1)^2k}{2}\ \text{when}\ n=2k,$$

$$\text{or}=\frac{(k+1)k^2}{2}\ \text{when}\ n=2k-1\,*.$$

Moreover, throughout the whole theory, as, for example, in certain marvellous formulæ of reduction which I have obtained concerning the maximum Δ denumerants, say of the diptych $[a^r . b^s . c^t \ldots]^2$, problems of partition of numbers occur and recur again and again, some of a known, others of a totally new kind. In this particular we see another point of resemblance between the theory of reducible cyclodes and that of geometric transformations.

(30) Annexed is a Table of the genera of diptychs† (the vacuous one not excepted) of the fourth class. I give it *subject to correction.* The calculations require great care to exclude repetitions or incompatible equalities and to guard against omissions.

It will be recollected that the sum of the terms is always the same for each *ala*; so that if the *alæ* agree in three places they must agree also in the fourth. When there exist binomial parallelisms, one only of the conjugates is exhibited in the Table, and in all cases only such (and in their simplest form) as are sufficient to define the genus. The mononomial parallel equalities are of course represented by the use of the same letter for the equal terms. The binomial parallelisms are indicated by affixing the same mark to all the terms which enter into each side of the equation constituting the expression of such parallelism.

The lines of division in the Table are introduced to give greater facility for exhaustive verification; the artificial groups of genera thus formed are

* The left-hand side of this equation, that is, $9\delta_1+4\delta_2+\delta_3+\delta_4-2\delta_5-\delta_6$ it will easily be seen may be expressed in terms of $\frac{n-4}{1:1:1:1:}$, $\frac{n-4}{1:1:2:}$, $\frac{n-4}{2:2:}$, $\frac{n-4}{1:3:}$, $\frac{n-4}{4:}$, where $\frac{n}{p:q:r:\&c.}$ means the number of ways of making up the number n with p, q, r, &c.; but the expression will not be a linear function of these quantities, inasmuch as δ_5, δ_6 introduce combinations of the second order. The term (-4) in the symbolical numerators arises from the fact of zero values of x, y, z, t being inadmissible.

† If we choose to regard the union of two molecules (alæ), each of like weight (amount) and containing a like number of atoms (elements), as a chemical duad, we see that it is possible for such a duad to be resolved in one or more, and sometimes in a great variety of ways, into two or more subordinate duads, satisfying the same law of number and weight; and it is really quite within the bounds of possibility that in establishing and determining diptychical genera and species, we are at the same time solving a problem of the Chemistry of the Future. The combination of two similar ("conformable") molecules (alæ) into a duad (diptych) conveys a sort of image of Brodie's idea of the combination of a substance such as oxygen with itself; and in fact the whole theory of diptychical partition and combination forcibly reminds one of that writer's profound morphological views of the nature of chemical being, action, and passion.

those in which respectively four, two, one, or no element contained (singly or repeatedly) in one *ala* are or is contained also in the other *ala*.

DIPTYCHS.

$a\ b\ c\ d$	$a\ b\ c\ d$
$a\ b\ c\ c$	$a\ b\ c\ c$
$a\ a\ c\ c$	$a\ a\ c\ c$
$a\ a\ a\ d$	$a\ a\ a\ d$
$a\ a\ a\ a$	$a\ a\ a\ a$
$\dot{a}\ \dot{b}\ c\ d$	$a\ b\ \dot{c}\ \dot{d}$
$\dot{a}\ \dot{b}\ c\ c$	$a\ b\ \dot{c}\ \dot{c}$

$a\ b\ c\ d$	$\alpha\ \beta\ c\ d$
$a\ b\ c\ c$	$\alpha\ \beta\ c\ c$
$\dot{a}\ \dot{b}\ c\ d$	$\dot{\alpha}\ \dot{\beta}\ c\ d$
$\dot{a}\ b\ c\ \dot{d}$	$\alpha\ \dot{\beta}\ \dot{c}\ d$
$a\ b\ c\ d$	$\alpha\ c\ c\ d$
$a\ b\ c\ d$	$c\ c\ c\ d$
$a\ b\ c\ d$	$c\ d\ c\ d$

$\dot{a}\ \dot{b}\ c\ c$	$\dot{\alpha}\ \dot{\beta}\ \alpha\ \beta$
$\dot{a}\ \dot{b}\ c\ c$	$\dot{\alpha}\ \dot{\alpha}\ \alpha\ \delta$
$\dot{a}\ \dot{b}\ a\ d$	$\dot{\alpha}\ \dot{\beta}\ \alpha\ \delta$
$\dot{a}\ \dot{b}\ a\ d$	$\dot{\alpha}\ \dot{\beta}\ \gamma\ \beta$
$\dot{a}\ \dot{b}\ a\ d$	$\dot{\alpha}\ \dot{\beta}\ \alpha\ \beta$
$\dot{a}\ \dot{b}\ a\ d$	$\dot{\alpha}\ \dot{\alpha}\ \alpha\ \delta$
$\dot{a}\ \dot{b}\ a\ b$	$\dot{\alpha}\ \dot{\beta}\ \alpha\ \beta$
$\dot{a}\ \dot{b}\ a\ b$	$\dot{\alpha}\ \dot{\alpha}\ \alpha\ \alpha$

DIPTYCHS.

$a\ b\ c\ d$	$\alpha\ \beta\ \gamma\ d$
$a\ b\ c\ d$	$\alpha\ \beta\ d\ d$
$a\ b\ c\ d$	$d\ d\ d\ d$
$a\ \dot{b}\ \dot{c}\ d$	$\dot{\alpha}\ \beta\ \gamma\ \dot{d}$
$\dot{a}\ a\ \dot{c}\ d$	$\dot{\alpha}\ \beta\ \gamma\ \dot{d}$
$\dot{a}\ \dot{a}\ a\ d$	$\dot{\alpha}\ \beta\ \gamma\ \dot{d}$
$\dot{a}\ a\ c\ \dot{d}$	$\dot{\alpha}\ \alpha\ \dot{\gamma}\ d$

$a\ b\ c\ d$	$\alpha\ \beta\ \gamma\ \delta$
$\dot{a}\ \dot{b}\ c\ d$	$\dot{\alpha}\ \dot{\beta}\ \gamma\ \delta$
$\dot{a}\ \dot{b}\ c\ d$	$\dot{\alpha}\ \dot{\beta}\ \alpha\ \delta$
$\dot{a}\ \dot{b}\ c\ d$	$\dot{\alpha}\ \dot{\beta}\ \alpha\ \beta$
$\dot{a}\ \dot{b}\ c\ d$	$\dot{\alpha}\ \dot{\alpha}\ \alpha\ \delta$
$\dot{a}\ \dot{b}\ c\ d$	$\dot{\alpha}\ \dot{\alpha}\ \alpha\ \alpha$
$\dot{a}\ \dot{b}\ c\ c$	$\dot{\alpha}\ \dot{\beta}\ \alpha\ \delta$

$\dot{a}\ \dot{a}\ a\ d$	$\dot{\alpha}\ \dot{\delta}\ \delta\ \delta$
$\dot{\acute{a}}\ \dot{b}\ \acute{c}\ d$	$\dot{\acute{\alpha}}\ \dot{\beta}\ \acute{\gamma}\ \delta$
$\dot{\acute{a}}\ \dot{b}\ \acute{c}\ d$	$\dot{\acute{\alpha}}\ \dot{\beta}\ \acute{\alpha}\ \delta$
$\dot{\acute{a}}\ \dot{b}\ \acute{c}\ d$	$\dot{\acute{\alpha}}\ \dot{\beta}\ \acute{\alpha}\ \delta$
$\dot{\acute{a}}\ \dot{b}\ \acute{c}\ c$	$\dot{\acute{\alpha}}\ \dot{\beta}\ \acute{\alpha}\ \delta$
$\dot{\acute{a}}\ \dot{b}\ \acute{c}\ c$	$\dot{\acute{\alpha}}\ \dot{\beta}\ \acute{\alpha}\ \beta$
$\dot{\acute{a}}\ \dot{b}\ \acute{a}\ d$	$\dot{\alpha}\ \dot{\acute{\beta}}\ \acute{\alpha}\ \delta$
$\dot{\acute{a}}\ \dot{b}\ \acute{a}\ d$	$\dot{\acute{\alpha}}\ \dot{\beta}\ \acute{\gamma}\ \beta$

In the last seven diptychs it will be observed double binomial parallelisms occur; thus, for example, in the last of all $a+b=\alpha+\beta$, $a+a=\alpha+\gamma$.

The total number above enumerated is 44, but not improbably one or more (most likely an odd number) have been overlooked. The first group

contains only one species to each genus; not so for the other groups. Thus, for example, $a\ b\ c\ d \mid \alpha\ \beta\ c\ d$ represents the two species,

$$\begin{array}{cccc|cccc} a & b & c & d & \alpha & \beta & c & d \\ a & b & c & d & \alpha & \alpha & c & d \end{array}$$

So, again, $a\ b\ c\ d \mid \alpha\ \beta\ \gamma\ d$ represents the several species,

$$\begin{array}{cccc|cccc} a & b & c & d & \alpha & \beta & \gamma & d \\ a & b & c & d & \alpha & \beta & \beta & d \\ a & b & c & d & \alpha & \alpha & \alpha & d \\ a & a & c & d & \alpha & \beta & \beta & d \\ a & a & c & d & \alpha & \alpha & \alpha & d \end{array}$$

and $a\ b\ c\ d \mid \alpha\ \beta\ \gamma\ \delta$ will represent, in like manner, the several species,

$$\begin{array}{cccc|cccc} a & b & c & d & \alpha & \beta & \gamma & \delta \\ a & b & c & d & \alpha & \beta & \gamma & \gamma \\ a & b & c & d & \alpha & \beta & \beta & \beta \\ a & b & c & d & \alpha & \alpha & \alpha & \alpha \\ a & b & c & c & \alpha & \beta & \gamma & \gamma \\ a & b & c & c & \alpha & \beta & \beta & \beta \\ a & b & c & c & \alpha & \alpha & \alpha & \alpha \\ a & b & b & b & \alpha & \beta & \beta & \beta \\ a & b & b & b & \alpha & \alpha & \alpha & \alpha \\ a & b & c & d & \alpha & \alpha & \gamma & \gamma \\ a & b & c & c & \alpha & \alpha & \gamma & \gamma \\ a & b & b & b & \alpha & \alpha & \gamma & \gamma \end{array}$$

But $a\ a\ c\ c \mid \alpha\ \alpha\ \gamma\ \gamma$ would belong to a different genus; for in such diptych we should have $2a + 2c = 2\alpha + 2\gamma$, and consequently $a + c = \alpha + \gamma$ a new parallel; and *à fortiori* $a\ a\ a\ a \mid \alpha\ \alpha\ \gamma\ \gamma$ belongs to a different genus. The algebraical denumerants will be the same for all the last-written twelve species, namely, $\Pi 3 . \Pi 4$, that is 144, the maximum value for the fourth class,—the arithmetical denumerants, on the other hand, being respectively 144, 72, 24, 6, 36, 12, 3, 4, 1, 36, 18, 6. Possibly it may turn out that the number of species may be given by as simple or a simpler function of the index of the class as or than that which expresses the number of genera. It seems very desirable to ascertain if either or both is or are amenable to algebraical or analytical quantification.

As Crystallography was born of a chance observation by Haüy of the cleavage-planes of a single fortunately fragile specimen, and the theory of Invariants owes its existence to a solitary individual accidentally encountered and put on record by Eisenstein, so out of the slender study of the Norwich Spiral has sprung the vast and interminable Calculus of Cyclodes, which strikes such far-spreading and tenacious roots into the profoundest strata of

denumeration, and, by this and the multitudinous and multifarious dependent theories which cluster around it, reminds one of the Scriptural comparison of the Kingdom of Heaven "to a grain of mustard-seed which a man took and cast into his garden, and it grew and waxed a great tree, and the fowls of the air lodged in the branches of it"*.

(31) The annexed computation of the denumerants of *diptychs* of the 3rd class and 8th and 9th orders respectively is given in illustration of articles (20) and (13).

The triadic partitionments of 8, it will be observed, are 1 1 6 : 1 2 5 : 1 3 4 : 2 2 4 : 2 3 3. Those of 9, 1 1 7 : 1 2 6 : 1 3 5 : 1 4 4 : 2 2 5 : 2 3 4 : 3 3 3. And of 7, 1 1 5 : 1 2 4 : 1 3 3 : 2 2 3.

In the D column the denumerants of the unconformable diptychs are *doubled*, in order to save the necessity of transposing the *alæ* in summing for the total denumerant [the $Q_{r,s}$ of Art. 13].

Diptychs		D
116	116	1
116	125	8
116	134	8
116	224	6
116	233	6
125	125	7
125	134	20
125	224	8
125	233	10
134	134	7
134	224	10
134	233	8
224	224	1
224	233	4
233	233	1
$\dfrac{\Pi 7\ \Pi 6}{\Pi 3\ \Pi 2\ \Pi 5\ \Pi 4} =$		105

* The three most remarkable and almost simultaneous births of time for the current year are, I think, Janssen's and Lockyer's hydrogenous solar chromosphere, Tyndall's indefinitely attenuated cometary matter, and the still more impalpable and shadowy product of cerebration embodied in diptychs with their *quasi*-chemical composition and parallels stretching between and connecting, as it were, with forces of affinity the atomic elements of the associated geminate molecules, in this sketch set forth. All three theories have originated alike in observation, followed up by processes of experiment and verification working with very different instruments, but having the same *primum mobile* in the human intelligence. On second thoughts I ought to tack on to this list of memorabilia, which must for ever make 1869 stand out in the Fasti of

Diptychs		D
115	115	1
115	124	8
115	133	4
115	223	6
124	124	7
124	133	10
124	223	8
133	133	1
133	223	4
223	223	1
$\frac{\Pi 6\ \Pi 5}{\Pi 3\ \Pi 2\ \Pi 4\ \Pi 3} =$		50*

Diptychs		D
117	117	1
117	126	8
117	135	8
117	144	4
117	225	6
117	234	12
117	333	2
126	126	7
126	135	20
126	144	10
126	225	8
126	234	20
126	333	4
135	135	7
135	144	10
135	225	10
135	234	20
135	333	2
144	144	1
144	225	6
144	234	8
144	333	2
225	225	1
225	234	8
225	333	2
234	234	7
234	333	2
333	333	0
$\frac{\Pi 8\ \Pi 7}{\Pi 3\ \Pi 2\ \Pi 6\ \Pi 5} =$		196

science, Capt. Andrew Noble's mechanical invention for measuring up to the millionth part of a second the rate of motion of a shot inside a cannon, and Dr Christian Wiener's wonderful realization in stereoscopic drawings of the Salmon-Cayley 27 lines on a cubic surface on the one hand, and on the other (Hermite's pupil, pupil worthy of his master) M. Camille Jordan's surprising discovery of their application to the trisection of Abelian functions. Surely with as good reason as had Archimedes to have the cylinder, cone, and sphere engraved on his tombstone might our distinguished countrymen leave testamentary directions for the cubic eikosi-heptagram to be engraved on theirs. Spirit of the Universe! whither are we drifting, and when, where, and how is all this to end?

* This Table has been already given, differently arranged, at page [677], where, however, the numbers corresponding to pairs in the D column are not, as here, doubled.

(32) The Tables following illustrate the values assigned to the special denumerants (Δ column) in the Tables of Arts. 21, 22. The diptychs being of the conformable kind, are, for greater simplicity, represented each by a single *ala*. The asterisk at the side distinguishes those diptychs in which one or more binomial parallel equalities exist.

Class = 8, Order = 28	
Ala	Δ
1.1.1.11	1
1.1.2.10	4
1.1.3.9	4
1.1.4.8	4
1.1.5.7	4
1.1.6.6	1
1.2.2.9	4
1.2.3.8	9
1.2.4.7	9
*1.2.5.6	7
1.3.3.7	4
*1.3.4.6	7
1.3.5.5	4
1.4.4.5	4
2.2.2.8	1
2.2.3.7	4
2.2.4.6	4
2.2.5.5	1
2.3.3.6	4
*2.3.4.5	7
2.4.4.4	1
3.3.3.5	1
3.3.4.4	1
$\dfrac{\{\Pi(5+2-1)\}^2}{\Pi 1\ \Pi 2\ \Pi 4\ \Pi 5}=$	90

Class = 10, Order = 30	
Ala	Δ
1.1.1.1.11	1
1.1.1.2.10	6
1.1.1.3.9	6
1.1.1.4.8	6
1.1.1.5.7	6
1.1.1.6.6	2
1.1.2.2.9	9
1.1.2.3.8	21
1.1.2.4.7	21
1.1.2.5.6	19
1.1.3.3.7	9
*1.1.3.4.6	19
1.1.3.5.5	9
1.1.4.4.5	9
1.2.2.2.8	6
*1.2.2.3.7	20
1.2.2.4.6	21
1.2.2.5.5	9
1.2.3.3.6	21
*1.2.3.4.5	39
1.2.4.4.4	6
*1.3.3.3.5	5
1.3.3.4.4	9
2.2.2.2.7	1
2.2.2.3.6	6
2.2.2.4.5	6
2.2.3.3.5	9
2.2.3.4.4	9
*2.3.3.3.4	5
3.3.3.3.3	0
$\dfrac{\Pi 6.\Pi 7}{\Pi 4.\Pi 5\,(\Pi 2)^2}=$	315

†Class = 10, Order = 32	
Ala	Δ
1.1.1.1.12	1
1.1.1.2.11	6
1.1.1.3.10	6
1.1.1.4.9	6
1.1.1.5.8	6
1.1.1.6.7	6
1.1.2.2.10	9
1.1.2.3.9	21
1.1.2.4.8	21
1.1.2.5.7	21
1.1.2.6.6	9
1.1.3.3.8	9
1.1.3.4.7	21
1.1.3.5.6	21
1.1.4.4.6	9
1.1.4.5.5	9
1.2.2.2.9	6
*1.2.2.3.8	20
1.2.2.4.7	21
*1.2.2.5.6	19
1.2.3.3.7	21
*1.2.3.4.6	41
1.2.3.5.5	21
*1.2.4.4.5	19
1.3.3.3.6	6
*1.3.3.4.5	20
1.3.4.4.4	6
2.2.2.2.8	1
2.2.2.3.7	6
2.2.2.4.6	6
2.2.2.5.5	2
2.2.3.3.6	9
*2.2.3.4.5	19
2.2.4.4.4	2
2.3.3.3.5	6
*2.3.3.4.4	8
3.3.3.3.4	1
$\dfrac{(\Pi 7)^2}{(\Pi 5\ \Pi 2)^2}=$	441

† That is, the cyclode is of such class and order, but the diptych is of class 5 and order 16, and so for the rest. I repeat that the diptychical class and order is the number of elements and their sum in *each* ala, and must be doubled to give the class and order of the corresponding cyclode.

Class =10 Order=34	
Ala	Δ
1.1.1.1.13	1
1.1.1.2.12	6
1.1.1.3.11	6
1.1.1.4.10	6
1.1.1.5.9	6
1.1.1.6.8	6
1.1.1.7.7	2
1.1.2.2.11	9
1.1.2.3.10	21
1.1.2.4.9	21
1.1.2.5.8	21
1.1.2.6.7	19
1.1.3.3.9	9
1.1.3.4.8	21
1.1.3.5.7	19
1.1.3.6.6	9
1.1.4.4.7	8
1.1.4.5.6	21
1.1.5.5.5	2
1.2.2.2.10	6
1.2.2.3.9	20
1.2.2.4.8	21
1.2.2.5.7	21
1.2.2.6.6	9
1.2.3.3.8	21
1.2.3.4.7	43
1.2.3.5.6	41
1.2.4.4.6	20
1.2.4.5.5	19
1.3.3.3.7	6
1.3.3.4.6	19
1.3.3.5.5	8
1.3.4.4.5	20
1.4.4.4.4	1
2.2.2.2.9	1
2.2.2.3.8	6
2.2.2.4.7	6
2.2.2.5.6	6
2.2.3.3.7	9
2.2.3.4.6	21
2.2.3.5.5	9
2.2.4.4.5	9
2.3.3.3.6	6
2.3.3.4.5	18
2.3.4.4.4	6
3.3.3.3.5	1
3.3.3.4.4	2
$\dfrac{\Pi 8\ \Pi 7}{\Pi 6\ \Pi 5\ (\Pi 2)^2} =$ Here $s=5$, $r=12$, $k=2$, $h=6$†	588

In the above Table, and those which follow, instead of using asterisks (danger-signals, so to say) as in the three Tables immediately preceding to indicate the existence of parallel equalities, I have employed the much more perfect method of connotation which actually exhibits such equalities to the eye.

† See second footnote, p. [669].

Class = 10 Order = 40	
Ala	Δ
1 1 1 1 16	1
1 1 1 2 15	6
1 1 1 3 14	6
1 1 1 4 13	6
1 1 1 5 12	6
1 1 1 6 11	6
1 1 1 7 10	6
1 1 1 8 9	6
1 1 2 2 14	9
1 1 2 3 13	21
1 1 2 4 12	21
1 1 2 5 11	21
1 1 2 6 10	21
1 1 2 7 9	21
1 1 2 8 8	9
1 1 3 3 12	9
1 1 3 4 11	21
1 1 3 5 10	21
1 1 3 6 9	21
1 1 3 7 8	21
1 1 4 4 10	9
1 1 4 5 9	21
1 1 4 6 8	21
1 1 4 7 7	9
1 1 5 5 8	9
1 1 5 6 7	21
1 1 6 6 6	2
1 2 2 2 13	6
$\dot{1}\ \dot{2}\ \dot{2}\ \dot{3}\ 12$	20

Class = 10 Order = 40	
Ala	Δ
1 2 2 4 11	21
1 2 2 5 10	21
1 2 2 6 9	21
$\dot{1}\ 2\ \dot{2}\ \dot{7}\ \dot{8}$	19
1 2 3 3 11	21
$\dot{1}\ \dot{2}\ \dot{3}\ \dot{4}\ 10$	43
1 2 3 5 9	45
$\dot{1}\ 2\ \dot{3}\ \dot{6}\ \dot{8}$	43
1 2 3 7 7	21
1 2 4 4 9	21
$\dot{1}\ \dot{2}\ \dot{4}\ \dot{5}\ 8$	41
$\dot{1}\ \dot{2}\ 4\ \dot{6}\ \dot{7}$	43
1 2 5 5 7	21
$\dot{1}\ \dot{2}\ 5\ \dot{6}\ 6$	19
1 3 3 3 10	6
1 3 3 4 9	21
$\dot{1}\ \dot{3}\ \dot{3}\ \dot{5}\ 8$	20
1 3 3 6 7	21
1 3 4 4 8	21
$\dot{1}\ \dot{3}\ 4\ \dot{5}\ \dot{7}$	43
$\dot{1}\ \dot{3}\ \dot{4}\ \dot{6}\ 6$	19
1 3 5 5 6	21
$\dot{1}\ \dot{4}\ \dot{4}\ 4\ \dot{7}$	5
1 4 4 5 6	21
1 4 5 5 5	6
2 2 2 2 12	1
2 2 2 3 11	6
2 2 2 4 10	6
2 2 2 5 9	6

Class = 10 Order = 40	
Ala	Δ
2 2 2 6 8	6
2 2 2 7 7	2
2 2 3 3 10	9
2 2 3 4 9	21
2 2 3 5 8	21
$2\ \dot{2}\ \dot{3}\ \dot{6}\ \dot{7}$	19
2 2 4 4 8	9
$2\ \dot{2}\ \dot{4}\ \dot{5}\ \dot{7}$	19
2 2 4 6 6	9
2 2 5 5 6	9
2 3 3 3 9	6
$\dot{2}\ \dot{3}\ \dot{3}\ \dot{4}\ 8$	20
2 3 3 5 7	21
2 3 3 6 6	9
2 3 4 4 7	21
$\dot{2}\ \dot{3}\ \dot{4}\ \dot{5}\ \dot{6}$	39
2 3 5 5 5	6
$\dot{2}\ \dot{4}\ 4\ \dot{4}\ \dot{6}$	5
2 4 4 5 5	9
3 3 3 3 8	1
3 3 3 4 7	6
3 3 3 5 6	6
3 3 4 4 6	9
3 3 4 5 5	9
$\dot{3}\ \dot{4}\ \dot{4}\ 4\ \dot{5}$	5
4 4 4 4 4	0
$\left(\frac{\Pi 9}{\Pi 2\ \Pi 7}\right)^2 =$	1296†
Here $s=5$, $r=15$, $k=2$, $h=7$	

† The *fall* in the arithmetical denumerant due to the existence of binomial parallelisms in the 4th or 5th diptychical class is easily recollected by aid of the following rule. Every such parallelism causes a fall of 2 units unless two elements in the parallel are identical, in which case the fall is of a single unit; but in applying this rule elements of the same name are not to be counted more than once over in the same parallel. Calling the two varieties of binomial

Anyone desirous of mastering the foregoing exposition (the cradle of a theory complementary to the existing theory of algebraical curves) will do well to possess himself of the ideas conveyed by the new or newly applied terms of art scattered up and down through the outline and resumed in the annexed stenograph: Continued Involutes; Cyclodes, their Pole and Parabolic function; Unicursal Relation; Norwich Spiral; Reducible Cyclodes and their Bifid functions; Algebraico-Diophantine condition; Symmetrical, Congeminate and Contrageminate forms; Compound Logarithmic Waves and Wavelets; Partitionments; Representative Diptychs, their Alæ conformable and unconformable; Monistic and Dualistic equation-systems; Algebraical and Arithmetical Denumeration; Zigzag Multiplication of continuous lines or columns; Pneumatic Analysis; Evaporation (partial and total); Consolidation; Reduction; Parallel Equalities; Mononomial, Binomial, and Polynomial Parallels (Reducible and Irreducible); Connotation; Order, Class, Genus, and Species of Cyclodes and Diptychs; Special and Total Denumerants. The quotation at the head of the paper is from an article by M. Réville in the *Revue des deux Mondes* for the present July, "sur la Science des Religions." I have since found that a strikingly similar passage occurs in Dr Mansel's Bampton Lectures:—"The only test *of truth* is *the harmonious consent of all human faculties.*"

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conjugation A and B respectively, it will be found that we may have B or A, or A and B, or A and A, or A and A and A coexisting in the same diptych of the 5th class, causing respectively falls of 1, 2, 3, 4, 6 units in the corresponding values of Δ. It may not unlikely be possible to give rules for assigning the effect of any polynomial parallel on the denumerant in all cases; and then the theory will be greatly simplified, as we shall have only to deal with diptychs of the form $a^r b^s c^t \ldots \mid \alpha^\rho \beta^\sigma \ldots$ freed from all latent parallelisms. I am virtually in possession of a complete theory of the special denumerants for diptychs of the form $[a^r . b^s . c^t \ldots]^2$ on the supposition of the absence of binomial or other polynomial parallel equalities. As regards the method of connotation for singling out the terms which enter into any parallel, a remark may be made which is of some importance, namely, that given such terms, and provided that the parallelism is *irreducible*, that is, not implied in parallelisms of a lower order, the resolution of the parallel into an equation can be effected in only one way. Hence, as we may always reject *reducible* parallels, the method of connotation is exempt from all ambiguity in its application to the single *alæ* of conformable diptychs. As to diptychs with the alæ both expressed, the method of connotation actually exhibits not only the parallels, but the equations which they respectively contain.

It is worth while to notice how gradually and slowly the final form of the theory is evolved. The first reducible cyclode is of the 2nd order. The plurality of such cyclodes only begins to show itself in the 6th order; the existence of non-symmetrical ones, in the 8th; congeminate pairs, in the 12th; reduction in the value of Δ (the special denumerant) owing to the existence of parallel equalities (other than identities), in the 16th. In like manner, I discovered in the theory of partitions that it is not until we have entered upon the discussion of ternary equation-systems that we can be said to have cleared out of the narrows and to be sailing on the open sea; for only then do the most essential features of the theory and its geometrical basis begin to disclose themselves. A very similar remark applies to my unpublished generalization of Gauss's method of interpolation, which I have extended to multiple integrals of all degrees of multiplicity. So, too, it is well known that the algebraical or geometrical theory of forms does not exhibit itself in its true colours until we have passed beyond the case of the second degree, which is quite a world within itself *conditionally unlimited*, incongruent as this association of epithets may sound to the unpurged ears of the Hamiltonian school of metaphysicians.

104.

THE STORY OF AN EQUATION IN DIFFERENCES OF THE SECOND ORDER.

[*Philosophical Magazine*, XXXVII. (1869), pp. 225—227.]

MY recent researches into the order of the various systems of equations which serve to determine the forms of reducible cyclodes have led me to notice an equation in the second order of differences which I imagine is new, and possesses a peculiarly interesting complete integral.

If we call
$$fx = (x^2 - a^2)^i (x^2 - b^2)^j (x^2 - c^2)^k \ldots,$$
and ($i, j, k, \ldots$ &c. being given) determine $a, b, c, \ldots$ &c. so as to make
$$(fx)^2 + (f'x)^2$$
a complete square, and if we suppose the indices $i, j, k, \ldots$ to consist of λ integers of one value, μ integers of a second value, ν of a third, and so on, the number of solutions of the problem will *in general* depend not on $i, j, k, \ldots$ but on the derived integers $\lambda, \mu, \nu, \ldots$; and we may denote the maximum value of this number by the type $[\lambda, \mu, \nu, \ldots]$*.

Now I have been able to establish the following theorem of derivation as a particular case of a more general one of which the clue is in my hands:—
$$[1, \lambda, \mu, \nu, \ldots] = [\lambda, \mu, \nu, \ldots] + \Sigma (\lambda^2 - \lambda) [1, \lambda - 2, \mu, \nu, \ldots]$$
$$+ 2\Sigma\lambda\mu [1, \lambda - 1, \mu - 1, \nu, \ldots].$$
Suppose now that $\lambda, \mu, \nu, \ldots$ all become unity, and that we call
$$[1, 1, 1, \ldots \text{ to } n \text{ terms}] = \Omega_n,$$
then the theorem above stated gives the relation
$$\Omega_n = \Omega_{n-1} + (n-1)(n-2)\,\Omega_{n-2}.$$

* For example, if $fx = (x^2 - a^2)^i (x^2 - b^2)^j (x^2 - c^2)^k (x^2 - d^2)^l$,
the type is [1, 1, 1, 1], of which the maximum value is 9; but if the sum of any two of the quantities i, j, k, l happens to become equal to the sum of the other two, the order sinks and is either 8 or 7; I am not quite certain which at present, although it is more probably the former.

But by virtue of the form of the equations for finding fx, I know independently that Ω_n is the product of n terms of the progression

$$1, 1, 2, 2, 3, 3, 4, \ldots$$

Hence we have one particular solution of the above equation in differences. To find the second, if we make Ω_1 and Ω_2, 1 and 2 respectively instead of 1, 1, it will be found that the nth term becomes the product of n terms of the analogous progression 1, 2, 2, 4, 4, 6, 6 .... Thus, then, we are in possession of the complete integral of the equation

$$u_{x+1} = u_x + (x^2 - x)\, u_{x-1},$$

namely
$$u_{2x} = C \,.\, 1^2 \,.\, 3^2 \,.\, 5^2 \ldots (2x-1)^2 + K \,.\, 2^2 \,.\, 4^2 \ldots (2x-2)^2\, 2x,$$
$$u_{2x+1} = C \,.\, 1^2 \,.\, 3^2 \,.\, 5^2 \ldots (2x-1)^2 (2x+1) + K \,.\, 2^2 \,.\, 4^2 \ldots (2x)^2.$$

Writing $u_x = 1 \,.\, 2 \,.\, 3 \ldots (x-1)\, v_x$, the above equation takes the remarkably simple form

$$v_{x+1} - v_{x-1} = \frac{v_x}{x} * .$$

The romance of algebra presents few episodes more wonderful than this specimen of the way in which the determination of the degree of an equation resulting from elimination can be made to contribute a new and by no means obvious fact to the Calculus of Differences.

* Whether taken under this or the original form, the equation will be found to lie *outside* the cases of integrable linear difference equations of the second order with linear or quadratic coefficients given by the late Mr Boole in his valuable treatise on finite differences. In the second form the solution ought by Laplace's method to be representable by a definite integral. Expressed under the more ordinary form the integral is as follows:

$$v_{2x} = C \frac{3 \,.\, 5 \,.\, 7 \ldots (2x-1)}{2 \,.\, 4 \,.\, 6 \ldots (2x-2)} + K \frac{2 \,.\, 4 \,.\, 6 \ldots (2x)}{1 \,.\, 3 \,.\, 5 \ldots (2x-1)},$$
$$v_{2x-1} = C \frac{3 \,.\, 5 \,.\, 7 \ldots (2x-1)}{2 \,.\, 4 \,.\, 6 \ldots (2x-2)} + K \frac{2 \,.\, 4 \,.\, 6 \ldots (2x-2)}{1 \,.\, 3 \,.\, 5 \ldots (2x-3)}.$$

105.

NOTE ON A NEW CONTINUED FRACTION APPLICABLE TO THE QUADRATURE OF THE CIRCLE.

[*Philosophical Magazine*, XXXVII. (1869), pp. 373—375.]

IN a recent note [p. 689, above] inserted by the author in the *Philosophical Magazine* it was virtually shown, and indeed becomes almost self-evident *as soon as stated*, that the equation $u_{x+1} = \frac{u_x}{x} + u_{x-1}$ possesses two particular integrals, α_x, β_x, which are the products of x terms of the respective progressions

$$[1,\ \ 1,\ \ \tfrac{3}{2},\ \ 1,\ \ \tfrac{5}{4},\ \ 1,\ \ \tfrac{7}{6}, \ldots];$$
$$[1,\ \ \tfrac{2}{1},\ \ 1,\ \ \tfrac{4}{3},\ \ 1,\ \ \tfrac{6}{5},\ \ 1, \ldots].$$

Now any continued fraction whose partial quotients are

$$\frac{1}{k},\ \frac{1}{k+1}, \ldots,\ \frac{1}{x}$$

will be equal to the ratio of some two particular values of u_x in the above equation, that is, of two linear functions of α_x, β_x; and in especial when $k=1$ it will be found very easily that this fraction is $\frac{\beta_x - \alpha_x}{\alpha_x}$.

But, on supposing x infinite, $\frac{\beta_x}{\alpha_x}$ becomes equal to the well-known factorial expression for $\frac{\pi}{2}$, viz. $\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\ldots$. Hence we may deduce the following value for $\frac{\pi}{2}$ under the form of a continued fraction, namely,

$$\frac{\pi}{2} = 1 + \cfrac{1}{1 + \cfrac{1}{2^{-1} + \cfrac{1}{3^{-1} + \cfrac{1}{4^{-1} + \ldots}}}} \textit{ ad infinitum.}$$

Reverting to pure integers, the above equality may be written as follows,

$$\frac{\pi}{2} = 1 + \frac{1}{1} + \frac{2}{1} + \frac{6}{1} + \frac{12}{1} + \frac{20}{1} \textit{ ad infinitum},$$

the denominators of the partial fractions being all units, and the numerators (after the first) the doubles of the natural series of triangular numbers 1, 3, 6, 10 .... This is obviously the simplest form of continued fraction for π that can be given, and yet, strange to say, has not, I believe, before been observed. Truly wonders never cease!

At first sight it might seem as if the above-stated continued fraction were incapable of teaching anything that cannot be got direct out of the Wallisian representation itself that has become transformed into it. Thus, for example, the convergent

$$1 + \frac{1}{1+}\,\frac{2}{1+}\,\frac{6}{1+}\,\frac{12}{1}, \text{ that is } \frac{64}{45},$$

is identical with the corresponding factorial product $\frac{2\,.\,2\,.\,4\,.\,4}{1\,.\,3\,.\,3\,.\,5}$. But I think a substantial difference does arise in favour of the continued fraction form, inasmuch as it indicates a certain obvious correction to be applied in order that the convergence may become more exact. For if we call

$$\frac{n(n+1)}{1+}\,\frac{(n+1)(n+2)}{1+} \dots \textit{ ad infinitum} = u_n,$$

we have $u_n = \frac{n^2+n}{1+u_{n+1}}$. This shows that u_n cannot remain finite when n becomes infinite; for then u_{n+1} would also be finite, and consequently u_n would be a finite fraction of infinity, which is a contradiction in terms.

Hence ultimately

$$u_n\,.\,u_{n+1} = n^2 + n, \text{ that is } u_n = n,$$

or, in other words,

$$\frac{1}{n^{-1}+}\,\frac{1}{(n+1)^{-1}+}\,\frac{1}{(n+2)^{-1}+} \dots \textit{ ad infinitum},$$

converges (and, it may be shown, always in an *ascending* direction) towards unity as its limit when n converges towards infinity. Thus we may write when n is very great,

$$\frac{\pi}{2} = 1 + \frac{1}{1+}\,\frac{2}{1+}\,\frac{6}{1+} \dots + \frac{n^2-n}{1+n}\text{*}.$$

* This comes to the same thing as saying that for the purposes of calculation the continued fraction should be always considered as ending with a numerator, 1, and not with a denominator,

For example, when $n=4$, $\frac{\pi}{2}$ approximately equals

$$1+\frac{1}{1+}\,\frac{2}{1+}\,\frac{6}{1+}\,\frac{12}{1+4}, \text{ that is } =\frac{128}{81}, \text{ or } 1{\cdot}5802,$$

and, when $n=5$, will be found to be $\frac{352}{225}$ or 1·5644. The uncorrected convergent corresponding to the former of these is, as we have seen, $\frac{64}{45}$, or 1·4222; and the next is $\frac{384}{225}$, or 1·7056, the true value of $\frac{\pi}{2}$ being 1·5708. The errors given by the uncorrected factorial values are ·1486 and ·1348 respectively (of course with opposite signs), whereas the errors corresponding to the corrected values are only ·0094 and ·0064; the approximation being thus more than fifteen and twenty-one times bettered for the fourth and fifth convergents respectively by aid of the correction.

$\frac{1}{k}$. For example, $1+\frac{1}{1+1}$, that is, $\frac{3}{2}$ is a good deal nearer to $\frac{\pi}{2}$ than $1+\frac{1}{1+2}$, that is, $\frac{4}{3}$, is; and so $1+\frac{1}{1}+\frac{1}{\frac{1}{2}+1}$, or $\frac{8}{5}$, is much nearer to it than $1+\frac{1}{1}+\frac{1}{\frac{1}{2}+3}$, that is, $\frac{16}{9}$, is.

By taking the mean between two such consecutive corrected convergents, or, still better, the mean between two such consecutive means, and so on, a few terms will serve to give a very close approximation indeed to the limit $\frac{\pi}{2}$.

106.

ON TWO REMARKABLE RESULTANTS ARISING OUT OF THE THEORY OF RECTIFIABLE COMPOUND LOGARITHMIC WAVES.

[*Philosophical Magazine,* XXXVII. (1869), pp. 375—382.]

THE fruitful investigations in which I have been for some time past engaged concerning reducible cyclodes and rectifiable compound logarithmic waves have led me *inter alia* to notice a problem of elimination which from its elegance and peculiarity is, I think, worthy of being offered in a detached form to the *Philosophical Magazine.*

Suppose any number of equations (to fix the ideas say four) of the form which follows:

$$\begin{aligned} U &= ax + by + cz + dt = 0, \\ V &= ax^3 + by^3 + cz^3 + dt^3 = 0, \\ W &= ax^5 + by^5 + cz^5 + dt^5 = 0, \\ \Omega &= ax^7 + by^7 + cz^7 + dt^7 = 0. \end{aligned}$$

If these be regarded as surfaces, they can only be made to intersect in one or another of a definite number of points.

For in the case of intersection we must evidently have

$$dt.\begin{vmatrix} 1 & 1 & 1 & 1 \\ x^2 & y^2 & z^2 & t^2 \\ x^4 & y^4 & z^4 & t^4 \\ x^6 & y^6 & z^6 & t^6 \end{vmatrix}, \text{ that is } dt\,\zeta(x^2, y^2, z^2, t^2) = 0,$$

ζ being the symbol which ex presses the product of the differences of the quantities which it affects. Hence

$$x \pm y = 0 \text{ or } x \pm z = 0 \text{ or } y \pm z = 0 \text{ or } x \pm t = 0 \text{ or } y \pm t = 0$$
$$\text{or } z \pm t = 0 \text{ or } t = 0.$$

Hence it will easily be seen by substitution and successive reduction that the points of intersection are confined to those hereinunder stated and their analogues, namely

$$x = \pm y = \pm z = \pm t,$$

$$x = \pm y = \pm z, \qquad t = 0,$$

$$x = \pm y, \quad z = 0, \quad t = 0,$$

$$x = 0, \qquad y = 0, \quad z = 0,$$

the total number of points in the group being

$$2^3 + 4 \,.\, 2^2 + 6 \,.\, 2 + 4, \text{ that is } \frac{3^4 - 1}{2};$$

and so in general for n such equations the number of possible points of intersection will be $\dfrac{3^n - 1}{2}$.

As regards the resultant, we have

$$\begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x^4 & y^4 & z^4 \end{vmatrix} \Omega + (\,.\,) W + (\,.\,) V + (\,.\,) U = dt\, \zeta (x^2, y^2, z^2, t^2).$$

Hence the resultant of U, V, W, Ω is the same as that of

$$U, V, W, dt \,.\, \zeta (x^2, y^2, z^2, t^2),$$

divided by the resultant of

$$U, V, W, \zeta (x^2, y^2, z^2),$$

that is, is the resultant of

$$U, V, W, dt (x^2 - t^2)(y^2 - t^2)(z^2 - t^2).$$

This enables us to see that the required resultant is the product of all the resultants of the systems that can be formed by the interchange of a, b, c after the pattern of the system

$$(a \pm d)\, x + (b \pm d)\, y + (c \pm d)\, z,$$

$$(a \pm d)\, x^3 + (b \pm d)\, y^3 + (c \pm d)\, z^3,$$

$$(a \pm d)\, x^5 + (b \pm d)\, y^5 + (c \pm d)\, z^5,$$

(the signs in the coefficients of the same column being alike, but independent as between column and column), multiplied by the resultant of

$$ax + by + cz,$$

$$ax^3 + by^3 + cz^3,$$

$$ax^5 + by^5 + cz^5,$$

multiplied by $\qquad d^{1\,.\,3\,.\,5}:$

and by continuing this process it is obvious that the required resultant will be made up exclusively of factors of the form

$$d^\lambda, \quad (d \pm c)^\mu, \quad (d \pm c \pm b)^\nu, \quad (d \pm c \pm b \pm a)^\pi.$$

So in general for n equations, it may be shown in like manner that the resultant is the product of factors of the form

$$(a_1 \pm a_2 \pm a_3 \pm \dots \pm a_i)^{u_{n,i}},$$

where $u_{n,i}$ is a function of n and i to be determined. But by aid of the method of reduction above indicated, and fixing his attention on those factors of the resultant only in which the single coefficient retained in the substituted equation appears, the intelligent reader will find no difficulty in ascertaining

$$(1) \quad \text{that } u_{n,1} = 1.3.5 \dots (2n-1),$$

$$(2) \quad \text{that } u_{n,i} = (i-1)\, u_{n-1,i-1}.$$

These two conditions furnish us with the following Table of double entry:—

$i=$	1,	2,	3,	4,	5,	6
$n=1$	1					
$=2$	1	1				
$=3$	3	1	2			
$=4$	15	3	2	6		
$=5$	105	15	6	6	24	
$=6$	945	105	30	18	24	120

which, of course, may be indefinitely extended. Thus, for example, when $n=3$, the resultant is

$$(abc)^3 (a^2-b^2)(a^2-c^2)(b^2-c^2)(a^4+b^4+c^4-2a^2b^2-2a^2c^2-2b^2c^2)^2.$$

The above investigation leads as a corollary to the following arithmetical theorem.

Call $1.3.5\dots(2x-1) = Q_x$ and $1 = Q_0$. Then

$$2x\frac{Q_{x-1}}{2} + 2x(2x-2)\frac{Q_{x-2}}{4} + 2x(2x-2)(2x-4)\frac{Q_{x-4}}{6} + \dots$$

$$+ 2x(2x-2)\dots 2 . \frac{Q_0}{2x} = \left(\frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\dots+\frac{1}{2x-1}\right) Q_x.$$

For example. If $x=4$,

$$8\frac{1.3.5}{2} + 8.6.\frac{1.3}{4} + 8.6.4.\frac{1}{6} + 8.6.4.2.\frac{1}{8}$$

$$= 60 + 36 + 32 + 48 = 176.$$

So, too,

$$3.5.7 + 1.5.7 + 1.3.7 + 1.3.5 = 176.$$

The value of $u_{n,i}$ is, of course, $\Pi(i-1)\, Q_{n-i}$.

There is a more elaborate system of $2n$ equations, the resultant of which can be made to depend on that of the system of n equations just ascertained. Thus, take $2n=6$, and consider the system

$$ax+by+cz+dt+eu+fv\,;\quad x+y+z+t+u+v\,;$$
$$ax^3+by^3+cz^3+dt^3+eu^3+fv^3;\quad x^3+y^3+z^3+t^3+u^3+v^3;$$
$$ax^5+by^5+cz^5+dt^5+eu^5+fv^5;\quad x^5+y^5+z^5+t^5+u^5+v^5;$$

the order of the resultant of this system in the letters a, b, c, d, e, f is obviously $1.3.5\,(1.3+1.5+3.5)$.

Now *pair* the six variables in every possible manner; the number of such pairs is $1.3.5$.

Let x, y, z, t, u, v be any one such set of pairs. Make

$$x+y=0,\quad z+t=0,\quad u+v=0\,;$$

then the latter set of three functions become zero, and the former three may be made zero with right assignments of x, z, t, provided the resultant of

$$\begin{aligned}&(a-b)\,x+(c-d)\,z+(e-f)\,u,\\&(a-b)\,x^3+(c-d)\,z^3+(e-f)\,u^3,\\&(a-b)\,x^5+(c-d)\,z^5+(e-f)\,u^5\end{aligned}$$

is zero. Hence the required resultant will *contain* the product of the resultants of the $1.3.5$ systems formed after the above pattern; and as this product will be of $1.3.5\,(1.3+1.5+3.5)$ dimensions in the constants, it must be not merely contained in, but identical with, the required resultant. Thus the new set of functions regarded as hyper-loci (like the former set) can only be made to intersect in one or another of a fixed group of points. Moreover, passing to the case of $2n$ equations, it is obvious that the resultant of such system will be made up exclusively of factors of the form

$$(a_1+a_2+\ldots+a_i-a_{i+1}-a_{i+2}-\ldots-a_{2i})^{J_{n,i}}\,*,$$

where $J_{n,i}$ is a function of n and i to be determined. The value of $u_{n,i}$, which has been found above, leads to this without difficulty. By an obvious method of calculation it may be shown that

$$\begin{aligned}J_{n,i}&=u_{n,i}\,\{1.3.5\ldots(2n-1)\}\,\frac{n.(n-1)\ldots(n-i+1)}{1.2\quad\ldots\quad i}\,2^{i-1}\\&\quad\div\left\{\frac{2n.(2n-1)\ldots(2n-2i+1)}{1.2\quad\ldots\quad 2i}\cdot\frac{1}{2}\cdot\frac{2i.(2i-1)\ldots(i+1)}{1.2\quad\ldots\quad i}\right\}\\&=2i\,.\,2^{i-1}\cdot\frac{\Pi n}{\Pi 2n}\cdot\frac{\Pi(2n-2i)}{\Pi(n-i)}\,\{\Pi(i-1)\}^2\,Q_{n-i}\,.\,Q_n\\&=\Pi(i-1)\,\Pi i\,(Q_{n-i})^2=i\,(u_{n,i})^2.\end{aligned}$$

* It will, of course, be understood that a_1, a_2, a_3, &c. are written above in place of a, b, c, &c.

We thus obtain the following Table for finding the *frequency* $J_{n,i}$ of any given form of factor:—

	$i=$ 1,	2,	3,	4,	5
$n=1$	1				
$=2$	1	2			
$=3$	9	2	12		
$=4$	225	18	12	144	
$=5$	$(105)^2$	450	108	140	2880

The resultant thus determined is the coefficient of the leading term of an equation of the degree $1^2 . 3^2 . 5^2 \ldots (2n-1)^2$, upon which depends the determination of a set of $2n$ quantities $\xi_1, \xi_2 \ldots, \xi_n$, so chosen as to make the arc of the curve whose equation is

$$y = a_1 \log (x^2 - \xi_1^2) + a_2 \log (x^2 - \xi_2^2) + \ldots + a_{2n} \log (x^2 - \xi_{2n}^2)$$

equal to

$$x + a_1 \log \frac{x - \xi_1}{x + \xi_1} + \ldots + a_{2n} \log \frac{x - \xi_{2n}}{x + \xi_{2n}},$$

$a_1, a_2, \ldots a_{2n}$ being $2n$ given unequal quantities. It follows from the above that the number of distinct solutions is $1^2 . 3^2 \ldots (2n-1)^2$, unless one group of i of the coefficients a and a second group of i other of them can be found such that the sum of the one group is equal to the sum of the other; in that case, and in that case only, the number of solutions undergoes a reduction. A similar conclusion can be extended to the case of an odd number $(2n+1)$ of the parameters (a), in which case the number of solutions is $1^2 . 3^2 \ldots (2n-1)^2 (2n+1)$, except when, as above, two sets of parameters can be found the same in number and equal in amount, in which case the number of solutions undergoes a reduction as before.

I mention these facts with the view of making it understood that the problems of elimination herein proposed and solved are not mere idle dreams and speculations of the fancy, but have a real ontological significance in connexion with a great Algebraico-Diophantine problem of the Integral Calculus.

P.S. Suppose ν to be any positive integer, even or odd, and that the curve or compound symmetrical logarithmic wave

$$y = \Sigma_{\theta=\nu}^{\theta=1} a_\theta \log (x^2 - \xi_\theta^2)$$

is to be made subject to the relation

$$\textit{arc minus abscissa} = \Sigma_{\theta=\nu}^{\theta=1} a_\theta \log \left(\frac{x - \xi_\theta}{x + \xi_\theta}\right).$$

Then the a coefficients (or form-parameters) being given, the ξ quantities (or asymptotic distances from the Y axis of the logarithmic wavelets) depend on the solution of an algebraical equation whose degree is the product of ν terms of the series

$$1,\ 1,\ 3,\ 3,\ 5,\ 5,\ 7,\ \ldots.$$

When $\nu = 2n$, the coefficient of the leading term of this equation is the resultant of the system, or rather double system, of $2n$ functions of $2n$ variables which has been already discussed.

When $\nu = 2n+1$, the coefficient of the leading term is the resultant of a system of $2n+1$ functions of $2n+1$ variables: $(n+1)$ of them of the form Σx, Σx^3, ... Σx^{2n+1}; n of them of the form Σax, Σax^3, ... Σax^{2n+1} respectively.

To obtain this last-named resultant we may pair the variables (leaving *one* out) in every possible way, then make the sum of each pair and also the solitary or unpaired one zero, and finally, substituting in the n equations last stated (which come down to the form of a system of n equations between n variables discussed at the outset of this paper), calculate its resultant*. The product of all the resultants so found will be the resultant required, as may be proved by counting its order in the given coefficients, which is easily ascertained to be

$$1\,.\,3\,.\,5\ldots(2n+1)\left(\frac{1}{1}+\frac{1}{3}+\ldots+\frac{1}{2n-1}\right)\{1\,.\,3\,.\,5\ldots(2n-1)\},$$

as it ought in order to be the complete resultant. It will be seen then that this complete resultant, like the former one, is still made up of linear factors of the form

$$(a_1+a_2+\ldots+a_i-a_{i+1}-a_{i+2}-\ldots-a_{2i}),$$

and it only remains to ascertain the *frequency* of each such factor. By a calculation precisely similar in nature to that indicated for the case of $\nu = 2n$, it will be found that for this case of $\nu = 2n+1$ the *frequency* in question

$$= \Pi(i-1)\,\Pi i\,.\,Q(n-i)\,Q(n-i+1).$$

For $\nu = 2n$ it has been already proved to be

$$\Pi(i-1)\,\Pi i\,\{Q(n-i)\}^2.$$

* Regarded as loci, the ν functions can only intersect in one or another of an invariable system of points independent of the particular values of the coefficients. The equations to any one of these points (from what has been shown in the text) will easily be seen to be of the form

$$\left\{\begin{matrix} x_1 = x_2 = \ldots = x_{2i} = -x_{2i+1} = -x_{2i+2} = \ldots = -x_{2j}, \\ x_{2j+1} = 0, \quad x_{2j+2} = 0, \ \ldots\ x_\gamma = 0. \end{matrix}\right\}$$

Hence by a simple enough combinatorial calculation it may be deduced that the number of these fixed possible points of intersection, or, so to say, ganglions of the system is $\frac{3^\gamma + (-1)^\gamma}{8} - \frac{1}{4}$, which is, of course, always an integer; or, more briefly, the ganglionic exponent is the integer part of $\frac{3^\gamma}{8}$.

Thus we obtain the complete double-entry Table of *Frequency* under-written:

$i=$	1,	2,	3,	4,	5,	6
$\nu=2$	1					
$=3$	1					
$=4$	1	2				
$=5$	3	2				
$=6$	9	2	12			
$=7$	45	6	12			
$=8$	225	18	12	144		
$=9$	1575	90	36	144		
$=10$	11025	450	108	144	2880	
$=11$	99225	3150	540	432	2880	
$=12$	893025	22050	2700	1296	2880	3628800

This Table, although obtained by two slightly varying processes according as ν is even or odd, forms, and ought to be regarded as, an organic whole.

To prevent misconception, I ought to add that when ν is sufficiently great, the compound symmetrical logarithmic wave $\Sigma a \log (x^2 - \xi^2)$ admits of other rectifiable cases besides those of the form $\Sigma a \log \left(\frac{x-\xi}{x+\xi}\right)$ above adduced.

It remains to study the relation between the frequency of each factor and the nature of the corresponding contact between the functions (regarded as *loci*) into whose resultant it enters. I have reason to hope that Dr Olaus Henrici, who has done such valuable work in the theory of Discriminants and Resultants, may be disposed to take up this interesting and pregnant question.

107.

NOTE ON THE THEORY OF A POINT IN PARTITIONS.

[*Edinburgh British Association Report* (1871), pp. 23—25.]

IN writing down all the solutions in positive integers of the indefinite *Equation of Weight*, $x + 2y + 3z + \ldots = n$, or, in other words, in exhibiting all the partitions of n any integer greater than zero, it may sometimes be useful to be provided with an easy test to secure ourselves against the omission of any of them. Such a test is furnished by the following theorem:—

$$\Sigma(1 - x + xy - xyz \ldots) = 0;$$

thus, for example, if $x + 2y + 3z + 4t + \ldots = 4$, the solutions are five in number, namely

(1) $y = 2$,

(2) $t = 1$,

(3) $x = 1$, $z = 1$,

(4) $x = 2$, $y = 1$,

(5) $x = 4$,

the values of the omitted variables in each solution being zero. The five corresponding values of $1 - x + xy \ldots$ are

$$1, \quad 1, \quad 0, \quad 1, \quad -3,$$

whose sum is zero.

The theorem may be proved immediately by expressing the denumerant (which is zero) of the simultaneous equations

$$\begin{Bmatrix} x + 2y + 3z + \ldots = n, \\ x + y + z + \ldots \quad = 0, \end{Bmatrix}$$

in terms of simple denumerants according to the author's general method, or by virtue of the known theorem,

$$(1-t)(1-t^2)(1-t^3)\ldots$$
$$=1-\frac{t}{(1-t)}+\frac{t^3}{(1-t)(1-t^2)}-\frac{t^6}{(1-t)(1-t^2)(1-t^3)}+\frac{t^{10}}{(1-t)(1-t^2)(1-t^3)(1-t^4)}+\ldots.$$

This gives at once the equation

$$\frac{1}{(1-t)(1-t^2)(1-t^3)\ldots}-\frac{t}{(1-t)^2(1-t^2)(1-t^3)\ldots}+\frac{t^3}{(1-t)^2(1-t^2)^2(1-t^3)\ldots}+\ldots=1.$$

Hence the coefficient of t^n in the above written series for all values of n other than zero is zero. But it will easily be seen that the coefficient of t^n in the first term is $\Sigma 1$, in the second term Σx, in the third Σxy, &c.; so that

$$\Sigma(1-x+xy\ldots)=0,$$

as was to be shown. Thus we have obtained for the problem of indefinite partition a new algebraical unsymmetrical test supplementing the well-known pair of transcendental symmetrical tests expressible by the equations

$$\Sigma\frac{\Pi(x+y+z\ldots)}{\Pi x\Pi y\Pi z\ldots}=2^{n-1},$$

$$\Sigma(-)^{x+y+z\ldots}\frac{\Pi(x+y+z\ldots)}{\Pi x\Pi y\Pi z\ldots}=0\,*.$$

The identity employed in the text is only a particular case of Euler's identity,

$$(1+tz)(1+t^2z)(1+t^3z)\ldots=1+\frac{tz}{(1-t)}+\frac{t^3z^2}{(1-t)(1-t^3)}+\ldots,$$

which is tantamount to affirming that the number of partitions of n into r distinct integers is the same as the number of partitions of n into any

* Subject of course to the condition that n is greater than 1. If $x, y, z, \ldots, \omega$ represents any solution in positive integers of the equation

$$x+2y+3z+\ldots+r\omega=r,$$

it is easy to see that

$$\Sigma(-)^{x+y+\ldots+}\frac{\Pi(x+y+\ldots+\omega)}{\Pi x\,\Pi y\ldots\Pi\omega}=1,\ -1,\text{ or }0,$$

according as n, in regard to the modulus $r+1$, is congruent to 0, 1, or neither to 0 nor 1, for the left-hand side of the equation is obviously the coefficient of x^n in the development of

$$\frac{1}{1+x+x^2\ldots+x^r},\ \text{that is}\ \frac{1-x}{1-x^{r+1}}.$$

On making $r=\infty$, this theorem becomes the one in the text. It obviously affords a remarkable pair of independent arithmetical quantitative criteria for determining whether or not one number is divisible by another.

integers none greater than r, in which all the integers from 1 to r appear once at least. It has not, I believe, been noticed that these two systems of partitions are conjugate to each other, each partition of the one system having a correspondent to it in the other. The mode of passing from any partition to its correspondent is by converting each of its integers into a horizontal line of units, laying these horizontal lines vertically under each other, and then summing the columns. Thus, for example, 3, 4, 5 will be first expanded horizontally into

$$\begin{array}{lllll} 1 & 1 & 1, & & \\ 1 & 1 & 1 & 1, & \\ 1 & 1 & 1 & 1 & 1, \end{array}$$

and then summed vertically into

$$3 \quad 3 \quad 3 \quad 2 \quad 1.$$

This is the method employed by Mr Ferrers to show that the number of partitions of n into r, or a less number of parts, is the same as the number of partitions of n into parts none greater than r, and is, in fact, only a generalization of the method of intuitive proof of the fact that

$$m \times n = n \times m,$$

the difference merely being that we here deal with a parallelogram separated into two conterminous parts by an irregularly stepped boundary—one filled with units, the other left blank, instead of dealing with one entirely filled up with units.

108.

ON AN ELEMENTARY PROOF OF SIR ISAAC NEWTON'S HITHERTO UNDEMONSTRATED RULE, GIVEN IN THE *ARITHMETICA UNIVERSALIS*, FOR THE DISCOVERY OF IMAGINARY ROOTS IN ALGEBRAICAL EQUATIONS.

[*Transactions of the Royal Irish Academy*, XXIV. (1871), pp. 247—252.]

I HAVE the honour to lay before the Royal Irish Academy a brief statement of the principal results I have obtained in an investigation which leads to a theorem, from a particular case of which Newton's celebrated and hitherto undemonstrated rule, published in the *Arithmetica Universalis*, flows as a corollary in precisely the same way as Des Cartes' rule can be made to flow from the theorem of Fourier.

Suppose $fx=0$ an ordinary algebraical equation of the nth degree divested of multiple roots, and let us consider the coupled series, or pair of progressions,

$$\left.\begin{matrix} f^n x, & f^{n-1}x, & f^{n-2}x, & \dots & f^1 x, & fx \\ G_n x, & G_{n-1}x, & G_{n-2}x, & \dots & G_1 x, & Gx \end{matrix}\right\}, \qquad \text{(A)}$$

where
$$f^r x \text{ means } \left(\frac{d}{dx}\right)^r fx,$$

and
$$G_r x = (f_r x)^2 - \lambda_r f_{r-1} x f_{r+1} x,$$

it being understood that

$$G_n x = (f_n x)^2, \quad Gx = (fx)^2.$$

Let us further suppose that λ_r satisfies the equation in differences

$$2 - \lambda_r = \frac{1}{\lambda_{r+1}};$$

and furthermore that λ_r is positive for all values of r from 1 to n, both inclusive.

When $f^r x = 0$ we have, as is well known,

$$f^r(x+\epsilon) = f^{r+1}x \,.\, \epsilon\,;$$

and in general, when

$$f^r x = 0, \quad f^{r+1}x = 0 \ldots f^{r+i-1}x = 0,$$

$$f^r(x+\epsilon) = f^{r+i}x \frac{\epsilon^i}{1\,.\,2 \ldots i}, \tag{B}$$

ϵ being supposed to be an infinitesimal. So similarly it will be found that when $G_r x = 0$, then

$$G_r(x+\epsilon) = \frac{\epsilon f^r x}{\lambda_{r+1} f^{r+1} x} G_{r+1} x\,;$$

and more generally, when

$$G_r x = 0, \quad G_{r+1} x = 0 \ldots G_{r+i-1} x = 0,$$

$$G_r(x+\epsilon) = \frac{f^r x}{\lambda_{r+1}\,.\,\lambda_{r+2} \ldots \lambda_{r+i} f^{r+i} x} G_{r+i} x \,.\, \frac{\epsilon^i}{1\,.\,2 \ldots i}. \tag{C}$$

Now let us understand by a *double permanence* in the coupled series (A) a double succession of associated terms, as $\begin{Bmatrix} f^r x & f^{r+1} x \\ G_r x & G_{r+1} x \end{Bmatrix}$ such that $\frac{f^{r+1}x}{f^r x}$ and $\frac{G_{r+1}x}{G_r x}$ are both positive; and let us study the effect produced upon the total number of such double permanences by increasing x continuously.

It is clear that no change in such number can take place, except by one or more of the terms in the upper or in the lower, or in both, simultaneously becoming zero.

$G_n x$ and Gx, and $f^n x$ being, the two former necessarily positive, and the third a constant, can never pass through zero. Let us suppose that, for a certain value of x, certain of the intermediate terms become zero, but that fx does not become zero.

The following cases arise and may be considered separately:—

(1) $f^r x$ may become zero alone without either of the adjacent terms doing so, in which case $G_r x$ by definition cannot become zero.

(2) $f^r x, f^{r+1} x, \ldots f^{r+i-1} x$ may all become zero simultaneously, in which case, i being supposed greater than unity, $G_r x$, $G_{r+1} x, \ldots G_{r+i-1} x$ will all become zero likewise.

(3) $G_r x$ may become zero without either of the adjacent terms doing so.

(4) $G_r x$, $G_{r+1} x, \ldots G_{r+i-1} x$, may all become zero simultaneously without the superior associated terms any of them vanishing.

On considering all these cases in conjunction with the equations (B), (C), it will be found that by the passage through zero double permanences may

be gained, but can never be lost, as x goes on increasing, and moreover that the number so gained is always even.

Now let us suppose that the last superior term fx becomes zero, that is that x becomes a root of the given equation, then it will be easily found that one double permanence is gained*.

Hence, if i real roots are included between λ and μ, and if the number of double permanences in the coupled series (A) when λ is written for x be called $P(\lambda)$, and the like quantity when μ is written for x be called $P(\mu)$; and if μ is greater than λ, we must have $P(\mu)-P(\lambda)=i+2k$, where k is zero or a positive integer. So that the number of real roots between λ, μ cannot exceed the difference between $P(\mu)$ and $P(\lambda)$.

It remains only to integrate the equation $2-\lambda_r=\dfrac{1}{\lambda_{r+1}}$; the general integral of this is easily found to be

$$\lambda_r=\frac{A+B(r-1)}{A+Br};$$

so that

$$G_r x=(f^r x)^2-\frac{A+B(r-1)}{A+Br}f^{r-1}x\,.\,f^{r+1}x;$$

the ratio $A:B$ being subject to the sole condition that $\dfrac{A+B(r-1)}{A+Br}$ shall be positive for all integer values of r not exterior to the limits 1, n, which condition is equivalent to the condition that A and $A+(n-1)B$ shall each have the same sign.

Let us suppose $A=n$, $B=-1$, then

$$\left.\begin{aligned} G_n x &= (f^n x)^2, \\ G_{n-1}x &= (f^{n-1}x)^2-\frac{2}{1}f^n x f^{n-2}x, \\ G_{n-2}x &= (f^{n-2}x)^2-\frac{3}{2}f^{n-1}x f^{n-3}x, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ G_1 x &= (f'x)^2-\frac{n}{n-1}f''x fx, \\ Gx &= (fx)^2. \end{aligned}\right\} \qquad \text{(D)}$$

Let fx be written under the form

$$c_0x^n+nc_1x^{n-1}+\frac{1}{2}n(n-1)c_2x^{n-2}+\ldots+nc_{n-1}x+c_n;$$

* This is on the supposition of the root in question being simple or unifold; if it is a multifold root, equivalent to i simple roots, i double permanences will be gained.

write down the coupled series

$$\begin{matrix} c_0, & c_1, & c_2, & \dots c_n \\ c_0^2, & c_1^2 - c_0c_2, & c_2^2 - c_1c_3, & \dots c_n^2 \end{matrix} \text{ say } \begin{matrix} c_0, & c_1, & c_2, & \dots c_n. \\ T_0, & T_1, & T_2, & \dots T_n. \end{matrix}$$

On consulting the *Arithmetica Universalis,* it will be found that Newton's complete theorem amounts to asserting that the number of negative roots in $fx = 0$ cannot exceed the number of double permanences in these associated series, and (which is a necessary inference from the former, and needs no separate proof, since to obtain it we need only to change x into $-x$ in the equation $fx = 0$) that the number of positive roots in the same cannot exceed the number of variation-permanences in such association; by a variation-permanence understanding a double succession in which the superior terms c_r, c_{r+1} form a variation, and the associated inferior terms T_r, T_{r+1} a permanence.

But it is easily seen that

$$c_0 = \frac{1}{\Pi n} f^n 0,$$

$$c_1 = \frac{1}{\Pi n} f^{n-1} 0,$$

$$c_2 = \frac{1.2}{\Pi n} f^{n-2} 0,$$

$$c_3 = \frac{1.2.3}{\Pi n} f^{n-3} 0,$$

$$\dots\dots\dots\dots\dots\dots$$

$$c_n = \frac{1.2.3 \dots n}{\Pi n} f 0;$$

and consequently,

$$T_0 = \frac{1}{(\Pi n)^2} (f^n 0)^2,$$

$$T_1 = \frac{1}{(\Pi n)^2} \left\{ (f^{n-1} 0)^2 - \frac{2}{1} f^n 0 f^{n-2} 0 \right\},$$

$$T_2 = \frac{(1.2)^2}{(\Pi n)^2} \left\{ (f^{n-2} 0)^2 - \frac{3}{2} f^{n-1} 0 f^{n-3} 0 \right\},$$

$$T_3 = \frac{(1.2.3)^2}{(\Pi n)^2} \left\{ (f^{n-3} 0)^2 - \frac{4}{3} f^{n-2} 0 f^{n-4} 0 \right\},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$T_n = \frac{(1.2.3 \dots n)^2}{(\Pi n)^2} f_0^2.$$

So that, using the symbol G in the special sense in which it is employed in equations (D), if we compare the two coupled systems

$$\left.\begin{matrix} c_0, & c_1, & c_2, \dots & c_n \\ T_0, & T_1, & T_2, \dots & T_n \end{matrix}\right\}, \quad \left.\begin{matrix} f^n 0, & f^{n-1} 0, & f^{n-2} 0, \dots f 0 \\ G_n 0, & G_{n-1} 0, & G_{n-2} 0, \dots G 0 \end{matrix}\right\},$$

it is evident that each term in the one association is a positive multiplier of the corresponding term in the other. But by the theorem previously established we know that the number of negative roots in fx, that is the number of roots between 0 and $-\infty$, cannot exceed the difference between the number of double permanences in the (f, G) couple, less the number of double permanences in this couple, when $-\infty$ is substituted in place of 0; but in the couple so modified, the superior terms present only variation successions, so that there can be no double permanences, and consequently the above assertion is tantamount to the more simple one, that the number of negative roots in fx cannot exceed the number of double permanences in the (f, G) couple, which are evidently identical with those in the (c, T) couple. This demonstrates Newton's complete rule.

More generally, if we form a double series precisely analogous to the (c, T) couple, except that the c's are derived from the equation $f(x+p)=0$ in lieu of the equation $fx=0$, and if we call the number of double permanences so resulting the number of such due to p, we have the more general theorem that the number of real roots passed over in ascending from p to q cannot exceed the number of double permanences due to q less the number of the same due to p, and can only fall short of such difference by some *even* number. This, however, it will be seen, is only a particular case of the general theorem where λ_r is of the form

$$\frac{A + B(r-1)}{A + Br}.$$

But furthermore it is proper to state, that although such is the most general form that according to what has been above demonstrated can be ascribed to λ_r, it follows from other considerations that it is *not* the most general.

109.

ON THE PARTITION OF AN EVEN NUMBER INTO TWO PRIMES.

[*Proceedings of the London Mathematical Society*, IV. (1871—73), pp. 4—6.]

IN one of his minor papers, Euler has enunciated, as a theorem resting entirely on intuition from a comparatively small number of instances, that every even number may be decomposed into a sum of two primes. The object of Mr Sylvester's communication was to obtain some measure of the probable number of ways in which such decomposition can be effected for any given number; if it can be shown to be probably greater than the square root of the number itself, it will follow, from generally admitted principles of the theory of chances, that the probability of the theorem being universally true above any assigned limit, if *proved* to be true up to that limit, may be represented by an infinite product of terms which will approach as near as we please to unity the higher the limit is taken.

The mere fact of the theorem, as Euler gave it, being proved up to 100,000,000 or any other number, however great, would leave the probability of its being universally true absolutely zero, just as the fact of the sun having risen 100,000,000 times would not contribute an atom of probability to the supposition that it would continue to rise for all time to come. In the case before us, on the contrary, the probability of the theorem being universally true by a sufficiently copious induction may be made to approach as near as we please to absolute certitude. The Author considers that he has established beyond the reach of reasonable doubt that the order of the magnitude which represents the mean probable value of the number of modes of effecting the resolution of a very large even number into two prime numbers is that of the square of the number

of primes inferior to the given number divided by the number itself, or, which (thanks to the discoveries of Legendre and Tchebicheff) we know to be the same thing, the number of the decompositions in question bears a finite ratio (assignable within limits) to the number to be decomposed divided by the square of its Napierian logarithm. If we agree provisionally to call preter-primes in respect to n those numbers which are prime themselves and also when subtracted from n leave prime remainders, the Author shows that the probable number of such preter-primes (that is, the most probable value attainable under our present conditions of knowledge) may be found approximately by multiplying the number of ordinary primes inferior to n by the product of a set of fractions depending in part on the order of the magnitude and in part on the constitution of the number n. If n is the double of a prime, the product in question is got by multiplying together all the quantities $\frac{p-2}{p-1}$, where p is every odd prime between unity and the square root of n; but if n itself contains any such primes among its factors, then the corresponding factors are to be omitted out of the product. We thus see that if two even numbers of considerable magnitude lie adjacent and tolerably near to each other, one of which is the double of a prime, but the other six times a prime, the number of preter-primes relative to the latter will be about twice as many as those relative to the former. For the purpose of greater simplicity of explanation, the formula of approximation has been stated above with less accuracy than it admits of being stated with; instead of the total number of odd primes being multiplied by the product of factors last described, those only should have been taken which are not intermediate between 2 and $\sqrt{n}$, and the result so modified should have been stated to be the probable value, not of the total number of preter-primes, but only of such of them (by far the larger number) as are not of the excluded class above described, nor, subtracted from n, give rise to remainders belonging to such class. The Author has found, by actual trial on an extensive scale, that the estimated values of the number of decompositions never differ by more than a moderate, and in some cases exceedingly slight, percentage from their actual values, determined by the use of Borchardt's tables.

The same methods enable him also to assign a probable value to the number of modes of resolving an odd number into the sum of one prime and the double of another, and in general leads to an approximate representation of the number of solutions in prime numbers of any system of linear equations of which the total number of solutions is limited, and even to resolve approximately such questions as that of determining how many prime numbers there are inferior to a given limit which are followed by prime numbers differing from them by any assigned interval.

Since the communication was made to the Mathematical Society, the Secretaries have been informed by Mr Sylvester that he has verified his results by quite a different method. The exact number of the solutions of the equation $x+y=n$ in prime numbers may be expressed algebraically by means of the method of Generating Functions in terms of the inferior primes to n. The expression will be found to consist of two parts—one a constant multiple of n, the other a function of the roots of unity corresponding to the several inferior primes and their combinations. The former non-periodic part may obviously be regarded as the mean value of the expression, and Mr Sylvester has found that it is identical with the value obtained by the method of averages previously employed.

110.

ON THE THEOREM THAT AN ARITHMETICAL PROGRESSION WHICH CONTAINS MORE THAN ONE, CONTAINS AN INFINITE NUMBER OF PRIME NUMBERS.

[*Proceedings of the London Mathematical Society*, IV. (1871—73), pp. 7, 8.]

THIS celebrated theorem is one of those which no one would think of doubting, but which are of extreme difficulty to prove. A pretended proof had been given by Legendre a good part of a century ago, and occupies a whole chapter in the *Théorie des Nombres*; but the first real demonstration was accomplished by Lejeune Dirichlet in his great memoir published in the Berlin *Transactions* for the year 1837.

The present communication is limited to the case of Arithmetical Progressions, proceeding according to the common difference 4 or 6. The fundamental theorem employed is an identical equation, on the one side of which are algebraical fractions of the form $\frac{x^p}{1-x^{2p}}$, where p represents any combination of the simple powers of any system of primes taken with the positive or negative sign, according as p contains an even or an odd number of factors, and on the other side simple powers of x, whose indices are all the odd numbers not containing any one of the given system of primes as a factor. In the case of progressions with the common difference 4, all the primes of the form $4q+3$ and their primary combinations figure as indices on the first side of the equation, and consequently the powers of x on the other side have for their indices combinations of factors of the form $4q+1$. By writing in place of x the square root of negative unity into x, it is shown instantaneously that if the number of primes of the form $4q+3$ were finite, a finite series of fractions converging to an infinite value as x approaches

to unity would be equal to another such series which would remain finite, which is of course absurd. The proof applicable to the case of progressions of the form $4q+1$ is not quite so simple: it depends on showing that, if their number were only i in number, it would be possible to have a rational integer function of the degree $2i$ in the logarithm of n greater than a finite multiple of n for a value of n unlimitedly great, which is known to be absurd.

A process precisely similar applies, *mutatis mutandis*, to the case of progressions of the form $6q+1$, $6q+5$, the sole difference being that, instead of substituting for x, x multiplied by the square root of negative unity, we must now substitute for it x multiplied successively by the two prime sixth-roots of unity, and subtract the results from one another. The method here successfully employed in the treatment of these elementary cases appears to differ fundamentally from Dirichlet's method, in regard of the circumstance that it deals with an infinitesimal variation in the value of the variable, whereas in Dirichlet's method the infinitesimal variation takes place in the index of the power of the variable.

APPENDIX.

Additional Notes to Prof. Sylvester's Exeter British Association Address

given in the author's Reprint of this address published in the volume *The Laws of Verse* (London, Longmans, 1870), together with the author's appendix to that volume

On the Incorrect Description of Kant's Doctrine of Space and Time common in English Writers,

a correspondence reprinted from *Nature* (Vol. I., 1869—70).

[*The references are to the pages of the present Volume.*]

p. 654, l. 38. The annexed instance of Mathematical euristic is, I think, from its intrinsic interest, worthy of being put on record. The so-called canonical representation of a binary quartic of the eighth degree I found to be a quartic multiplied by itself, together with a sum of powers of its linear factors, just as for the fourth degree it was known to be a quadric into itself, together with a sum of powers of its factors; but for a sextic a cubic multiplied into itself, with a tail of powers as before, was not found to answer. To find the true representation was like looking out into universal space for a planet desiderated according to Bode's or any other empirical law. I found my *desideratum* as follows: I invented a catena of morphological processes which, applied to a quadric or to a quartic, causes each to reproduce itself: I then considered the two quadrics and two quartics to be noumenally distinguishable (one as an automorphic derivative of the other) although phenomenally identical. The same catena of processes applied to the cubic gave no longer an identical but a distinct derivative, and the product of the two I regarded as the analogue of the before-mentioned square of the quadric or of the quartic. This product of a cubic by its derivative so obtained together with a sum of powers of linear factors of the original cubic, I found by actual trial to my great satisfaction satisfied the conditions of canonicity, and it was thus I was led up to the desired representation, which will be found reproduced in one of Prof. Cayley's memoirs on Quantics, and in Dr Salmon's lectures on Modern Algebra. Here certainly induction, observation, invention, and experimental verification all played their part in contributing to the solution of the problem. I discovered and developed the whole theory of canonical binary forms for odd degrees, and, as far as yet made out, for even degrees too, at one evening sitting, with a decanter of port wine to sustain nature's flagging energies, in a back office in Lincoln's Inn Fields. The work was done, and well done, but at the usual cost of racking thought—a brain on fire, and feet feeling, or feelingless, as if plunged in an ice-pail. *That night we slept no more.* The canonisant of the quartic (its cubic covariant) was the first thing to offer itself in the inquiry. I had but to think the words "Resultant of Quintic and its Canonisant," and the octodecadic skew invariant would have fallen spontaneously into my lap. By quite another mode of consideration, M. Hermite subsequently was led to the discovery of this, the key to the innermost sanctuary of invariants—so hard is it in Euristic to see what lies immediately before one's eyes. The disappointment weighed deeply, far too deeply, on my mind, and caused me to relinquish for long years a cherished field of meditation; but the whirligig of time brings about its revenges. Ten years later this same Canonisant gave me the upper hand of my honoured predecessor and guide, M. Hermite, in the inquiry (referred to at the end of this address) concerning the invariantive criteria of the constitution of a Quintic

with regard to the real and imaginary. By its aid I discovered the essential character of the famous amphigenous surface of the ninth order, and its bicuspidal unicursal section of the fourth order (otherwise termed the bicorn), as may be seen in the third part of my Trilogy, printed in the *Philosophical Transactions* [p. 376, above].

p. 654, l. 41. I was under the conviction that a passage to that effect from Lagrange had been cited to me some years ago by M. Hermite of the Institute of France; on applying to him on the subject, I received the following reply:

"Relativement à l'opinion que suivant vous j'aurais attribuée à Lagrange, je m'empresse de vous informer qu'il ne faut aucunement, à ma connaissance, l'en rendre responsable. Nous nous sommes entretenus du rôle de la *faculté d'observation dans les études que nous avons poursuivies de concert pendant bien des années*, et c'est alors, sans doute, que je vous aurai conté une anecdote que je tiens de M. Chevreul. M. Chevreul, allant à l'Institut dans la voiture de Lagrange, a été vivement frappé du sentiment de plaisir avec lequel ce grand géomètre lui faisait voir, dans un travail manuscrit, la beauté extérieure et artistique, si je peux dire, des nombreuses formules qui y figuraient. Ce sentiment nous l'avons tous éprouvé en faisant, avec sincérité, abstraction de l'idée analytique dont les formules sont l'expression écrite. Il y a là, n'est-il point vrai, un imperceptible lien qui rattache au monde de l'art le monde abstrait de l'algèbre et de l'analyse, et j'ôserai même vous dire que je crois a des sympathies réelles, qui vous font trouver un charme dans les notations d'un auteur et des répulsions qui éloignent d'un autre, par l'apparence seule des formules."

I am, however, none the less persuaded that on one or more than one occasion, M. Hermite, speaking of Lagrange, expressed to me, if not as I supposed on Lagrange's, then certainly on his own high authority, "that the faculty of observation was no less necessary for the successful cultivation of the pure mathematical than of the natural sciences." I am glad also to notice that Lagrange was able to accommodate a friend *dans sa voiture*. England has much to learn from France and Russia as to the proper mode of treating its greatest men.

p. 655, l. 9. It is very common, not to say universal, with English writers, even such authorised ones as Whewell, Lewes, or Herbert Spencer, to refer to Kant's doctrine as affirming space "to be a form of thought," or "of the understanding." This is putting into Kant's mouth (as pointed out to me by Dr C. M. Ingleby), words which he would have been the first to disclaim, and is as inaccurate a form of expression as to speak of "the plane of a sphere," meaning its surface or a superficial layer, as not long ago I heard a famous naturalist do at a meeting of the Royal Society. Whoever wishes to gain a notion of Kant's leading doctrines in a succinct form, weighty with thought, and free from all impertinent comment, should study Schwegler's *Handbook of Philosophy*, translated by Stirling. He will find in the same book a most lucid account of Aristotle's doctrine of matter and form, showing how matter passes unceasingly upwards into form, and form downwards into matter; which will remind many of the readers of *Nature* of the chain of depolarisations and repolarisations which are supposed to explain the decomposition of water under galvanic action, eventuating in oxygen being thrown off at one pole and hydrogen at the other (it recalls also the high algebraical theories in which the same symbols play the part of operands to their antecedents and operators to their consequents): at one end of the Aristotelian chain comes out πρώτη ὕλη, at the other πρῶτον εἶδος. We have, then, only to accept and apply the familiar mathematical principle of the two ends of infinity being one and the same point (like the extremities of a divided and extended ring), and the otherwise immovable stumbling-block of duality is done away with, and the universe reintegrated in the wished-for unity. For this

corollary, which to many will appear fanciful, neither Aristotle nor Schwegler is responsible. We perfectly understand how in perspective the latent polarities of any point in a closed curve (taken as the object) may be developed into and displayed in the form of a duad of *quasi* points or half-points at an infinite distance from each other in the picture. In like manner we conceive how *actuality* and *potentiality* which exist indistinguishably as one in the *absolute* may be projected into seemingly separate elements or moments on the plane of the human understanding. Whatever may be the merits of the theory in itself, this view seems to me to give it a completeness which its author could not have anticipated, and to accomplish what Aristotle attempted but avowedly failed to effect, viz. the complete subversion of the "Platonic Duality," and the reintegration of matter and mind into one.

p. 655, l. 19. I know there are many, who, like my honoured and deeply lamented friend the late eminent Prof. Donkin, regard the alleged notion of generalised space as only a disguised form of algebraical formulisation; but the same might be said with equal truth of our notion of infinity in algebra, or of impossible lines, or lines making a zero angle in geometry, the utility of dealing with which as positive substantiated notions no one will be found to dispute. Dr Salmon, in his extension of Chasles' theory of characteristics to surfaces, Mr Clifford, in a question of probability (published in the *Educational Times*), and myself in my theory of partitions, and also in my paper on Barycentric Projection in the *Philosophical Magazine* [p. 358, above], have all felt and given evidence of the practical utility of handling space of four dimensions, as if it were conceivable space. Moreover, it should be borne in mind that every perspective representation of figured space of four dimensions is a figure in real space, and that the properties of figures admit of being studied to a great extent, if not completely, in their perspective representations. In philosophy, as in aesthetic, the highest knowledge comes by faith. I know (from personal experience of the fact) that Mr Linnell or Madame Bodichon can distinguish purple tints in clouds where my untutored eye and unpurged vision can perceive only confused grey. If an Aristotle, or Descartes, or Kant, assures me that he recognises God in the conscience, I accuse my own blindness if I fail to see with him. If Gauss, Cayley, Riemann, Schläfli, Salmon, Clifford, Kronecker, have an inner assurance of the reality of transcendental space, I strive to bring my faculties of mental vision into accordance with theirs. The positive evidence in such cases is more worthy than the negative, and actuality is not cancelled or balanced by privation, as matter plus space is none the less matter. I acknowledge two separate sources of authority—the collective sense of mankind, and the illumination of privileged intellects. As a parallel case, I would ask whether it is by demonstrative processes that the doctrine of limits and of infinitely greats and smalls, has found its way to the ready acceptance of the multitude; or whether, after deducting whatever may be due to modified hereditary cerebral organisation, it is not a consequence rather of the insensible moulding of the ideas under the influence of language which has become permeated with the notions originating in the minds of a few great thinkers? I am assured that Germans, even of the non-literary class, such as ladies of fashion and novel readers, are often appalled by the hebetude of their English friends in muddling up together, as if they were nearly or quite the same thing, the reason and the understanding, in doing into English the words Vernunft and Verstand, thereby confounding distinctions now become familiar (such is the force of language) to the very milkmaids of Fatherland.

As a public teacher of mere striplings, I am often amazed by the facility and absence of resistance with which the principles of the infinitesimal calculus are accepted and assimilated by the present race of learners. When I was young, a boy of sixteen or seventeen who knew his infinitesimal calculus would have been almost pointed at in

the streets as a prodigy, like Dante, as a man who had seen hell. Now-a-days, our Woolwich cadets at the same age, talk with glee of tangents and asymptotes and points of contrary flexure and discuss questions of double maxima and minima, or ballistic pendulums, or motion in a resisting medium, under the familiar and ignoble name of *sums*.

p. 657, l. 15. Induction and analogy are the special characteristics of modern mathematics, in which theorems have given place to theories, and no truth is regarded otherwise than as a link in an infinite chain. "Omne exit in infinitum" is their favourite motto and accepted axiom. No mathematician now-a-days sets any store on the discovery of isolated theorems, except as affording hints of an unsuspected new sphere of thought, like meteorites detached from some undiscovered planetary orb of speculation. The form, as well as matter, of mathematical science, as must be the case in any true living organic science, is in a constant state of flux, and the position of its centre of gravity is liable to continual change. At different periods in its history defined, with more or less accuracy, as the science of number or quantity, or extension or operation or arrangement, it appears at present to be passing through a phase in which the development of the notion of Continuity plays the leading part. In exemplification of the generalising tendency of modern mathematics, take so simple a fact as that of two straight lines or two planes being incapable of including "a space." When analysed this statement will be found to resolve itself into the assertion that if two out of the four triads that can be formed with four points lie respectively *in directo*, the same must be true of the remaining two triads; and that if two of the five tetrads that can be formed with five points lie respectively *in plano*, the remaining three tetrads (subject to a certain obvious exception) must each do the same. This, at least, is one way of arriving at the notion of an unlimited rectilinear and planar schema of points. The two statements above made, translated into the language of determinants, immediately suggest as their generalised expression my great "Homaloidal Law," which affirms that the vanishing of a certain specifiable number of minor determinants of a given order of any matrix (i.e. rectangular array of quantities) implies the simultaneous evanescence of all the rest of that order. I made (*inter alia*) a beautiful application of this law (which is, I believe, recorded in Mr Spottiswoode's valuable treatise on Determinants, but where besides I know not) to the establishment of the well-known relations, wrung out with so much difficulty by Euler, between the cosines of the nine angles, which two sets of rectangular axes in space make with one another. This is done by contriving and constructing a matrix such that the six known equations connecting the nine cosines taken both ways in sets of threes shall be expressed by the evanescence of six of its minors; the simultaneous evanescence of the remaining minors given by the Homaloidal Law will then be found to express the relations in question (which, Euler has put on record, it drove him almost to despair to obtain), but which are thus obtained by a simple process of inspection and reading off, without any labour whatever. The fact that such a law, containing in a latent form so much refined algebra, and capable of such interesting immediate applications, should present itself to the *observation* merely as the extended expression of the ground of the possibility of our most elementary and seemingly intuitive conceptions concerning the right line and plane, has often filled me with amazement to reflect upon.

p. 657, l. 17. As the prerogative of Natural Science is to cultivate a taste for observation, so that of Mathematics is, almost from the starting-point, to stimulate the faculty of invention.

p. 658, l. 41. Is it not the same disregard of principles, the same *indifference to truth for its own sake* which prompts the question, "Where's the good of it?" in reference to speculative science, and "Where's the harm of it?" in reference to white lies and pious

frauds? In my own experience I have found that the very same class of people who delight to put the first question are in the habit of acting upon the denial implied in the second. *Abit in mores incuria.*

p. 658, l. 43. This theorem still awaits proof; it is stated, I believe, in Euler's correspondence with Goldbach: I re-discovered it in ignorance of Euler's having mentioned it, in connection with a theory of my own concerning cubic forms. The evidence in its favour is *induction* of the undemonstrative or purely accumulative kind, and it may or may not turn out eventually to be true. As a most learned scholar who heard this address given at Exeter remarked to me, not many days ago, it is certainly by no process of deduction that we make out that five times six is thirty. I mention this, because I know some, who agree, or did agree, with Professor Huxley's published opinions about mathematics, are under the impression that the higher processes of mind in mathematics only concern "the aristocracy of mathematicians": on the contrary, they lie at the very foundations of the subject. There are besides, and in abundance, mathematical processes which only by a forced interpretation can be brought under the head of demonstration, whether deductive or inductive, and really belong to a sort of artistic and constructive faculty, such for example as evaluating definite integrals, or making out the best way one can the number of distinct branches and the general character of each branch of a curve from its algebraical equation.

p. 660, l. 8. I am happy to be able to add that this gentleman who, when the above lines were printed, was at the bottom of the Staff of the Mathematical Instructors at Woolwich, has been since appointed, with the full concurrence of his colleagues, on the nomination of the Governor, Sir John Lintorn Simmons, K.C.B., to succeed me as Professor of Mathematics at the Royal Military Academy.

p. 660, l. 20. *Complete* in the sense of *universal*, more *than perfect or complete* in the ordinary sense. Two criteria are absolutely fixed; but in addition to these two an additional criterion or set of criteria must be introduced to make the system of conditions sufficient. The number of such set may be either one or whatever number we please, and into such one or into each of the set (if more than one) an indefinite number of arbitrary parameters (limited) may be introduced. Now the geometrical construction I arrive at contains implicitly the totality of all these infinitely varied forms of criteria, or sets of criteria, and without it the existence and possibility of such variety in the shape of the solution could never have been anticipated or understood. My truly eminent friend M. Charles Hermite (Membre de l'Institut), with all the efforts of his extraordinary analytical power, and with the knowledge of my results to guide him, has only been able by the non-geometrical method to arrive at one form of solution consisting of a third criterion absolutely definite and destitute of a single variable parameter. As is well known, I have made a very important use of a criterion of the same form as M. Hermite's, but containing one arbitrary parameter (limited). The subject will be found resumed from the point where I left it, and pursued in considerable detail by Prof. Cayley, in one of his more recent memoirs on Quantics in the *Philosophical Transactions.* M. Hermite it was who first surprised Invariantists (l'Eglise Invariantive, as we are sometimes styled) by an *à priori* demonstration that the nature of the roots or factors of quartics could in general be found by means of invariantive criteria. This was known to be possible up to the *fourth* order of binary quantics, and impossible for the *fourth.* M. Hermite showed that this negation which seemed to stop the way to further progress was an exceptional case; that whereas for the second, third, fifth, sixth, and all higher degrees the thing could be done, for the fourth alone it was impossible: as regards linear Quantics, the question does not arise. I look upon this failure of a law

for one term in the middle of an infinite progression as an unparalleled *miracle of arithmetic*, far more real and deeper seated than the one alluded to by Mr Babbage in connection with the discontinuous action of a supposed machine in his ninth Bridgwater Treatise.

p. 660, l. 22. So I found, as a pure matter of observation, that allineation (*alignement*) in ornamental gardening—i.e. the method of putting trees in positions to form a very great number or the greatest number possible of straight rows, of which a few special cases only had been previously considered as detached porismatic problems, forms part of a great connected theory of the pluperfect points on a cubic curve, those points, of which the nine points of inflection and Plücker's twenty-seven points may serve as the lowest instances.

On the Incorrect Description of Kant's Doctrine of Space and Time common in English Writers *.

In the very remarkable contribution by Professor Sylvester (*Nature*, No. 9) this sentence occurs: "It is very common, not to say universal, with English writers, even such authorised ones as Whewell, Lewes, or Herbert Spencer, to refer to Kant's doctrine as affirming space to be a 'form of thought' 'or of the understanding.' This is putting into Kant's mouth (as pointed out to me by Dr C. M. Ingleby) words which he would have been the first to disclaim."

It is not on personal grounds that I wish to rectify the misconception into which Dr Ingleby has betrayed Professor Sylvester. When objections are made to what I have written, it is my habit either silently to correct my error, or silently to disregard the criticism. In the present case I might be perfectly contented to disregard a criticism which any one who even glanced at my exposition of Kant would see to be altogether inexact; but as misapprehensions of Kant are painfully abundant, readers of Kant being few, and those who take his name in vain being many, it may be worth while to stop *this* error from getting into circulation through the channel of *Nature*. Kant assuredly did teach, as Professor Sylvester says, and as I have repeatedly stated, that space is a form of intuition. But there is no discrepancy at all in also saying that he taught space to be a "form of thought," since every student of Kant knows that intuition without thought is mere sensuous *impression*. Kant considered the mind under three aspects, Sensibility, Understanding, and Reason. The *à priori* forms of Sensibility, which rendered Experience possible, were Space and Time: these were forms of thought, conditions of cognition. It was by such forms of thought that he reoccupied the position taken by Leibnitz in defending and amending the doctrine of innate ideas, namely, that knowledge has another source besides sensible experience—the *intellectus ipse*.

While, therefore, any one who spoke of space as a "form of the understanding" would certainly use language which Kant would have disclaimed, Kant himself would have been surprised to hear that space was not held by him as a "form of thought."

George Henry Lewes.

January 3.

The following paragraphs, I believe, faithfully render sundry passages of Kant's writings:

"Objects are given to us by means of sense (Sinnlichkeit), which is the sole source of intuitions (Anschauungen); but they are thought by the understanding, from which arise conceptions (Begriffe)." (*Kritik*, p. 55, Hartenstein's edition.)

* From *Nature*, Vol. I. (1869—70). See [p. 655, l. 9].

"The understanding is the faculty of thought. Thought is knowledge by means of conception." (*Ibid.* p. 93.)

"The original consciousness of space is an intuition *à priori*, and not a conception (Begriff)." (*Ibid.* p. 60.)

"Space is nothing else than the form of all the phenomena of the external senses; that is, it is the subjective condition of sense, under which alone external intuition is possible for us." (*Ibid.* p. 61.)

"Our nature is such, that intuition can never be otherwise than sensual (sinnlich); that is, it only contains the modes in which we are affected by objects. On the other hand, the power of thinking the object of sensual intuition, is the understanding. Neither of these faculties is superior to the other. Without sense, no object would be given us, and without understanding none would be thought. Thoughts without contents are empty, intuitions without conceptions (Begriffe) are blind." (*Ibid.* p. 82.)

"Time and space are 'mere forms of sense'" (Formen unserer Sinnlichkeit, *Prolegomena*, p. 33) and "mere forms of intuition." (*Kritik*, p. 76.)

With these passages before one, there can be no doubt that that thorough and acute student of Kant, Dr Ingleby, was perfectly right when he said that Kant would have repudiated the affirmation that "space is a form of thought." For in these sentences, and in many others which might be cited, Kant expressly lays down the doctrine that thought is the work of the understanding, intuition of the sense; and that space, like time, is an intuition. The only "forms of thought" in Kant's sense, are the categories.

T. H. HUXLEY.

January 14.

I do not believe Professor Sylvester has been betrayed, as Mr G. H. Lewes asserts, into any misconception of this matter by me.

When Kant, at the outset, says, "Alles Denken aber muss sich, es sei geradezu oder im Umschweife, vermittelst gewisser Merkmale, zuletzt auf Anschauungen...beziehen," it would take the veriest dunderhead not to see that all forms of intuition must be, indirectly at least, forms of thought. I never dreamed of disputing so obvious a position. But I object to the phrase, "forms of thought," as designating Space and Time, on the ground of precision. They are *peculiarly* forms of general Sense, and not forms of Thought *as Thought*. Kant, I believe, eschewed the phrase in that sense, and, for all I see, might for the same reason have disclaimed it.

C. M. INGLEBY.

ILFORD, *January* 14.

It is not *my* habit "when objections are made to what I have written, silently to correct my error or silently disregard the criticism." If the objections are well founded, I think it due to the cause of truth to make a frank confession of error, and in the opposite case to reply to the objections.

With reference, then, to Mr Lewes's strictures in *Nature's* last number, I beg to say that Dr Ingleby has "betrayed" me into no error. If I have fallen into error, it is with my eyes open, and after satisfying myself by study of Kant, that to speak of Space and Time, whether as forms of understanding, or as forms of thought, is an unauthorised and misleading mode of expression. Space and Time are forms of sensitivity or intuition. The categories of Kant (so essentially in this point differing from those of Aristotle) do not contain Space and Time among them, and are properly called forms of understanding or thought.

To the existence of thought the operation of the understanding is a necessary preliminary.

Sensibility and intuition are antecedent to any such operation.

Can Mr Lewes point to any passage in Kant where Space and Time are designated *forms of thought?* I shall indeed be surprised if he can do so—as much surprised as if Mr Todhunter or Mr Routh, in their Mechanical Treatises, were to treat *energy* and *force* as convertible terms. To such a misuse of the word energy it would be little to the point to urge that *force without energy is a mere potential tendency*. It is just as little to the point in the matter at issue, for Mr Lewes to inform the readers of *Nature* that *intuition without thought is mere sensuous impression.*

Dr Ingleby has rendered, in my opinion, a very great service to the English reading public, by drawing attention to so serious and prevalent an error as that of confounding the categories (the proper forms of thought *as thought*) with Space and Time, the forms of intuition, the sentinels, so to say, who keep watch and ward outside the gates of the Understanding.

J. J. SYLVESTER.

ATHENAEUM CLUB, *January* 15.

Although I do not feel myself called upon to modify in the least what was said in my former letter on this subject, the three letters which appear to-day in answer to it are too important to be left unnoticed.

The case is briefly this: In the *History of Philosophy* I had to expound Kant's doctrine, and to criticise it, not only in itself, but in reference to the great question of the origin of knowledge. In the pages of exposition I *uniformly* speak of Space and Time as forms of Intuition; no language can be plainer. I also mark the distinction between Sensibility and Understanding, as that of Intuition and Thought. After enumerating the Categories, I add, "In those Categories Kant finds the pure forms of the Understanding. They render Thought possible."

But when, ceasing to expound the system, I had to criticise it, and especially to consider it in reference to the great question; there was no longer any need to adhere to a mode of expression which would have been obscure and misleading. I therefore *uniformly* class Space and Time *among the forms of Thought*, connecting them with the doctrine of Necessary Truths and Fundamental Ideas, which, according to the *à priori* school, are furnished ready-made—brought by the Mind as its native dowry, not evolved in it through Experience.

Now the question is, Have I put language into Kant's mouth which he would disclaim, or is such language misleading? That Kant would have said the language was not what he had employed, I freely admit; but that he would have disclaimed it as misrepresenting his meaning, I deny. I was not bound to follow his language when the task of exposition was at an end; but only bound not to translate his opinions into language which would distort them.

In classing Space and Time *among* the Forms of Thought, I classed them *beside* the Categories of the Understanding and the Ideas of Reason, that is, the purely intellectual conditions existing *à priori* in the Mind. The Mind is said by Kant to be endowed with three faculties—Sensibility, Understanding, and Reason. The activity of the Mind is threefold—Intuitive Thought, Conceptive or Discursive Thought, and Regulative Thought. There could not be an equivoque in my using the word Thought in its ordinary philosophical acceptation as expressive of all mental activity whatever, exclusive of mere

sensation; although Kant assigns a more restricted meaning in his technical use of the word, that is, what we call Logic. And that Kant *meant* nothing opposed to the ordinary interpretation is obvious. It is obvious because, as I said in my former letter, Intuition without Thought is mere sensuous impression. Mr Sylvester demurs to this, so I will show it in a single citation:—"In the transcendental Aesthetic," says Kant, "we will first isolate Sensibility by separating from it all that the Understanding through its concepts thinks therewith, so that nothing but empirical Intuition remains. Secondly, we will lop off from this empirical Intuition everything relating to sensation (*Empfindung*); so that thereby nothing will remain but pure Intuition and the mere form of phenomena, which is the one thing that Sensibility can furnish *à priori*. By this investigation it will appear that there are two pure forms of sensuous Intuition which are *à priori* principles of Cognition." (*Kritik*, § 1, ed. Hartenstein, p. 61.)

Mr Sylvester correctly says, that Intuition and Thought are not convertible terms. But he is incorrect in assuming that they differ as potential and actual; they differ as species and genus; therefore, whatever is a form of Intuition, though not a form of Logic, must be a form of Thought; unless intuitive Thought be denied altogether. How little Kant denied it is evident in every section of his work. In asserting that Space and Time as Intuitions belong to the subjective constitution of the Mind—*subjectiven Beschaffenheit unseres Gemüths* (p. 62)—he expresses this; but it is unequivocally expressed in the following definition:—"A perception, when it refers solely to the subject, as a modification of its states, is *sensation*, an objective perception is *cognition*: *this is either Intuition or Concept*, 'intuitus vel conceptus.'" (*Kritik*, p. 294.) Is not thought implied in cognition? Again:—"The proposition 'I think' is an undetermined empirical Intuition, that is, Perception; consequently, it proves that Sensation, which belongs to Sensibility, must lie at the basis of this proposition....I do not mean thereby that the 'I' in the 'I think' is an empirical representation (*Vorstellung*), on the contrary, it is *purely intellectual, because it belongs to thought in general*. But without some empirical representation which would give Thought its material there could be no such act of Thought as the 'I think'" (p. 324, *note*).

"Man is always thinking," says Hegel, "even when he has nothing but intuitions"—*denkend ist der Mensch immer auch wenn er nur anschaut*. (*Encyclop.* § 24.)

If, because Kant has a restricted use of the term Thought, all who venture on the more ordinary use are said to misrepresent his philosophical meaning, I must call upon those who criticise this laxity to refrain henceforth from speaking of Reason as Thought, since Kant no less excluded Reason from the province of the Understanding. If "the only forms of thought, in Kant's sense, are the Categories," this sweeps away Reason on the one side, as it sweeps away Sensibility on the other; and Ideas are not more correctly named Thoughts than Intuitions are. Kant, it is true, speaks of the concepts of Reason, and defines an Idea to be a "Vernunftbegriff" (page 294); but Kant, equally and in a hundred places, speaks of the "concept of Space" (Begriff des Raumes). The truth is, as already intimated, that in spite of his technical restriction of Thought to the formation of concepts, he recognised intuitive and regulative Thought no less than discursive Thought; nor would his system have had any coherence without such a recognition. Why does he call his work the *Critik of Pure Reason*, unless he intended to display the common intellectual ground of Sensibility, Understanding, and Reason? and does not the word Thought, in ordinary philosophical language mean this activity of the Intellect? When, by Sir W. Hamilton, Dr Whewell, Mr Spencer, and myself, the phrase Forms of Thought is used, does not every reader understand it as meaning Forms of intellectual activity?

In conclusion, I affirm that in the ordinary acceptation of the term Thought—the activity of the Mind—Space and Time as forms of Intuition are forms of Thought, conditions of mental action; and to suppose that because Kant's language is different, his meaning is misrepresented by classing forms of Intuition among the forms of Thought is to misunderstand Kant's doctrine and its purpose.

GEORGE HENRY LEWES.

January 22.

Dr Ingleby, I should think, is quite entitled to say not only that Kant might, but that he would, have disclaimed the phrase Form of Thought as applied to Space or Time taken simply. The remark of Mr Lewes, that "intuition without thought is mere sensuous impression,"—or, as it might have been put, that phenomena of sense (constituted such in the forms of Space and Time) must further be thought under Categories of Understanding, before they can be said to be known or to become intellectual experience—cannot be a sufficient reason for making a Form of Thought proper out of a Form of Intuition.

There is, nevertheless (and Mr Lewes does not fail to suggest it), a sense in which, when taken along with the Categories of the Understanding, and with or without the Ideas of the Reason, the Forms of Intuition may be spoken of as Forms of Thought: Thought being understood, with the same extension that Kant himself gives to Reason in the title (not the body) of his work, as equivalent to faculty of Knowledge in general. It is in this sense that Kant calls all the forms alike, *à priori* principles of Knowledge; and the ambiguity of the word Thought is so well recognised that the English writers arraigned by Prof. Sylvester take no great liberty, when for their purpose, which commonly is the discussion of the general question as to the origin of Knowledge, they talk generally of Kant's "Forms of Thought." If, indeed, any of them ever speaks of Space as a "form of the Understanding," which was part of the original charge, the case is very different; Kant being so careful with his *Verstand*. But Mr Lewes, at least, would never be caught speaking thus, even though his main reason for merging Intuition in Thought might seem to justify this also.

G. CROOM ROBERTSON.

UNIVERSITY COLLEGE, *January* 22.

You will perhaps permit me to make a remark on a controversy at present going on in your columns. There has seldom, I believe, been a grosser or more misleading perversion of the Critical Philosophy than ascribing to Kant the view that Space and Time are in any meaning of the terms "forms of thought." One of his chief grounds of complaint against Leibnitz is, that the latter "intellectualised these forms of the sensibility" (Meiklejohn's translation of the *Critick*, p. 198): and lest the import of this assertion should be mistaken, he explicitly tells us that "Space and Time are not merely forms of sensuous intuition, but intuitions themselves" (Meiklejohn's trans., p. 98): that is, *sensuous* intuitions, as he has been just before asserting that all human intuitions must be. It is precisely on this distinction of pure sensibility and pure thought that Kant founds the possibility of Mathematics—a science which could never be derived from a mere analysis of the concepts employed, but only from the construction of them in intuition. He ridicules, for example, the idea of attempting to deduce the proposition, "Two right lines cannot enclose a space," from the mere concepts or notions of a straight line and the number two. "All your endeavours," says he, "are in vain, and you find yourself compelled to have recourse to intuition, as in fact Geometry always does."

(Meiklejohn, p. 39: see also his long contrast of Mathematical and dogmatical methods in the beginning of the *Methodology.*) And not only is Kant's Mathematical theory founded on this distinction, but his Physical theory also, since it is only by means of pure intuition that he connects pure thought with sensations (see the *Schematism*, and still more the *General Remark on the System of Principles*, Meiklejohn, pp. 174—7); and when he fails to make out this connection he regards the Ideas of Pure Reason as possessed of no objective validity (Transcendental Dialectic). In the first edition of the *Critick* he went still further, and in his remarks on the Second Paralogism of Rational Psychology he speaks of "that something which lies at the basis of external phenomena, which so *affects* our *sense* as to give it the representations of *space*, matter, *form*, &c." And while he abbreviated his discussion in the second edition, he tells us in his preface that he found nothing to alter in the views put forward in the previous one.

I might quote whole pages of the *Critick* in proof of these views, but I ought rather to apologise for writing so much after the letters which you have already published. I believe the mistakes as to Kant's doctrine of Space and Time, his refutation of Idealism, and his discussion of the Antinomies of the Pure Reason, are almost without a parallel in the History of Philosophy.

W. H. Stanley Monck.

Trinity College, *January* 22.

In answer to my invitation, Mr Lewes now "freely admits that Kant nowhere speaks of Space and Time as 'Forms of Thought,'" but still contends that "Kant would not have disclaimed such language, as misrepresenting his meaning." As well might he argue that although Euclid never uses the word *epipedon* (our English word *plane* or *plain*), to signify a curved surface (ἐπιφάνεια), he would not have remonstrated against the use of the term *cylindrical epipedon* or conical *epipedon*, to denote the surface of a cylinder or cone, in a professed exposition or criticism of his Elements of Geometry, because in common life we speak of rough or undulating plains, or because a plane admits of being bent into the shape of a cylindrical or conical surface. I think the ladies who are getting up their Planes and Solids at St George's Hall would be of a different opinion from Mr Lewes in this matter, and with good reason on their side.

Mr Lewes, reiterating a statement contained in his previous letter, goes out of his way to affirm that he "uniformly speaks of Space and Time as forms of Intuition in his pages of exposition" of Kant's doctrine in his *History of Philosophy*. Were the fact so, it would not in any material degree excuse the inaccuracy of subsequently styling them "Forms of Thought"; and, moreover, the real point at issue is not Mr Lewes's general accuracy or inaccuracy, but whether a mode of speech which he, along with others, employs, is right in itself and ought to be persisted in.

However, as Mr Lewes has thought fit to put in a sort of plea in mitigation of former wrong-doing, I have taken the trouble of looking through his *exposition* and *criticism* of Kant in his *History* (ed. 1867) and in no single instance have I come upon the phrase *forms of intuition* applied to Space and Time, either in the one or the other; although he states he has *uniformly spoken of them* as such in the former. I have marked the word *intuitions* as occurring once, and *forms of sensibility* several times, but *forms of intuition* never. If *form of sensibility* is as good to use as *form of intuition*, *form of understanding* ought to be as good as *form of thought*; but Mr Lewes owns that the former is indefensible, whilst he avers that the latter is correct. If Mr Lewes has ever called Space and Time *forms of intuition* in the *History*, it will be easy for him to set me right by quoting the passage where the phrase occurs, although that circumstance would not in any

degree better his own position, and still less excuse the assertion of his *uniform* use of the term.

If Mr Lewes cannot quote correctly from his own writings, it will surprise nobody that he misquotes the language of an opponent. He repeats, "Intuition without thought is mere sensuous impression," and adds, "Mr Sylvester demurs to this." My words are (*Nature*, Jan. 13, 1870): "To such a misuse of the word energy it would be little to the point to urge that *force without energy is mere potential tendency*. It is just as little to the point in the matter at issue for Mr Lewes to inform the readers of *Nature* that *intuition without thought is mere sensuous impression*." So that, according to Mr Lewes, to say that a proposition is *little to the point* is *demurring to its truth*.

I should not hesitate to say if some amiable youth wished to entertain his partner in a quadrille with agreeable conversation, that it would be *little to the point*, according to the German proverb, to regale her with such information as how

"Long are the days of summer-tide,
And tall the towers of Strasburg's fane,"

but should be surprised to have it imputed to me on that account that I demurred to the proposition of the length of the days in summer, or the height of Strasburg's towers.

In another passage, Mr Lewes gives me credit for "saying correctly that Intuition and Thought are not convertible terms"—a platitude I never dreamed of giving utterance to; but that I am "incorrect in assuming that they differ as potential and actual"—words which, or the like of which, in any sort or sense, never flowed from my pen. Surely this is not fair controversy, to misquote the words and allegations of an opponent. It seems to me too much like fighting with poisoned weapons. I decline to continue the contest on such terms; and, passing over Mr Lewes's very odd statement about *species* and *genus* with reference to Intuition and Thought, shall conclude with expressing my surprise at his and Mr G. C. Robertson's confident assumption that Kant uses in the title of his book *pure reason* in a far wider sense than in the body of his work, simply because to arrive at the Pure Reason he has to *go through* the Critick of the Sensibility and of the Understanding. If in a history of the Reign of Queen Victoria the author should find it expedient to go back to the times of the Norman and Saxon conquests, would it be right to infer therefrom that he used in his title-page the name Victoria in a generalised sense, to include not only her most Gracious Majesty, but also the Tanner's daughter and Princess Rowena?

Perhaps by this time many of the Naturalistic readers of the journal who regard the human intelligence as forming no part of the scheme of Nature, wish Space at the bottom of the sea; but the more the subject is canvassed, and the greater the number of English authorities brought forward to back up Mr Lewes in wresting the words of Kant from their proper scientific signification, the higher meed of praise seems to me to accrue to Dr Ingleby for stemming the tide of depravation, and banishing, as I feel confident this discussion will have the effect of doing, from the realm of English would-be philosophy, such a loose and incautious way of talking as that of giving to Space and Time the designation which the Master has appropriated to the categories of his system, and to them alone.

J. J. Sylvester.

P.S.—I should be doing injustice to the very sincere sentiments of respect I entertain for Mr Lewes's varied and brilliant attainments (which constitute him a kind of link between the material and spiritual sides of Nature), and of gratitude for the pleasure the perusal of his *History of Philosophy* has afforded me, were I to part company with

him without disclaiming all acrimony of feeling, if perchance any too strident tones should have seemed to mingle with my enforced reply. In naming him in the original offending footnote (the fountain of these tears), my purpose was simply to emphasise the necessity of protesting against what seemed to me an unsound form of words, *apropos* of Kant, which went on receiving countenance from such and so eminent writers as himself and the others named; and I should be false to my own instincts did I not at heart admire the courageous spirit with which, almost unaided and alone (like a good, I meant to say valiant knight of old), he has done his best to defend his position and maintain his ground against all oppugners.

J. J. S.

I am quite willing to leave the readers of *Nature* and the students of Kant to decide on the propriety, in English philosophical discourse, of calling Space and Time "forms of Thought," the more so as Sir W. Hamilton—a great stickler for philosophic precision—uses the term in that sense and would have been surprised to hear that he had misrepresented Kant in so doing. My opponents persist in limiting the term Thought to the restricted meaning given to it in Kant's terminology, which, in English, is restricting it to Conception or Judgment: on this ground they might deny that Imagination or Recollection could be properly spoken of as Thought. Throughout I have accepted Thought as equivalent to mental activity in general and the "forms of Thought" as the conditions of such activity. The "forms of Thought" are the forms which the thinking principle (Kant's *pure* Reason) brings with it, antecedent to all experience. The thinking principle acts through three distinct faculties: Sensibility (Intuition), Understanding (Conception), and Reason (Ratiocination): to suppose Thought absent from Intuition, is to reduce Intuition to mere sensuous impression. Therefore, whatever is a form of Intuition must be a form of Thought.

The following passage from Mr Mahaffy's valuable translation of Kuno Fischer's work on Kant, may here be useful: "Sensibility and understanding are cognitive faculties differing not in degree but in kind, and form the *two original faculties of the human mind*"... The general problem of a Critick of the Reason "is subdivided into two particular objects, as human Reason is into two particular faculties of knowledge. The first object is the investigation of the sensibility; the second, that of the understanding. The first question is, How is rational knowledge possible through sensibility? The second question, How is the same knowledge possible through the understanding?" (pp. 4, 5.)

Those who maintain that it is improper to speak of Space and Time as forms of Thought, must either maintain that Kant held Sensibility *not* to be a faculty of the Mind (thinking principle); or that the term Thought is *not*, in English discourse, a correct expression for the activity of the thinking principle. I believe that the student will agree with me in saying that, although Kant restricted the term Thought to what we call Conception or Judgment, he understood by the activity of the mental faculties (Pure Reason) what we understand by Thought.

It is not, however, to continue this discussion that I again trespass on your space; but to reply to the personal part of Mr Sylvester's letter. He charges me with misquoting myself and with misquoting him. I said that, in my exposition, Space and Time were uniformly spoken of as forms of Intuition and I say so still. Mr Sylvester has taken the trouble of reading that exposition without taking the trouble of understanding it; he declares that he "has marked the word intuition as occurring once and forms of sensibility several times; but forms of intuition never." His *carefulness* may be estimated by the fact that the word intuition occurs *four* times on the two pages: his *comprehension*

by the fact that it is perfectly indifferent whether Sensibility or Intuition be the term employed, since sensibility is the faculty and Intuition the action of that faculty. Mr Sylvester, not understanding this, says, "If form of sensibility is as good to use as form of intuition, form of understanding ought to be as good as form of thought; but Mr Lewes owns that the former is indefensible, whilst he avers that the latter is correct." Considering that this passage occurs in a letter which charges me with unfair misquotation, it is curious. So far from owning that the former is "indefensible," it is what I declare to be true; and, with regard to the latter, though I do think a form of Understanding is a form of Thought, my statement was altogether *away* from it, namely, that Space and Time as forms of Sensibility, would be incorrectly spoken of as forms of the Understanding.

With regard to the alleged misquotation of his own words, which he characterises as unfair and as "too much like fighting with poisoned weapons," it was a charge which both astonished and pained me. There are few things for which I have a bitterer contempt than taking such unfair advantages of an adversary. I beg to apologise to Professor Sylvester for any misrepresentation which, unintentionally, I may have been guilty of. But, in accepting his denial of the construction I placed upon his language, I must still say that, after re-reading his letter I am at a loss to see what other construction it admits of, that has any bearing on the dispute, and that he has not expressed his meaning with sufficient clearness. Intuition and Thought are there compared with Force and Energy as terms "not convertible"; Force is detached from Energy as potential from actual, and Intuition without Thought is made to hold an analogous position. Here is the passage; let the reader judge:

"Can Mr Lewes point to any passage in Kant where Space and Time are designated *forms of thought*? I shall indeed be surprised if he can do so—as much surprised as if Mr Todhunter or Mr Routh in their Mechanical Treatises were to treat *energy* and *force* as convertible terms. To such a misuse of the word energy it would be little to the point to urge that *force without energy is a mere potential tendency*. It is just as little to the point, in the matter at issue, for Mr Lewes to inform the readers of *Nature* that *intuition without thought is mere sensuous impression*."

Is it to use "poisoned weapons" to interpret this as assuming that Intuition and Thought differ as potential and actual? I repeat that, since Mr Sylvester disclaims the interpretation, my only course is to apologise for it; but, after his own misinterpretations of me, he will not, I hope, persist in attributing mine to a desire to take an unfair advantage. If I make no reply to the other points raised in the various letters it is in order not to prolong the discussion.

George Henry Lewes.

I do not know whether Mr Sylvester and Dr Ingleby will be satisfied with Mr Lewes' letter in yours of the 27th. I am not, and I think, in defending his former mistake, Mr Lewes has fallen into additional errors.

It is undoubtedly fair to translate an author into your own language before criticising him, provided you found no criticism on the language that you have put into his mouth. But this I think Mr Lewes has done. He accuses Kant of inconsistency in speaking of pure *à priori* cognitions, when, on his own system, pure thought only supplies one element to these cognitions, the other being derived from sense or intuition. Now (not to insist here that Kant constantly uses the term cognition in a wider sense than that which Mr Lewes insists on fastening upon him), this criticism is evidently invalidated by the simple remark that Kant admits pure intuitions as well as pure concepts, and explains the

nature of mathematics, as a system of *à priori* cognitions, by the fact that its object-matter consists of nothing but pure intuitions.

Mr Lewes now informs us that Kant's Intuition and Thought "differ as species and genus." According to Kant they differ in kind; and Leibnitz was as wrong in making sensibility a species of thought as Locke was in making Thought a species of sensibility. Space and Time, Mr Lewes adds, are forms of "mental activity" and, therefore, are properly termed "forms of Thought," in the meaning of the latter term which is usually current in this country. If they were forms of mental activity they would be forms of Thought, according to Kant, likewise; for the criterion by which Kant distinguishes between Intuition and Thought (under which term he includes both the understanding proper and the reason proper) is that, in the former, the mind is passive (receptive) while, in the latter, it is spontaneously active; and it is precisely on this ground—the passive reception of them by the mind—that he refers Space and Time to Sensibility rather than Thought. This is repeatedly brought out in the Transcendental Deduction of the Categories. See in particular Sections 11 (Meiklejohn, p. 80) and 18 (Meiklejohn, p. 90).

I think if Mr Lewes will turn to the preface to the first edition of the *Critick*, he will see that the transcendental logic only (and perhaps I might limit it to the transcendental dialectic) grapples *directly* with the problem indicated by the title of the book. The Aesthetic is a preliminary inquiry, which proves afterwards of great use; but is not to be considered as a Critick of Pure Reason in this particular department. His using the term "concept" of space, is certainly confusing; but its explanation, I think, is to be found in a passage in the "Transcendental Exposition" of this "concept" (Meiklejohn, p. 25), where he says, "It must be *originally* intuition, for from a *mere* conception no propositions can be deducted which go out beyond the conception, and this happens in geometry." In the preceding page he similarly qualifies his statement that Space is an intuition. "No conception *as such*," he says, "can be so conceived as if it contained, within itself, an infinite multitude of representations." We may *now* have a concept as well as an intuition of Space and Time; but the intuition was the original form of the idea, and it is to the intuition that we must always have recourse in mathematics when we wish to discover a new truth.

I think, if Mr Lewes will again read over the Transcendental Aesthetic and the parts of the Transcendental Analytic which are closely related to it, he will see that Kant never designates the *original* representations of space and time "concepts," or refers their origin to "pure reason."

W. H. Stanley Monck.

Trinity College, Dublin, *January* 29.

[To the foregoing correspondence from *Nature*, Prof. Sylvester adds, in *The Laws of Verse*, the following remarks.]

In order that the reader may judge of the correctness of the assertions made by Mr Lewes in his concluding letter, and his general fairness in controversy, I request attention to the annexed *catena* of passages drawn from the above correspondence*.

No. 1. The Author.

"It is very common, not to say universal, with English writers, even such authorised" (I meant to say authoritative) "ones as Whewell, Lewes or Herbert Spencer, to refer to Kant's doctrine as affirming space to be 'a form of thought,' or 'of the understanding.' This is putting into Kant's mouth words which he would have been the first to disclaim."

* The words in SMALL CAPITALS are in ordinary print in the original passages.

No. 2. Mr G. H. Lewes.

(*a*) "Kant assuredly did teach as Professor Sylvester says, and as I HAVE REPEATEDLY STATED, THAT SPACE IS A FORM OF INTUITION."

(*β*) "Every student of Kant knows that intuition without thought is mere sensuous *impression*."

(*γ*) "While therefore anyone who spoke of space as 'A FORM OF THE UNDERSTANDING' WOULD CERTAINLY USE LANGUAGE WHICH KANT WOULD HAVE DISCLAIMED, Kant himself would have been surprised to hear that space was not held by him as a form of thought."

[(*a*) In no one single instance in his fifty pages of exposition and criticism has Mr Lewes ever once stated that *Space is a Form of Intuition.*]

No. 3. The Author.

"Can Mr Lewes point to any passage in Kant where Space and Time are designated *forms of thought?* I shall indeed be much surprised if he can do so—as much surprised as if Mr Todhunter or Mr Routh in their mechanical treatises were to treat *energy* and *force* as convertible terms. To such a misuse of the word energy it would be as little to the point to urge that *force without energy is mere potential tendency.* It is just AS LITTLE TO THE POINT in the matter at issue for Mr Lewes to inform the readers of *Nature* that *intuition without thought is mere potential tendency.*"

No. 4. Mr G. H. Lewes.

"In the pages of exposition *I uniformly speak of Space and Time as forms of intuition*; no language can be plainer." [In no one single instance does Mr Lewes so speak of Space or Time.]

(*a*) "Mr Sylvester correctly says that intuition and thought are not convertible terms. (*β*) But he is incorrect in affirming that they differ as potential and actual."

[These are words put into my mouth by Mr Lewes, which I disclaim as Kant would have disclaimed the words put into his. I nowhere have stated the truism (*a*). I nowhere have affirmed the absurdity (*β*).]

No. 5. Mr G. Croom Robertson.

"If indeed any of them ever speaks of space as a 'form of the understanding,' which was part of the original charge, the case is very different, Kant being so careful with his *Verstand.* But Mr Lewes at least would never be caught speaking thus."

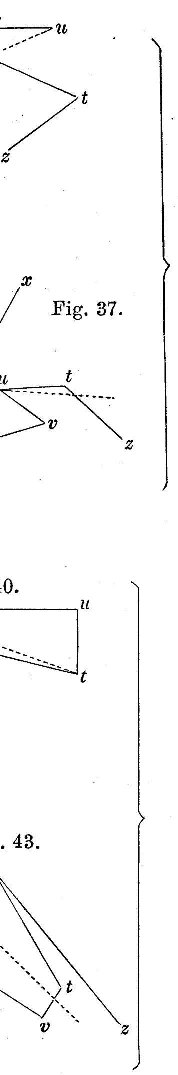
No. 6. The Author.

(*a*) "If form of sensibility is as good to use as form of intuition, form of understanding ought to be as good to use as form of thought (*β*), but Mr Lewes owns that the former is indefensible whilst he avers that the latter is correct."

[In proof of (*β*) above see (*γ*) of No. 2. (*a*) above evidently implies the proportion:

sensibility : intuition :: understanding : thought.

The first and third terms representing faculties, the second and fourth the actions of those faculties respectively.]



. 43.

Perspective Regions of Normal Orders.

Fig. 45.

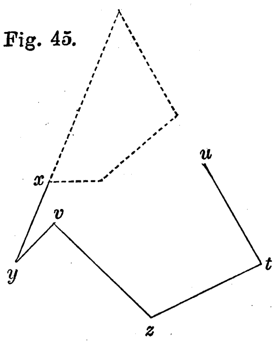


Fig. 46.

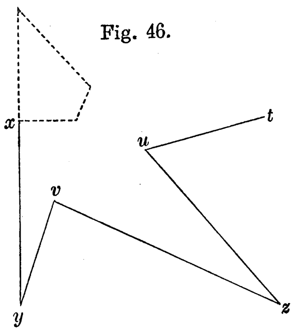


Fig. 50.

(1)

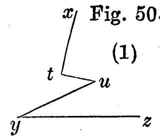


Fig. 51.

(2)

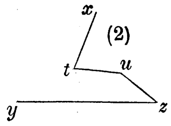
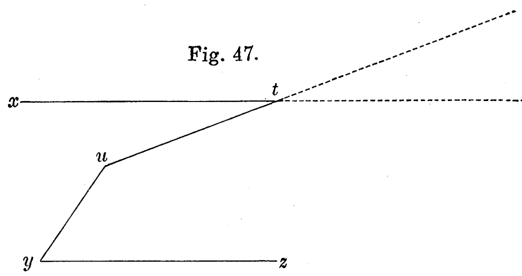
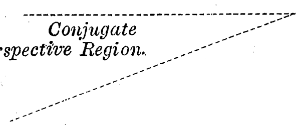
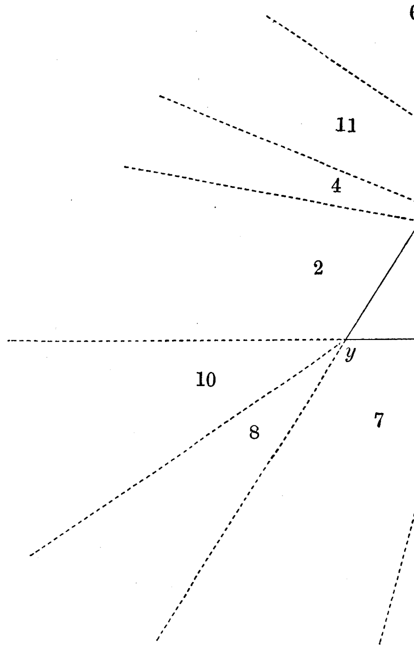


Fig. 47.







Crossed Lines or Lines of Abnormal Order.

Fig. 48.

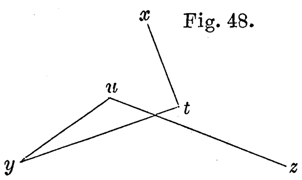


Fig. 49.

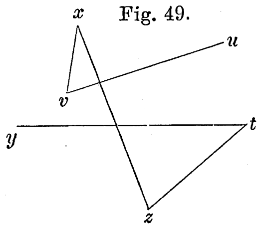


Fig. 55.

(6)

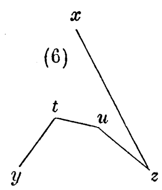


Fig. 56.

(7)